Linear Programming, tropical convexity, and repeated games

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Based on joint work with Akian, Guterman (IJAC 2012) and Allamigeon, Benchimol, Joswig arXiv:1308.0454, arXiv:1309.5925, arXiv:1405.4161.

Connections between:

- linear programming (simplex algorithm, interior points)
- mean payoff games

through tropical geometry.

Some open questions in linear programming

A linear program is an optimization problem:

$$\min c \cdot x;\ Ax \leqslant b,\ x \in \mathbb{R}^n \ ,$$

where $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$.
3 methods

- simplex algorithm (Dantzig), sometimes exponential time, efficient
- ellipsoid (Khachyan), polynomial time, inefficient
- interior points (Karmakar...), polynomial time, efficient

Smale's problem for LP

Question

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polynomial time: $=$ execution time bounded by a polynomial $P(m, n, L)$, $L =$ number of bits to code the A_{ij}, b_i, c_j (sum of their $\log_2$'s).

$\neq$ *strongly polynomial*: number of arithmetic operations bounded by a polynomial $P(m, n)$, **and** the size of operands of arithmetic operations is bounded by a polynomial in L

The simplex method

Principle: iterate over adjacent vertices (basic points) of the polyhedron while improving the objective function

$$c^\top v^1 \geqslant c^\top v^2 \geqslant \ldots \geqslant c^\top v^N$$

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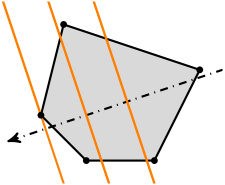

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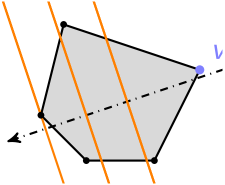

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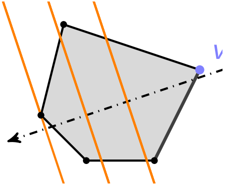

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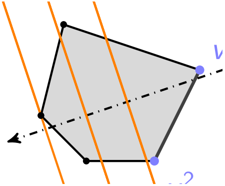

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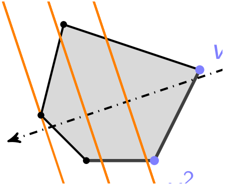

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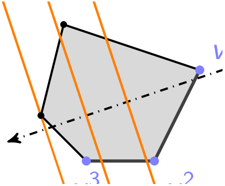

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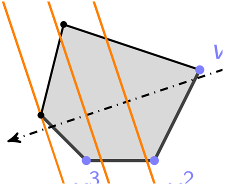

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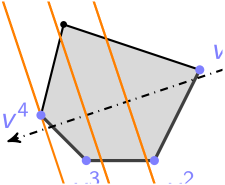

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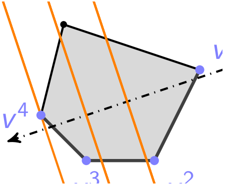

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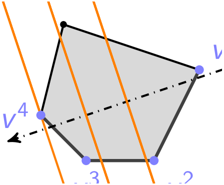

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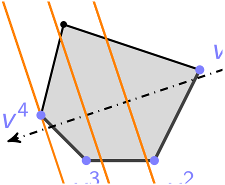

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Example: Dantzig's rule, steepest edge rule, Bland's rule, randomized rules, *etc*

Complexity of the simplex method?

the number of iterations depends on choice of the pivoting rule and every iteration (pivoting from a basic point to the next one) can be done with a strongly polynomial complexity

$\Longrightarrow$ if the number N of iterations is polynomial (in m and n), the overall complexity is *strongly polynomial*.

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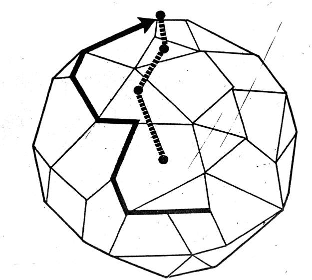
- for most (all?) pivoting rules, there are some counterexamples with super polynomial number of iterations (Klee-Minty cube, *etc*)
- is there a pivoting rule ensuring that the number of iterations in the worst case is polynomially bounded?
- or, equivalently, is there a strongly polynomial simplex algorithm?

Ideal version of Dantzig's problem:

Conjecture (Hirsch)

Any two vertices of the graph of a polytope with m facets in dimension n can be joined by a path of length $m - n$ at most.

Disproved by F. Santos (2012, Ann. Math.), but the conjecture fails only by one unit. Weaker forms of the conjecture, like length $=$ poly(m), still hold.

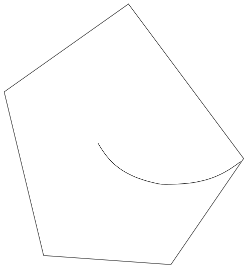


Interior points

For all $\mu > 0$, consider the *barrier problem*

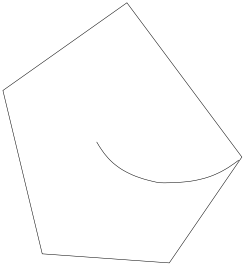
$$\min \mu^{-1} c \cdot x - \sum_{j=1}^{n} \log x_j - \sum_{i=1}^{m} \log w_i, \; Ax + w = b, x > 0, w > 0$$

log strictly concave $\implies$ optimal solution $x(\mu)$ is unique.
$\mu \mapsto x(\mu)$ is the central path. $x(0)$ is the solution of the LP.



Inductive step of interior points:

- $x \leftarrow \text{Newton}(x, \mu)$;
- reduce μ not too quickly so that x remains in an appropriate attraction bassin of Newton's method. Size of attraction bassin is bounded by the inverse of a condition number (Shub-Smale).



Several path following methods. Complexity bounded by the length of the path in a degenerate Riemannian metric (locally inverse of condition number) or by a special curvature integral (Sonnevend).

Continuous analogue of Hirsch's conjecture

An intrinsic complexity measure of the central path is:

$$\text{total curvature} = \int_0^\ell \|\kappa(\tau)\| d\tau$$

$$\kappa = \Phi''$$

$$\Phi = \text{central path parametrized by arclength}$$

Conjecture (Deza, Terlaky and Zinchenko, 2008)

The total curvature of the central path of a polytope with m facets in dimension n is bounded by $O(m)$.

Dedieu, Malajovich, and Shub showed that the total curvature *averaged* over the 2^m LP's with sign conditions $\pm s_i \leqslant 0$ is $O(n)$.
Dedieu and Shub first conjectured that the total curvature is $O(n)$.
Contradicted by Deza, Terlaky, Zinchenko, redundant Klee-Minty cube.

The mean payoff problem

Question (Gurvich, Karzanov, Khachyan 88)

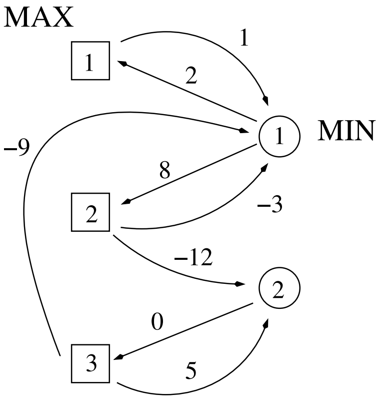
Is there a polynomial time algorithm to solve a mean payoff deterministic game?

Mean payoff (deterministic) games

$G = (V, E)$ bipartite graph. r_{ij} price of the arc $(i, j) \in E$.

"Max" and "Min" move a token. The player making the move receives from the other player the paiement written on the arc.

v_i^k value of MAX, initial state (i, MIN).



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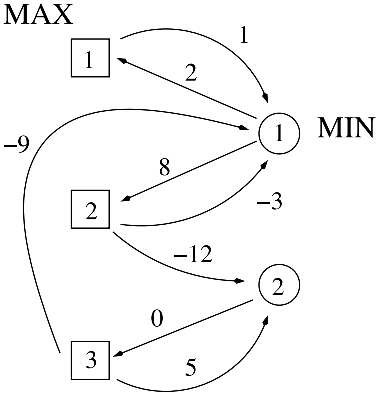
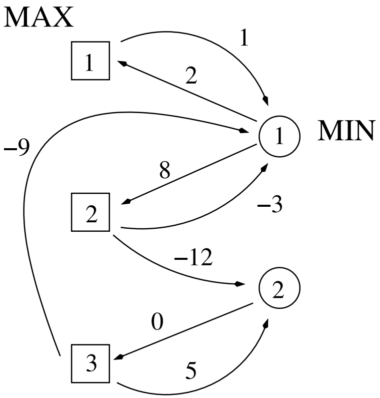


$$\lim_k v^k/k = (-1, 5)$$

v_i^k value of MAX, initial state (i, MIN).

$$v_1^k = \min(-2 + 1 + v_1^{k-1}, -8 + \max(-3 + v_1^{k-1}, -12 + v_2^{k-1}))$$
$$v_2^k = 0 + \max(-9 + v_1^{k-1}, 5 + v_2^{k-1})$$



$$\lim_k v^k/k = (-1, 5)$$

Shapley operator

$$v^k = T(v^{k-1}),\ v^0 = 0, \qquad T : \mathbb{R}^n \to \mathbb{R}^n$$

$$[T(x)]_j = \min_{i\in[m]} \Big(- r_{ji} + \max_{l\in[n]}(r'_{il} + x_l)\Big)$$

Mean payoff vector

$$\chi(T) = \lim_{k\to\infty} T^k(0)/k = \lim_{k\to\infty} v^k/k \ .$$

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T is order preserving, additively homogeneous $\Rightarrow$ sup-norm nonexpansive:

$$x \leqslant y \implies T(x) \leqslant T(y)$$
$$T(\alpha + x) = \alpha + T(x), \qquad \forall \alpha \in \mathbb{R}$$
$$\| T(x) - T(y)\| \leqslant \|x - y\|$$

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Proof: introduce a discount factor $\alpha < 1$, let $v_\alpha = T(\alpha v_\alpha)$ be the discounted value, $\lim_{\alpha \to 1^-}(1-\alpha)v_\alpha$ exists (a bounded semi-algebraic function of one variable has a limit) and is equal to $\chi(T)$.

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Is there a polynomial time algorithm to solve the following problem for deterministic games: is state i winning for MAX? i.e., does $\chi_i(T) \geqslant 0$ *?*

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- policy iteration performs well, but super-polynomial counter example by Friedmann 10

Theorem (Allamigeon, Benchimol, SG, Joswig arXiv:1309.5925)

Any strongly polynomial simplex algorithm equipped with a combinatorial pivoting rule yields a strongly polynomial algorithm to solve mean payoff games.

Combinatorial pivoting rule: the choice of the next vertex depends on (A, b, c) only through signs of minors of the matrix

$$\begin{pmatrix} A & b \\ c^\top & 0 \end{pmatrix}$$

This includes signs of reduced costs. Bland's rule is combinatorial.

$\rightarrow$ positive answer to Dantzig implies positive answer to Gurvich, Karzanov, Khachyan.

The central path can be tortuous

Theorem (Allamigeon, Benchimol, SG, Joswig arXiv:1405.4161)

There is a family of linear programs with $3r + 4$ inequalities in dimension $2r + 2$ where the central path has a total curvature greater than $2^r/(3r)$.

This disproves the continuous analogue of Hirsch's conjecture of Deza, Terlaky, Zinchenko.

The proof uses max-plus or tropical algebra, actually, tropical linear programming.

Max-plus or tropical algebra

In an exotic country, children are taught that:

$$“a + b” = \max(a, b) \qquad “a \times b” = a + b$$

So

- $“2 + 3” =$

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$$\text{“}\begin{pmatrix} 7 & 0 \\ -\infty & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}\text{”} =$$

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$$\text{``}\begin{pmatrix} 7 & 0 \\ -\infty & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}\text{''} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$$

Max-plus / tropical semiring

$$\mathbb{R}_{\mathsf{max}} = (\mathbb{R} \cup \{-\infty\}, \mathsf{max}, +)$$

- Cuninghame-Green 1960- OR (scheduling, optimization)
- Vorobyev ∼65 ... Zimmerman, Butkovic; Optimization
- Maslov ∼ 80'- ... Kolokoltsov, Litvinov, Samborskii, Shpiz... Quasi-classic analysis, variations calculus
- Simon ∼ 78- ... Hashiguchi, Leung, Pin, Krob, ... Automata theory
- Gondran, Minoux ∼ 77 Operations research
- Cohen, Quadrat, Viot ∼ 83- ... Olsder, Baccelli, S.G., Akian discrete event systems, optimal control, idempotent probabilities, linear algebra
- Nussbaum 86- Nonlinear analysis, dynamical systems, also related work in linear algebra, Friedland 88, Bapat ˜94
- Kim, Roush 84 Incline algebras
- Fleming, McEneaney ∼00- max-plus approximation of HJB
- Del Moral ∼95 Puhalskii ∼99, idempotent probabilities.

Since 2000' in pure maths, tropical geometry: Viro, Mikhalkin, Passare, Sturmfels ..., recent work by Connes, Consani

The sister algebra: min-plus

$$\text{“}a + b\text{”} = \min(a, b) \qquad \text{“}a \times b\text{”} = a + b$$

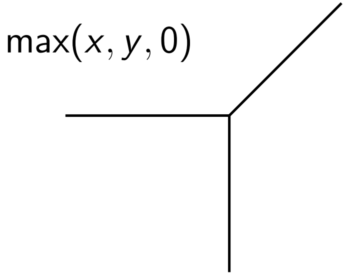
- $\text{“}2 + 3\text{”} = 2$
- $\text{“}2 \times 3\text{”} = 5$

Some elementary tropical geometry

A tropical line in the plane is the set of (x, y) such that the max in

$$\text{“}ax + by + c\text{”}$$

is attained at least twice.



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$$\max(a + x, b + y, c)$$

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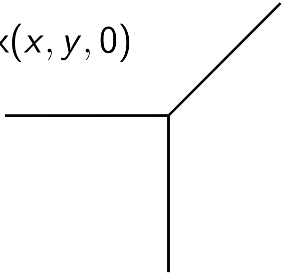
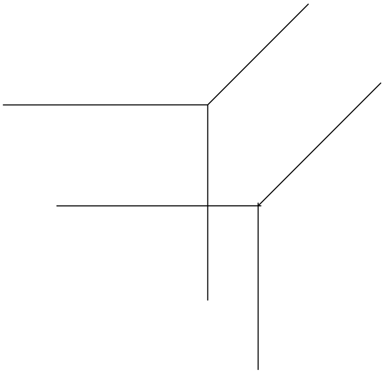
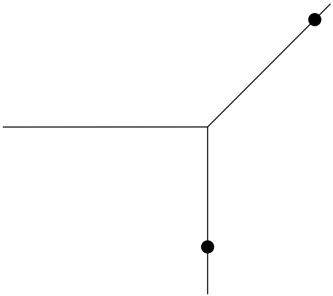
$\max(x, y, 0)$



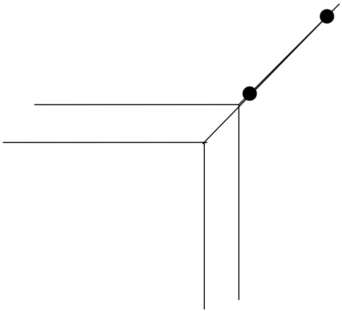
Two generic tropical lines meet at a unique point



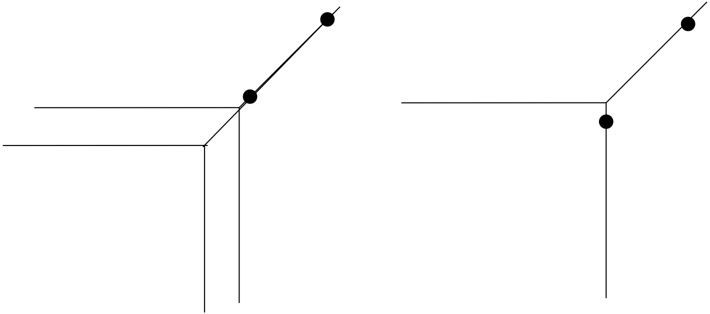
By two generic points passes a unique tropical line



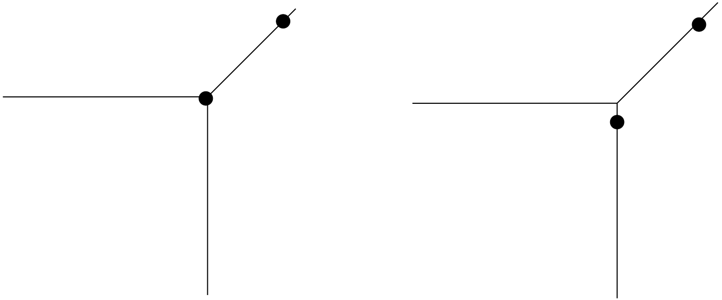
non generic case



non generic case resolved by perturbation



non generic case resolved by perturbation



Nonarchimedean valuation point of view

Let $\mathbb{C}\{\{t\}\}$ denote the field of Puiseux series, equipped with the valuation $\operatorname{val} s = -$ smallest exponent of s, $\mathbb{C}\{\{t\}\} \to \mathbb{R}_{\max}$;

E.g., $\operatorname{val}(t^{-1/2} - t + 7t^{3/2} + \dots) = 1/2$

$\operatorname{val}(z_1 + z_2) \leqslant \max(\operatorname{val}(z_1), \operatorname{val}(z_2))$, with equality when leading coeffs dont cancel (e.g. if $\operatorname{val}(z_1) \neq \operatorname{val}(z_2)$).

$\operatorname{val}(z_1 z_2) = \operatorname{val}(z_1) + \operatorname{val}(z_2)$

The rational points of tropical hyperplanes are images of hyperplanes of $(\mathbb{C}\{\{t\}\})^n$ by the valuation, see:

Theorem (Kapranov)

Given $p = \sum_\alpha p_\alpha z^\alpha \in \mathbb{C}\{\{t\}\}[z_1, \ldots, z_n]$, *and* $Z \in \mathbb{Q}^n$,

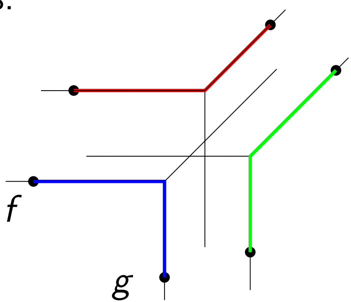
$$\exists z \in (\mathbb{C}\{\{t\}\})^n, \qquad p(z) = 0, \qquad Z = \operatorname{val} z$$

iff

$$\max_\alpha \operatorname{val} p_\alpha + \langle \alpha, Z \rangle \text{ attained twice}$$

Restriction to $\mathbb{Q}$ can be avoided by working with Puiseux series with real exponents (Hahn / Hardy / van den Dries / Markwig), definable in $\mathbb{R}_{an,*}$ o-minimal model (Van Den Dries, Alessandrini).

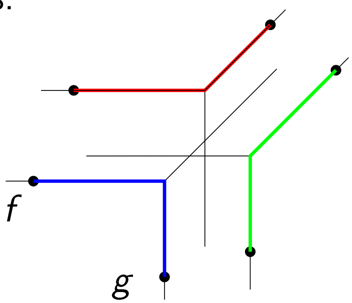
Tropical segments:



$$[f, g] := \{\text{“}\lambda f + \mu g\text{”} \mid \lambda, \mu \in \mathbb{R} \cup \{-\infty\},\ \text{“}\lambda + \mu = 1\text{”}\}.$$

(The condition “$\lambda, \mu \geqslant 0$” is automatic.)

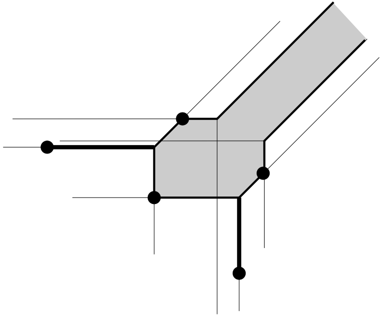
Tropical segments:



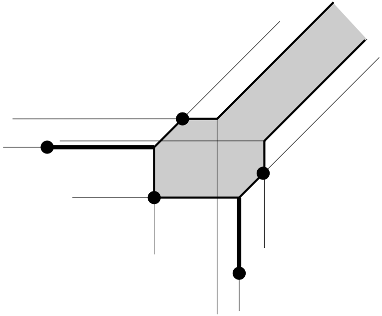
$[f,g] := \{\, \sup(\lambda + f, \mu + g) \mid \lambda, \mu \in \mathbb{R} \cup \{-\infty\},\ \max(\lambda,\mu) = 0\}.$

(The condition $\lambda, \mu \geqslant -\infty$ is automatic.)

Tropical convex set: $f, g \in C \implies [f, g] \in C$



Tropical convex set: $f, g \in C \implies [f,g] \in C$



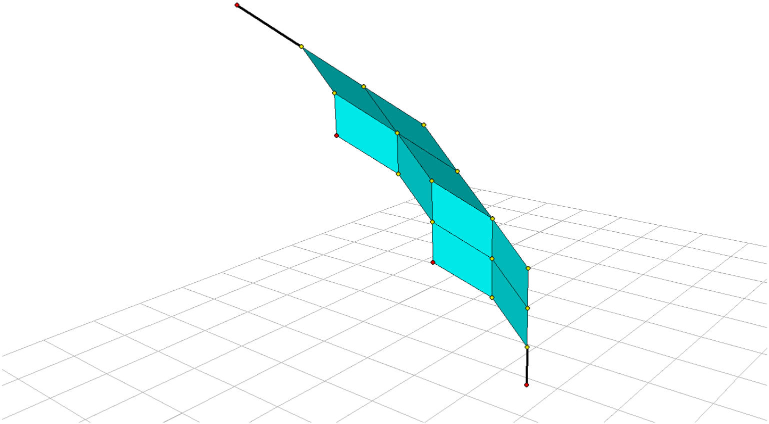
Tropical convex cone: ommit "$\lambda + \mu = 1$", i.e., replace $[f,g]$ by $\{\sup(\lambda + f, \mu + g) \mid \lambda, \mu \in \mathbb{R} \cup \{-\infty\}\}$

Homogeneization

A convex set C in $\mathbb{R}^n_{\max}$ is a cross section of a convex cone $\hat{C}$ in $\mathbb{R}^{n+1}_{\max}$,

$$\hat{C} := \{(\lambda + u, \lambda) \mid u \in C, \lambda \in \mathbb{R}_{\max}\}$$

A tropical polytope with four vertices



Structure of the polyhedral complex: Develin, Sturmfels

Tropical half-spaces

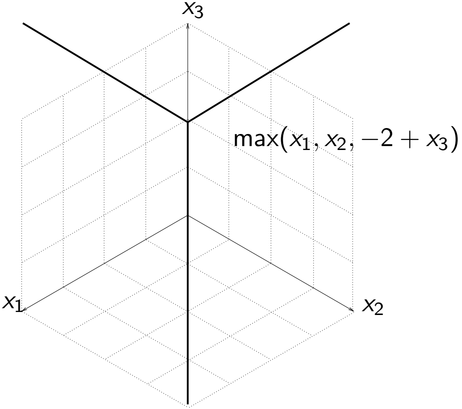
Given $a, b \in \mathbb{R}^n_{\max}$, $a, b \not\equiv -\infty$,

$$H := \{x \in \mathbb{R}^n_{\max} \mid \text{“}ax \leqslant bx\text{”}\}$$

Tropical half-spaces

Given $a, b \in \mathbb{R}^n_{\max}$, $a, b \not\equiv -\infty$,

$$H := \{x \in \mathbb{R}^n_{\max} \mid \max_{1 \leqslant i \leqslant n} a_i + x_i \leqslant \max_{1 \leqslant i \leqslant n} b_i + x_i\}$$



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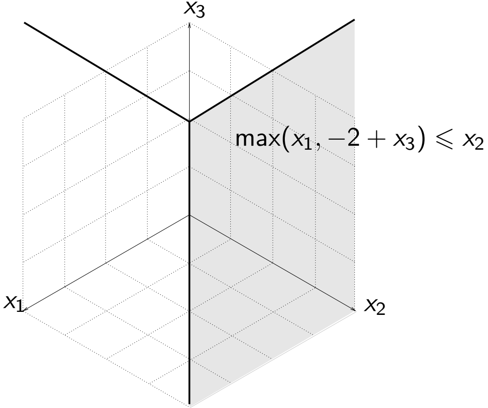

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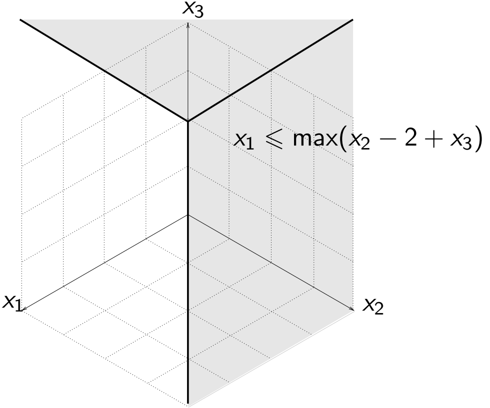

Tropical half-spaces

Given $a, b \in \mathbb{R}^n_{\max}$, $a, b \not\equiv -\infty$,

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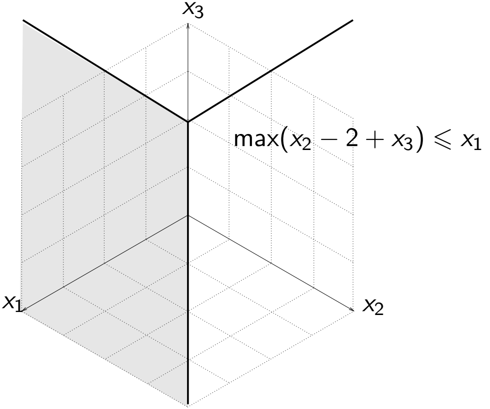
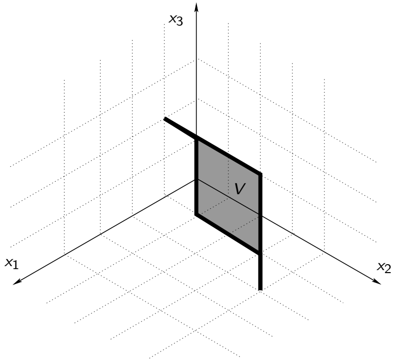


Tropical polyhedral cones

can be defined as intersections of finitely many half-spaces



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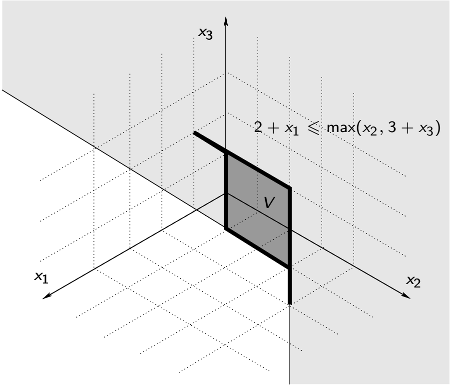

Tropical polyhedral cones

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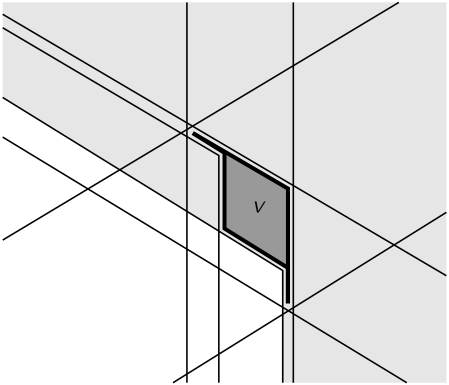
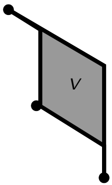


Tropical polyhedral cones

or as the tropical linear combinations of finitely many vectors

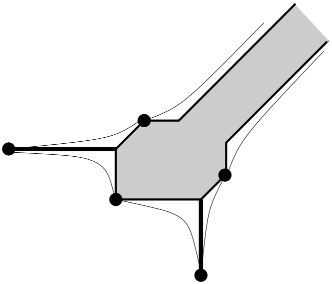


Viro's log-glasses, related to Maslov's dequantization

$$a +_h b := h \log(e^{a/h} + e^{b/h}), \qquad h \to 0^+$$

$$\max(a, b) \leqslant a +_h b \leqslant h \log 2 + \max(a, b)$$

Tropical convex sets are deformations of classical convex sets



$$\sum_i t^{a_i} x_i \geqslant \sum_i t^{b_i} x_i \to \max_i a_i + X_i \geqslant \max_i b_i + X_i,$$

$$X_i = \log x_i / \log t, \qquad t \to \infty.$$

Tropical linear programming

Problem (Feasibility, conical form)

Given $A, B \in \mathbb{R}_{\max}^{m\times n}$, *is there a vector* $x \not\equiv -\infty$ *such that* "$Ax \leqslant Bx$".

Problem (Tropical LP)

Given $A, C \in \mathbb{R}_{\max}^{m\times n}$, $b, d, f \in \mathbb{R}_{\max}^{m}$,

$$\min \text{“}f^\top x\text{”}, \ \text{“}Ax + b \geqslant Cx + d\text{”}, \qquad x \in \mathbb{R}_{\max}^{n}$$

Theorem (Akian, SG, Guterman, IJAC 2012)

- *Mean payoff games are equivalent to arrangements of tropical half-spaces (i.e., inequalities "$Ax \leqslant Bx$").*
- *State i is winning iff there is a vector x such that*

$$\text{“}Ax \leqslant Bx\text{”}, \ x_i \neq -\infty \ .$$

Corollary

Mean payoff games are equivalent to feasibility problems in tropical linear programming.

By homogeneity, $x_i \neq -\infty$ can be replaced by $x_i \geqslant 0$.

Consider the tropical polyhedral cone

$$C = \bigcap_{1 \leqslant i \leqslant m} H_i$$

where $(H_i)_{1\leqslant i\leqslant m}$ is a family of tropical half-spaces.

$$H_i : \text{“} A_i x \leqslant B_i x \text{”}$$

Consider the tropical polyhedral cone

$$C = \bigcap_{1 \leqslant i \leqslant m} H_i$$

where $(H_i)_{1\leqslant i\leqslant m}$ is a family of tropical half-spaces.

$$H_i : \max_{1\leqslant j\leqslant n} a_{ij} + x_j \leqslant \max_{1\leqslant k\leqslant n} b_{ik} + x_k , \qquad a_{ij}, b_{ik} \in \mathbb{R} \cup \{-\infty\}$$

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Let:

$$[T(x)]_j = \min_{1 \leqslant i \leqslant m} -a_{ij} + \max_{1 \leqslant k \leqslant n} b_{ik} + x_k \ .$$

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Let:

$$[T(x)]_j = \min_{1\leqslant i\leqslant m} -a_{ij} + \max_{1\leqslant k\leqslant n} b_{ik} + x_k \ .$$

$$x \leqslant T(x) \iff \max_{1\leqslant j\leqslant n} a_{ij} + x_j \leqslant \max_{1\leqslant k\leqslant n} b_{ik} + x_k, \ \forall 1 \leqslant i \leqslant m \ .$$

$$H_i : \max_{1\leqslant j\leqslant n} a_{ij} + x_j \leqslant \max_{1\leqslant k\leqslant n} b_{ik} + x_k$$

$$[T(x)]_j = \min_{1\leqslant i\leqslant m} -a_{ij} + \max_{1\leqslant k\leqslant n} b_{ik} + x_k \ .$$

Interpretation of the game

- State of MIN: variable x_j, $j \in \{1, \dots, n\}$
- State of MAX: half-space H_i, $i \in \{1, \dots, m\}$
- In state x_j, Player MIN chooses a tropical half-space H_i with x_j in the LHS
- In state H_i, player MAX chooses a variable x_k at the RHS of H_i
- Payment $-a_{ij} + b_{ik}$.

Assume that "$Ax \leqslant Bx$" for some x such that $x_i \neq -\infty$.

Assume that "$Ax \leqslant Bx$" for some x such that $x_i \neq -\infty$.
Then $T(x) \geqslant x$

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WLOG $0 \geqslant x$ (homogeneity).

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Then $T(x) \geqslant x$
WLOG $0 \geqslant x$ (homogeneity).

$$T \text{order preserving} \implies T^k(0) \geqslant T^k(x) \geqslant x$$

and so

$$\chi_i(T) = \lim_{k\to\infty} [T^k(0)]_i/k \geqslant \lim_{k\to\infty} x_i/k = 0$$

Assume that "$Ax \leqslant Bx$" for some x such that $x_i \neq -\infty$.
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I.e., i is winning for MAX.

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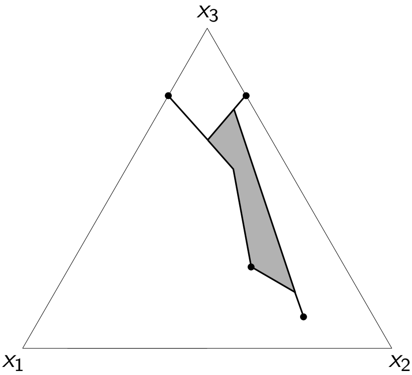
and so

$$\chi_i(T) = \lim_{k\to\infty} [T^k(0)]_i/k \geqslant \lim_{k\to\infty} x_i/k = 0$$

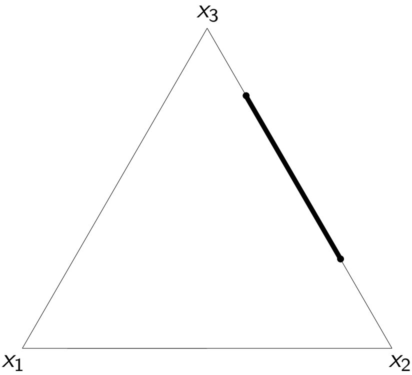
I.e., i is winning for MAX.
Conversely, asssume i is winning for MAX. We use an invariant half-line (Kohlberg theorem):

$$\exists v, \eta,\ T(v + s\eta) = v + (s+1)\eta, \qquad \forall s > 0$$

Then, $\eta = \chi(T)$, and x is constructed from v.



states 1,2,3 winning



states 2,3 winning

Tropical simplex

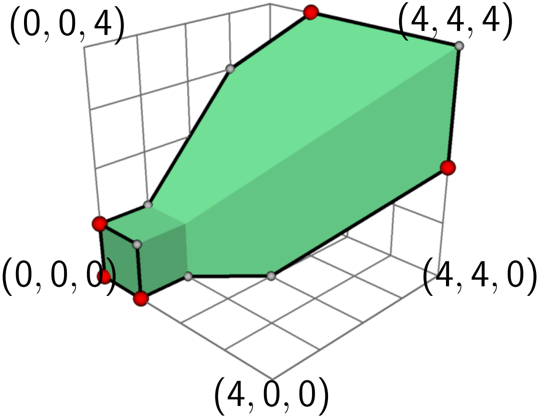
A tropical LP

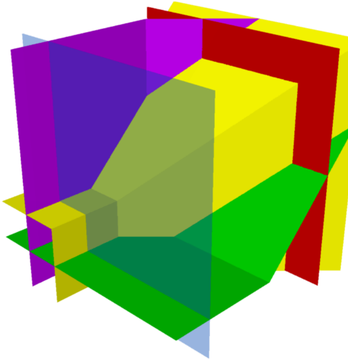
$$\min \text{“}f \cdot x\text{”}; \quad \text{“}Ax + c \leqslant Bx + d\text{”}$$

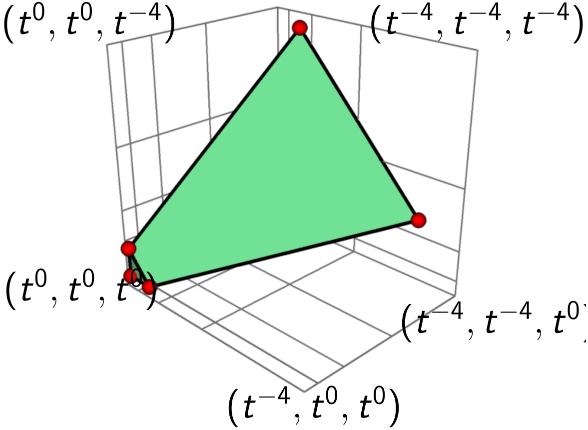
$A, B \in \mathbb{R}_{\max}^{m\times n}$, $b, c \in \mathbb{R}_{\max}^{m}$, $f \in \mathbb{R}_{\max}^{n}$, the inequalities "$x \geqslant 0$" being included in "$Ax + c \leqslant Bx + d$", can be lifted to a classical LP over Hahn series

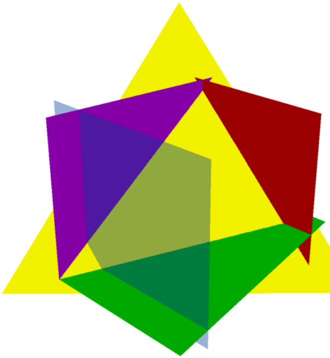
$$\min \mathbf{f} \cdot \mathbf{x}; \quad \mathbf{Ax} + \mathbf{c} \leqslant \mathbf{Bx} + \mathbf{d}$$

$\mathbf{A}, \mathbf{B} \in \mathbb{K}^{m\times n}$, $\mathbf{b}, \mathbf{c} \in \mathbb{K}^{m}$, $\mathbf{f} \in \mathbb{K}^{n}$,
meaning that $\operatorname{val} \mathbf{A} = A$, $\operatorname{val} \mathbf{B} = B$, etc. Recall that $\operatorname{val} 7t^{-1/2} - 1 + t^{1/2} + 7t + \cdots = 1/2$.


(0, 0, 4)
(4, 4, 4)
(0, 0, 0)
(4, 4, 0)
(4, 0, 0)




(t^0, t^0, t^{-4})
(t^{-4}, t^{-4}, t^{-4})
(t^0, t^0, t^0)
$(t^{-4}, t^{-4}, t^0$
(t^{-4}, t^0, t^0)



The simplex algorithm makes sense over any (totally) ordered field, in particular Hahn series.

Question

Can we compute the image of the path followed by the simplex algorithm over Hahn series by the valuation, working only "tropically" (use only the information of the valuation, dont use arithmetic operations on the series).

Question

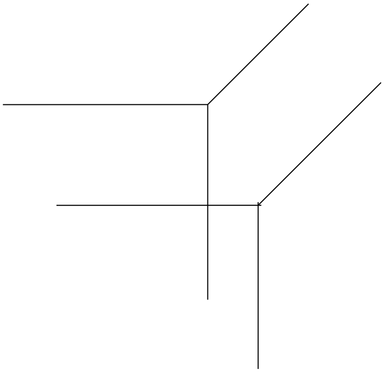
What are tropical basic points?

A point of “$Ax + c \leqslant Bx + d$” is expected to be (tropically) *basic* if it saturates n “independent” inequalities.

What does independent mean?

This can be formalized using the tropical Cramer theorem.

Two generic tropical lines meet at a unique point



- Every n affine tropical hyperplanes of $\mathbb{R}^n_{\max}$ in general position meet at a unique point. Richter-Gebert, Sturmfels, Theobalt, 05 complex version, real version in Max Plus, 90, Akian, SG, Guterman 09, 13.

x being in such an intersection reads

$$\text{“}Mx + g = 0\text{”}, \qquad M \in \mathbb{R}^{n\times n}_{\max}, g \in \mathbb{R}^n_{\max} \ .$$

One has $x_i = \text{“}D^{-1}D_i\text{”}$ where $D = \text{“}\det M\text{”}$ and $D_i = \text{“}\det M_i\text{”}$; replace column i of M by g to get M_i.

How are Cramer det defined?

$$\det A = \text{``}\sum_{\sigma} \operatorname{sgn} \prod_{i} A_{i\sigma(i)}\text{''} = \max_{\sigma} \sum_{i} A_{i\sigma(i)}$$

This is an optimal assignment problem ($O(n^3)$ time). General position means that there is only one optimal permutation.

The sign of the determinant is the sign of the optimal permutation.

(Tropical Cramer determinant revisited, arXiv1309.6298)

Exercise

Solve

$$\max(2+x,y,3) \text{ reached 2 times}$$
$$\max(x,y,2) \text{ reached 2 times}$$

$$D = \begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix} = 2$$

$$D_x = \begin{vmatrix} 3 & 0 \\ 2 & 0 \end{vmatrix} = 3 \qquad D_y = \begin{vmatrix} 2 & 2 \\ 0 & 2 \end{vmatrix} = 4$$

$$x = \text{“}D_x/D\text{”} = 1,\ y = \text{“}D_y/D\text{”} = 2\ .$$

Assume that the data are in general position. This can be defined in terms of tropical Cramer subdeterminants of "$(A + B, c + d)$".

A **tropical basic point** is obtained by saturating n inequalities.

Theorem (Allamigeon, Benchimol, SG, Joswig arXiv:1308.0454)

The valuation of the path of the simplex algorithm over Hahn series can be computed tropically (with a compatible pivoting rule). One iteration takes $O(n(m + n))$ *time.*

Tropical Cramer determinants = opt. assignment used to compute reduce costs.

Example of compatible pivoting rule. A rule is **combinatorial** if any entering/leaving inequalities are functions of the history (sequence of bases) and of the signs of the minors of the matrix

$$M = \begin{pmatrix} \text{“}A - B\text{”} & \text{“}c - d\text{”} \\ f^\top & \text{“}0\text{”} \end{pmatrix} .$$

(eg signs of reduced costs).

Corollary (Allamigeon, Benchimol, SG, Joswig arXiv:1309.5925)

If any combinatorial rule in classical linear programming would run in strongly polynomial time, then, mean payoff games could be solved in strongly polynomial time.

Sketch of Proof

1. Mean payoff games are equivalent to tropical linear programs (Akian, SG, Guterman)
2. Tropical linear programs can be lifted to a subclass of classical linear programs over Hahn series.
3. The set of runs (sequences of bases) of the classical simplex algorithm equipped with a combinatorial pivoting rule is independent of the real closed field.
   Being a run is a first order property, apply Tarski's theorem.
4. Can simulate the classical simplex on Hahn series tropically, every pivot being strongly polynomial (Allamigeon, Benchimol, SG, Joswig)

A key technical difficulty is to relax the general position condition.

$\rightarrow$ work with higher valuation groups

Replace field of series $\mathbb{K}$ in the parameter t with real exponents by $\mathbb{R}[[t^G]]$, G totally ordered group

$$G = (\mathbb{R}^k, +, \leqslant_{\mathrm{lex}})$$

$\mathbb{R}[[t^G]]$ is sent to G by the valuation.

$\mathbb{R}[[t^G]]$ is known to be real closed, Ribenboim.

Tropicalization of interior points ?

Given a positive $\mu \in \mathbb{R}$, the *barrier problem* is

$$\begin{aligned} \text{minimize} \quad & \frac{c^\top x}{\mu} - \sum_{j=1}^{n} \log(x_j) - \sum_{i=1}^{m} \log(w_i) \\ \text{subject to} \quad & Ax + w = b, \; x > 0, w > 0. \end{aligned} \tag{1}$$

$$\begin{aligned} Ax + w &= b \\ -A^\top y + s &= c \\ w_i y_i &= \mu \quad \text{for all } i \in [m] \\ x_j s_j &= \mu \quad \text{for all } j \in [n] \\ x, w, y, s &> 0 \; . \end{aligned} \tag{2}$$

For any $\mu > 0$, $\exists ! \; (x^\mu, w^\mu, y^\mu, s^\mu) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^n$. The *central path* is the image of the map $\mathcal{C}_{A,b,c} : \mathbb{R}_{>0} \to \mathbb{R}^{2m+2n}$ which sends a positive real number μ to the vector $(x^\mu, w^\mu, y^\mu, s^\mu)$.

The tropical central path

Assume now that $\mathbf{A}(t), \mathbf{b}(t), \mathbf{c}(t)$ have entries in $\mathbb{K}$ (absolutely converging series in t, with real exponents).

The tropical central path is the log-limit:

$$\mathcal{C}^{\mathrm{trop}} : \lambda \mapsto \lim_{t \to +\infty} \log_t \mathcal{C}(t, \lambda) \ . \tag{3}$$

The poinwise limit does exist since $\mathcal{C}(\cdot, \lambda)$ is definable in a polynomially bounded o-minimal structure.

Theorem

The family of maps $(\log_t \mathcal{C}(t,\cdot))_t$ *converges uniformly on any closed interval* $[a,b] \subset \mathbb{R}$ *to the tropical central path* $\mathcal{C}^{\mathrm{trop}}$.

Theorem (Allamigeon, Benchimol, SG, Joswig arXiv:1405.4161)

Let $(\mathbf{x}^{\boldsymbol{\mu}}, \mathbf{w}^{\boldsymbol{\mu}})$ *be the point on the primal central path of the linear program* $\mathrm{LP}(\mathbf{A}, \mathbf{b}, \mathbf{c})$ *at* $\boldsymbol{\mu} \in \mathbb{K}$ *with* $\boldsymbol{\mu} > 0$*, and let* $\boldsymbol{\nu}$ *be that LP's optimal value. Then* $\mathrm{val}(\mathbf{x}^{\boldsymbol{\mu}}, \mathbf{w}^{\boldsymbol{\mu}})$ *is the greatest element of* $\mathrm{val}(\boldsymbol{\mathcal{P}}^{\boldsymbol{\mu}})$ *where*

$$\boldsymbol{\mathcal{P}}^{\boldsymbol{\mu}} := \{(\mathbf{x}, \mathbf{w}) \in \mathbb{K}^{n+m} \mid \mathbf{A}\mathbf{x} + \mathbf{w} = \mathbf{b},\ \mathbf{c}\mathbf{x} \leqslant \boldsymbol{\nu} + (n+m)\boldsymbol{\mu},\ \mathbf{x} \geqslant 0,\ \mathbf{w} \geqslant 0\} .$$

$$
\begin{aligned}
\mathbf{x}_1 + \mathbf{x}_2 &\leqslant 2 \\
t\mathbf{x}_1 &\leqslant 1 + t^2\mathbf{x}_2 \\
t\mathbf{x}_2 &\leqslant 1 + t^3\mathbf{x}_1 \\
\mathbf{x}_1 &\leqslant t^2\mathbf{x}_2 \\
\mathbf{x}_1, \mathbf{x}_2 &\geqslant 0 \; .
\end{aligned}
\tag{4}
$$

Its value $\mathrm{val}(\mathcal{P})$ is the tropical set described by the inequalities:

$$
\begin{aligned}
\max(x_1, x_2) &\leqslant 0 \\
1 + x_1 &\leqslant \max(0, 2 + x_2) \\
1 + x_2 &\leqslant \max(0, 3 + x_1) \\
x_1 &\leqslant 2 + x_2 \; .
\end{aligned}
\tag{5}
$$

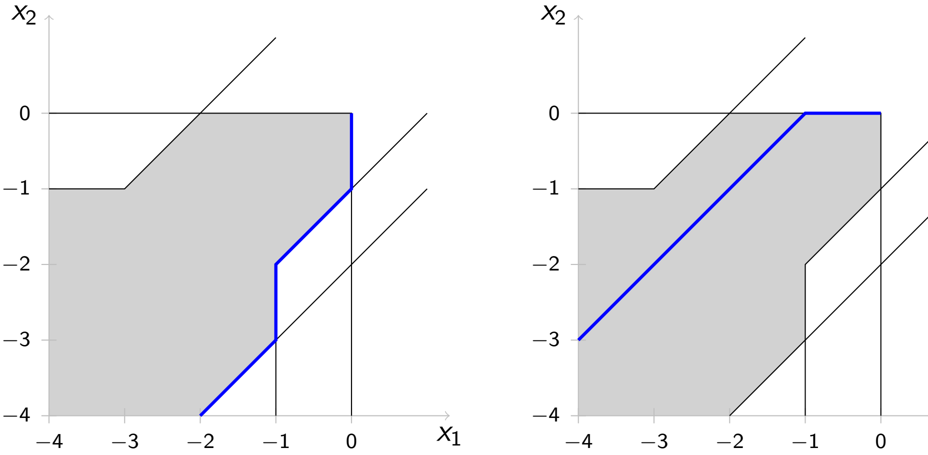


Figure: Tropical central paths on the Hardy polyhedron (4) for the objective function $\min \mathbf{x}_1$ (left) and $\min t\mathbf{x}_1 + \mathbf{x}_2$ (right).

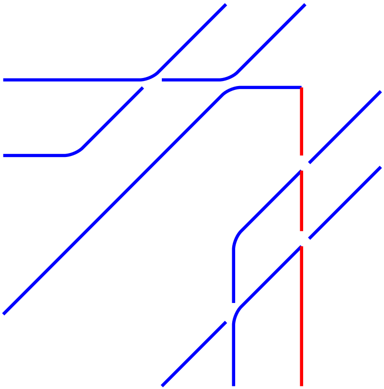


Figure: Tropical central paths on the full-dimensional cells included in the positive orthant induced by the arrangement of hyperplanes associated with (4); for the objective function $\min t\mathbf{x}_1 + \mathbf{x}_2$ (in blue) and $\max t\mathbf{x}_1 + \mathbf{x}_2$ (in red). The parts of the paths that lie on the boundaries are slightly shifted inside their respective cell.

Bezem, Nieuwenhuis and Rodríguez-Carbonell (2008) constructed a class of tropical linear programs for which an algorithm of Butkovič and Zimmermann (2006) exhibits an exponential running time.

On this example, the tropical central path passes through an exponential number of basic points.

By dequantization, $\mathcal{C}(t, \cdot)$ for t large enough, yields a counter example to the continuous analogue of the Hirsch conjecture.

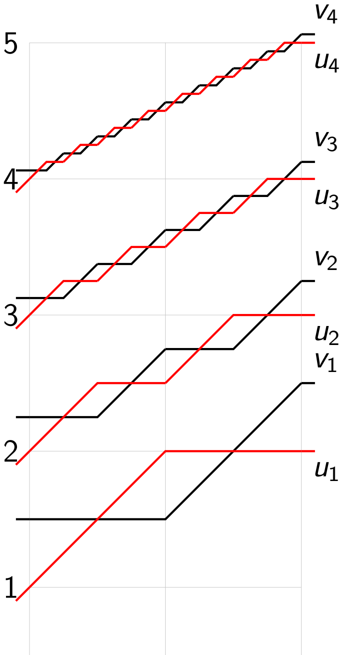
The counter example

$$
\begin{array}{lll}
\min & \mathbf{v}_0 & \\
\text{s.t.} & \mathbf{u}_0 \leqslant t & \\
& \mathbf{v}_0 \leqslant t^2 & \\
& \mathbf{v}_i \leqslant t^{1-\frac{1}{2^i}}(\mathbf{u}_{i-1} + \mathbf{v}_{i-1}) & \text{for } 1 \leqslant i \leqslant r \\
& \mathbf{u}_i \leqslant t\mathbf{u}_{i-1} & \text{for } 1 \leqslant i \leqslant r \\
& \mathbf{u}_i \leqslant t\mathbf{v}_{i-1} & \text{for } 1 \leqslant i \leqslant r \\
& \mathbf{u}_r \geqslant 0,\ \mathbf{v}_r \geqslant 0 &
\end{array}
$$

r

The tropical central path is determined by the following dynamical systems

$$
\begin{aligned}
u_0 &= 1 \\
v_0 &= \min(2, \lambda) \\
v_i &= 1 - \frac{1}{2^i} + \max(u_{i-1}, v_{i-1}) \\
u_i &= 1 + \min(u_{i-1}, v_{i-1})
\end{aligned}
$$



v_4
5
u_4
v_3
4
u_3
v_2
3
u_2
v_1
2
u_1
1

Concluding remarks

- Complexity of tropical LP = deterministic mean payoff games is open
- Transfer theorem: some classes of pivoting rules for the simplex algorithm also solve mean payoff games. Worst case number of iterations is smaller (or equal) for mean payoff games as there are fewer instances.
- Use higher value groups $G = \mathbb{R}^k$, $\mathbb{K} = \mathbb{R}[[t^G]]$.
- Tropical interior points can be as bad as tropical simplex.
- Are tortuous central paths really an obstruction to a "strong-polynomialization" of interior point / Shub-Smale homotopy methods ?

Thank you !