Identifiability analysis and parameter estimation in PDE models

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2. Identifiability of ODEs from Input-output polynomials
   - Case of regular functions
   - Case of piece-wise functions

3. The spatio-temporal chikungunya model
   - Presentation of the model
   - Identifiability
   - Identification

4. Conclusion and perspectives
Question

From measurements of the output(s) of the model, is it possible to estimate uniquely the parameter vector? If the answer is YES, then the model is said identifiable.

Two simple examples ...

**Example 1:**

\[
\begin{align*}
\dot{x}_1 &= k_{12}(x_2 - x_1) - \frac{k_{\nu} x_1}{1 + x_1}, \quad x_1(0) = x_{10} \\
\dot{x}_2 &= k_{21}(x_1 - x_2), \quad x_2(0) = 0 \\
y &= x_1,
\end{align*}
\]

**Example 2:**

\[
\begin{align*}
\dot{x}_1 &= u, \quad x_1(0) = 0 \\
\dot{x}_2 &= x_1 + k \ u, \quad x_2(0) = 0 \\
y &= x_2
\end{align*}
\]

(1)

Tools

✓ Similarity method (S. Vajda),

✓ Method of invariants (M. Petitot),

✓ Input-output method based on the Rosenfeld-Groebner algorithm (implemented in Maple by F. Boulier, CRIStAL) and based on differential algebra approach (Kolchin and al., 1973)
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Example 2:
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\dot{x}_1 &= u, \quad x_1(0) = 0 \\
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Tools
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⇒ Identifiability
⇒ Parameter estimation
Identifiability of ODEs from Input-output polynomials

Case of regular functions

\[ \Gamma^p \left\{ \begin{array}{l}
\dot{x}(t, p) = f(x(t, p), u(t), p), \\
y(t, p) = h(x(t, p), p).
\end{array} \right. \]  

✓ \( x(t, p) \in \mathbb{R}^n \): state variables at time \( t \),
✓ \( y(t, p) \in \mathbb{R}^m \): output vector at time \( t \),
✓ \( u(t) \in \mathbb{R}^r \): input vector at time \( t \),
✓ \( f, h \): real functions, analytic on \( M \) (an open set of \( \mathbb{R}^n \)),
✓ \( p \in U_P \): vector of parameters, \( U_P \subset \mathbb{R}^p \): an a priori known set of admissible parameters.
Formalization

Controlled models \((u \neq 0)\) WITHOUT initial condition; \((\tilde{x}, \tilde{y})\) = unique set of solutions

- The model is **globally identifiable** if there exists an input \(u\) such that, for all \(p \in \mathcal{U}_p\), one gets
  \[
  \tilde{y}(t, p) \neq \emptyset, \\
  \tilde{y}(t, p) \cap \tilde{y}(t, \bar{p}) \neq \emptyset, \forall t \geq 0, \bar{p} \in \mathcal{U}_p
  \]
  \[
  \Rightarrow p = \bar{p}.
  \]

- The model is **locally identifiable** if it is globally identifiable in an open neighborhood \(v(p) \subset \mathcal{U}_p\) of \(p\).
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  \tilde{y}(t, p) \cap \tilde{y}(t, \bar{p}) &\neq \emptyset, \quad \forall \ t \geq 0, \ \bar{p} \in \mathcal{U}_p 
  \end{align*}
  \]

- The model is **locally identifiable** if it is globally identifiable in an open neighborhood \(v(p) \subset \mathcal{U}_p\) of \(p\).

Controlled model \((u \neq 0)\) WITH initial conditions; \((x, y) = \text{unique solution}\)

- The model is **globally identifiable** if there exists an input \(u\) such that, for all \(p, \bar{p} \in \mathcal{U}_p\), there exists \(t_1 > 0\) such that if for all \(t \in [0, t_1]\), the equalities \(y(t, p) = y(t, \bar{p})\) implies that \(p = \bar{p}\).

- The model is **locally identifiable** if it is globally in an open neighborhood \(v(p) \subset \mathcal{U}_p\) of \(p\).
Method in the case of ODE’s systems:

\[
\begin{align*}
    \dot{x}(t, p) &= f(x, u, p), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^q, \quad p \in \mathbb{R}^p \\
    y(t, p) &= h(x, p) \in \mathbb{R}^m.
\end{align*}
\]  

(4)

1. Use the Rosenfeld-Groebner algorithm with the elimination order \([p] \prec [y, u] \prec [x] :\)

\[
    C(p) = \{\dot{p}_1, \ldots, \dot{p}_p, P_1(y, u, p), \ldots, P_m(y, u, p), Q_1(x, y, u, p), \ldots, Q_n(x, y, u, p)\}.
\]

This set is called the characteristic presentation (general case).

Example 1:

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\begin{align*}
    \dot{x}_1 &= k_{12}(x_2 - x_1) - \frac{k_{12}x_1}{1 + x_1}, \\
    \dot{x}_2 &= k_{21}(x_1 - x_2), \\
    y &= x_1,
\end{align*}
\]

\[
    C(p) = \{\dot{k}_{12}, \dot{k}_\nu, \dot{k}_{21}, k_{12} y^2 \dot{y} + k_{21} k_\nu y^2 + k_{21} y^2 \dot{y} + 2 k_{12} y \dot{y} + 2 k_{21} k_\nu y + 2 k_{21} y \dot{y} + y + 2 y \dot{y} + y, \\
    x_1 - y, k_{12} x_2 y - k_{12} y^2 + k_{12} x_2 - k_{12} y - k_\nu y - y \dot{y} - y\}.
\]
Identifiability of ODEs from Input-output polynomials

Case of regular functions

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This set is called the characteristic presentation (general case of regular chains).

2. Identifiability study done from the input-output polynomials

\[
P_i(y, u, p) = m_i^0(y, u) + \sum_{k=1}^{q} \gamma_k^i(p)m_k^i(y, u) = 0. \hspace{1cm} (5)
\]

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\[
P(y) = \ddot{y} + 2y \dot{y} + y^2 \dot{y} + (k_{12} + k_{21})(y^2 \dot{y} + 2y \dot{y}) + k_{21} k_\nu (y^2 + y) + (k_{12} + k_{21} + k_\nu) \dot{y}
\]
Method in the case of ODE’s systems:

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P_i(y, u, p) = m_0^i(y, u) + \sum_{k=1}^{q} \gamma_k^i(p)m_k^i(y, u) = 0.
\] (6)

Afterwards, 1 observation \(\Rightarrow i = 1\).
Identifiability of ODEs from Input-output polynomials

Case of regular functions

\[ P(y, u, p) = m_0(y, u) + \sum_{k=1}^{q} \gamma_k(p) m_k(y, u) = 0, \]

\((\gamma_k(p))_{1 \leq k \leq q}\) is called the exhaustive summary.

**Proposition**

If \((m_k(y, u))_{1 \leq k \leq q}\) are linearly independent then the model is globally identifiable at \(p\) if for all \(\bar{p} \in U_P\)

\[ \forall k = 1, \ldots, q, \gamma_k(\bar{p}) = \gamma_k(p) \Rightarrow p = \bar{p}. \] (7)
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y &= x_1,
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\]

\[ P(y) = \ddot{y} + 2 y \dot{y} + y^2 \ddot{y} + (k_{12} + k_{21}) (y^2 \dot{y} + 2 y \dot{y}) + k_{21} k_\nu (y^2 + y) + (k_{12} + k_{21} + k_\nu) \dot{y} \]

Let \(p = (k_{12}, k_{21}, k_\nu)\).

Exhaustive summary = \((\gamma_k(p))_{1 \leq k \leq 4}\) = \((k_{12} + k_{21}, k_{21} k_\nu, k_{12} + k_{21} + k_\nu)\).

1. \(\det(y^2 \dot{y} + 2 y \dot{y}, y^2 + y, \dot{y}) = -2 \dot{y} (-\dot{y}^4 + (3 y^2 \ddot{y} + 5 y \dot{y} + 2 \ddot{y}) \dot{y}^2 + y \dddot{y} (y + 1)^2 \dddot{y} - 3 y \ddot{y}^2 (y + 1)^2) \neq 0\)
2. \(\gamma_k(p) = \gamma_k(\bar{p}) \Rightarrow p = \bar{p}\) (Groebner algorithm).

The model is identifiable at \(p\).
Identifiability of ODEs from Input-output polynomials

Case of regular functions

\[ P(y, u, p) = m_0(y, u) + \sum_{k=1}^{q} \gamma_k(p) m_k(y, u) = 0, \]

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If \((m_k(y, u))_{1 \leq k \leq q}\) are linearly independent then the model is globally identifiable at \(p\) if for all \(\bar{p} \in U_p\)

\[ \forall k = 1, \ldots, q, \gamma_k(\bar{p}) = \gamma_k(p) \Rightarrow p = \bar{p}. \quad (7) \]

**Remark**

✓ If \(\phi(p) = (\gamma_k(p))_{k=1,\ldots,q}\), (7) consists in verifying that \(\phi\) is injective.

✓ We recall that the Wronskian of the sequence of functions \((\phi_1, \ldots, \phi_s)\) is defined by:

\[ \text{Wronskian} = \text{Det}(\phi_1, \ldots, \phi_s) = \begin{vmatrix} \phi_1 & \cdots & \phi_s \\ \dot{\phi}_1 & \cdots & \dot{\phi}_s \\ \vdots & \cdots & \vdots \\ \phi_1^{(s-1)} & \cdots & \phi_s^{(s-1)} \end{vmatrix}. \quad (8) \]

✓ The functions \((m_k(y, u))_{k=1,\ldots,q}\) are linearly independent if the functional determinant is not identically equal to zero. It is sufficient to find a time point at which the Wronskian is non-zero.
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Case of regular functions

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**Example 2:**

\[
\begin{align*}
\dot{x}_1 &= u, & x_1(0) &= 0 \\
\dot{x}_2 &= x_1 + k u, & x_2(0) &= 0 \\
y &= x_2
\end{align*}
\] (9)

where \(k\) is the parameter to identified.

The IO polynomial: \(P(y, u) = \ddot{y} - u - k \dot{u}\).

✓ If \(u\) is not a piece-wise constant input, then the model is identifiable.

✓ If \(u\) is a constant input, then the model is not identifiable:

\[ \Rightarrow \text{how to integrate initial conditions?} \]
The identifiability when initial conditions are considered:

\[
\Gamma_p = \begin{cases} 
\dot{x}(t, p) = f(x(t, p), p), x(0, p) = x_0 \\
y(t, p) = h(x(t, p), p).
\end{cases}
\] (10)

\(f\) and \(h\) are supposed to be rational and analytical.

\[P(y, u, p) = m_0(y, u) + \sum_{k=1}^{q} \gamma_k(p) m_k(y, u) = 0.\]

**Proposition**

Let \(l\) the highest order derivative in the polynomial \(P\). If \((m_k(y, u))_{1 \leq k \leq q}\) are linearly independent then the model is globally identifiable at \(p\) if for all \(\bar{p} \in \mathcal{U}_p\)

\[
\left\{ \begin{array}{l}
\forall k = 1, \ldots, q, \gamma_k(\bar{p}) = \gamma_k(p) \\
1 \leq s \leq l - 1, y^{(s)}(0^+, p) = y^{(s)}(0^+, \bar{p})
\end{array} \right. \Rightarrow p = \bar{p} \] (11)

Moreover, if the coefficient of \(y^{(l)}\) in \(P\) is not equal to 0 at \(t = 0\) then the reciprocal is true.

**Example 2:** when \(u\) is constant input

\[
\left\{ \begin{array}{l}
\dot{x}_1 = u, x_1(0) = 0 \\
\dot{x}_2 = x_1 + k u, x_2(0) = 0 \\
y = x_2
\end{array} \right. \Rightarrow \left\{ \begin{array}{l}
P(y, u) = \ddot{y} - u, \\
\dot{y}(0) = k u(0^+)
\end{array} \right. \] (12)

Identifiability: \(\gamma_k(\bar{p}) = \gamma_k(p) \Rightarrow k = \bar{k}.\)
Aim:
- consider functions not necessary regular
- propose a method in a more general case: systems of PDE (derivatives in function of $t$ and $x$)
⇒ distribution framework to obtain identifiability
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Example 2: when $u$ is piece-wise constant input

\[
\begin{align*}
\dot{x}_1 &= u, & x_1(0) &= 0 \\
\dot{x}_2 &= x_1 + k u, & x_2(0) &= 0 \\
y &= x_2
\end{align*}
\]  

where

\[
\begin{align*}
u(t) := \begin{cases} 
0 & \text{if } t \in [0, t_1[ \\
u_2 & \text{if } t \in ]t_1, t_2],
\end{cases}
\end{align*}
\]  

where $u_2 \in \mathbb{R}$ and $u_2 > 0$.

IO polynomial: $P(y, u, k) = \ddot{y} - u - ku$.

\[
\begin{align*}
T_{\ddot{y}} &= T_y^{(2)} - y(t_1^+)\delta_{t_1}' - ku_2\delta_{t_1}, \\
T_{\dot{u}} &= T'u - u_2\delta_{t_1}
\end{align*}
\]  

In the distribution sens: $T_{P(y,u,k)} = T'' - T_u - kT'u$.

If $y(t, k) = y(t, \bar{k})$, then

\[
T_{P(y,u,k)} - T_{P(y,u,\bar{k})} = (k - \bar{k})T'u = 0 \Rightarrow k = \bar{k}
\]
Aim:
- consider functions not necessary regular
- propose a method in a more general case: systems of PDE (derivatives in function of $t$ and $x$)
⇒ distribution framework to obtain identifiability

Remarks
- Considering the IO polynomial in the distribution sense can turn an unidentifiable model into an identifiable one.
- Extension of this method to more complex examples.
103 countries affected by the virus (March 2015)
Increase of i) reported cases of chikungunya in the population ii) number of countries with local transmission (December 2013-March 2015)

✓ The model is focusing on human → mosquito → human type transmissions and ignoring vertical or within-species transmissions.

✓ Two systems: the vector population model, the transmission model

Problem: this system does not take into account the spatial mobility of humans and mosquitoes!

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✓ Two systems: the vector population model, the transmission model

Problem: this system does not take into account the spatial mobility of humans and mosquitoes!

Our assumptions:

1. Small areas (like islands) with sufficiently similar weather (temperature/precipitation), socioeconomic, demographic distributions. The movement of humans is assumed random described as Brownian random motion.

2. General homogeneity assumptions of the region (region of operation such that the values of parameters are assumed to be the same).

3. The displacements of mosquitoes are negligible compared to those of humans.

$\Rightarrow$ extend the system of ODEs to a system of PDEs for modeling a random spatial mobility in adding diffusion terms.
The transmission model:

- $S_H$, susceptible humans, $I_H$, infected humans and $R_H$, recovered humans
- $\beta_m > 0$ (resp. $\beta_H > 0$) is the infectious contact rate between susceptible mosquitoes and infected humans (resp. susceptible humans and vectors),
- $b_H > 0$ is the human birth rate, $\gamma > 0$ is the human recovery rate.

\[
\begin{align*}
S'_H(t, x) &= -(b_H + \beta_H l_m(t, x))S_H(t) + b_H + a_1 \Delta S_H(t, x), \\
I'_H(t, x) &= \beta_H l_m(t, x)S_H(t, x) - (\gamma + b_H)I_H(t, x) + a_2 \Delta I_H(t, x), \\
l'_m(t, x) &= -(s_L u(t, x) + \beta_m I_H(t, x))l_m(t, x) + \beta_m I_H(t, x),
\end{align*}
\]
All the parameters are not directly accessible: $\beta_H, \beta_m, d_1, d_2$.

Observed variables:

- $u$

- $S_H$ and $I_H$: the concerned region before the epidemic is susceptible and, periodically, the number of new infections can be estimated by authorities (INVS: French Institute for Health Care).

\[
\begin{align*}
S'_H(t, x) &= -(b_H + \beta_H I_m(t, x))S_H(t, x) + b_H + d_1 \Delta S_H(t, x), \\
I'_H(t, x) &= \beta_H I_m(t, x)S_H(t, x) - (\gamma + b_H)I_H(t, x) + d_2 \Delta I_H(t, x), \\
I'_m(t, x) &= -(s_L u(t, x) + \beta_m I_H(t, x))I_m(t, x) + \beta_m I_H(t, x), \\
y_1 &= S_H, \; y_2 = I_H
\end{align*}
\]
The spatio-temporal chikungunya model

Presentation of the model

\[
\begin{align*}
(S_H(t, x))' &= -(b_H + \beta_H l_m(t, x)) S_H(t, x) + b_H + d_1 \Delta S_H(t, x), \\
(l_H(t, x))' &= \beta_H l_m(t, x) S_H(t, x) - (\gamma + b_H) l_H(t, x) + d_2 \Delta l_H(t, x), \\
(l_m(t, x))' &= -(s_L u(t, x) + \beta_m l_H(t, x)) l_m(t, x) + \beta_m l_H(t, x), \\
\gamma_1 &= S_H, \quad \gamma_2 = l_H
\end{align*}
\]

\[
X := L^2(\Omega; \mathbb{R}^2) \times C(\Omega; \mathbb{R}), \quad X_+ := \{(u_1, u_2, u_3) \in X \mid u_1, u_2, u_3 \geq 0\}, \quad U = (S_H, l_H, l_m)
\]

\[
\partial_t U(t, p) + B(p)U(t, p) = g(\zeta, U, p), \quad t \in [0, \infty),
\]

\[
y(t, p) = h(\zeta, U), \quad t \in [0, \infty),
\]

completed with \( U(t = 0, p) = U_0 \).

The notations are the following:

- \( p = (\beta_m, \beta_H, d_1, d_2) \)
- \( B_1 := -d_1 A + b_H, \quad B_2 := -d_2 A + \gamma + b_H \) in \( L^2(\Omega) \) where \( A \) is the realization of the Laplacian in \( L^2(\Omega) \) with Neumann homogeneous boundary condition on \( \partial \Omega \), \( B_3 = I \).
- \( B = \text{diag}(B_1, B_2, B_3) \) in \( H^2_N(\Omega)^2 \times C(\bar{\Omega}) \).
- \( g(U) := (-\beta_H u_1 u_3 + b_H, \beta_H u_1 u_2, -(s_L u + \beta_m u_3 - 1)u_1 + \beta_m u_3) \) is a rational function with respect to \( u, U \) and \( p \) and satisfies a Lipschitz condition
- \( h = (h_1, h_2) = (y_1, y_2) \) where \( h_i, i = 1, 2 \), are rational functions with respect to \( u, U \) and \( p \)
The spatio-temporal chikungunya model

Presentation of the model

\( \Gamma_p \)

\[
\begin{align*}
S'_H(t, x) &= -(b_H + \beta_H l_m(t, x)) S_H(t, x) + b_H + d_1 \Delta S_H(t, x), \\
I'_H(t, x) &= \beta_H l_m(t, x) S_H(t, x) - (\gamma + b_H) I_H(t, x) + d_2 \Delta I_H(t, x), \\
l'_m(t, x) &= -(s_L u(t, x) + \beta_m I_H(t, x)) l_m(t, x) + \beta_m I_H(t, x), \\
y_1 &= S_H, \; y_2 = I_H
\end{align*}
\]

\( X := L^2(\Omega; \mathbb{R}^2) \times C(\Omega; \mathbb{R}), \; X_+ := \{(u_1, u_2, u_3) \in X \mid u_1, u_2, u_3 \geq 0\}, \; U = (S_H, I_H, l_m) \)

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\begin{align*}
\partial_t U(t, p) + B(p) U(t, p) &= g(\zeta, U, p), \; t \in [0, \infty), \\
y(t, p) &= h(\zeta, U), \; t \in [0, \infty),
\end{align*}
\]

completed with \( U(t = 0, p) = U_0 \).

We have the following theorem which states that the solution is global (Theorem 4.4 of A. Yagi* (p. 207)).

**Theorem: existence, unicity, non-negativity**

If there is a survival of mosquitoes adults, for any \((S^H_0, I^H_0, l^m_0) \in X_+, \; (\Gamma_p)\) possesses a unique non-negative global solution such that

\[
\begin{align*}
S_H, \; l_H &\in C([0, \infty); H^0_N(\Omega)) \cap C^1((0, \infty); L^2(\Omega)), \\
l_m &\in C^1([0, \infty); C(\Omega; \mathbb{R})).
\end{align*}
\]

The spatio-temporal chikungunya model

Identifiability

Let \( u_1 \) and \( u_2 \) the solution of (\( \tilde{\Gamma}_p \)) in \( C([0, \infty); H^2_N(\Omega)) \cap C^1((0, \infty); L^2(\Omega)) \).

**Proposition: Identifiability of \( d_i \)**

Assume there exists \( t_1 > 0 \) such that, for all \( t \in [0, t_1] \),

1. there exists \( (\lambda_i)_{i=1,2} \in (\mathbb{R}^*)^k \) such that \( \sum_{i=1}^{2} \lambda_i (g_i(\zeta(t), U(t), p) - \partial_t u_i) \) is independent of \( p \)
2. the functions \( u_1(t, \cdot), u_2(t, \cdot) \) are linearly independent up to an additive constant in \( \Omega \).

Then, (\( \tilde{\Gamma}_p \)) is identifiable at \( d_1 \) and \( d_2 \).
The spatio-temporal chikungunya model

Identifiability

\[
\begin{aligned}
S'_H(t, x) &= -(b_H + \beta_H l_m(t, x))S_H(t, x) + b_H + d_1 \Delta S_H(t, x) \\
I'_H(t, x) &= \beta_H l_m(t, x)S_H(t, x) - (\gamma + b_H)I_H(t, x) + d_2 \Delta I_H(t, x) \\
l'_m(t, x) &= -(s_L u(t, x) + \beta_m l_H(t, x))l_m(t, x) + \beta_m l_H(t, x)
\end{aligned}
\]

(21)

Proposition 2

Let \( S_H \) and \( I_H \) solutions of (21) in \( C([0, \infty); H^2_N(\Omega)) \cap C^1((0, \infty); L^2(\Omega)) \).

Suppose that for all \( t \in [0, t_1] \), \( S_H(t, \cdot) \) and \( I_H(t, \cdot) \) are linearly independent up to an additive constant in \( \Omega \).

Then, the model is identifiable at \( d_1, d_2 \).

Proof

Sum of (21)-(a) and (21)-(b):

\[
d_1 \Delta y_1(t, x) + d_2 \Delta y_2(t, x) = \partial_t y_1(t, x) + \partial_t y_2(t, x) + b_H(y_1(t, x) - 1) + (\gamma + b_H)y_2(t, x).
\]

1. The right hand-side does not depend on the unknown parameters \( \beta_m, \beta_H, d_1, d_2 \)

2. \( y_1(t, \cdot) = S_H(t, \cdot) \) and \( y_2(t, \cdot) = I_H(t, \cdot) \) are linearly independent.

Conclusion: the model is identifiable at \( d_1 \) and \( d_2 \).

Remark: \( P_1 := d_1 \Delta y_1(t, x) + d_2 \Delta y_2(t, x) - \partial_t y_1(t, x) - y_2(t, x) - b_H(y_1(t, x) - 1) - (\gamma + b_H)y_2(t, x) \) is a IO polynomial.
The spatio-temporal chikungunya model

Identifiability

\[
\begin{align*}
S'_H(t, x) &= -(b_H + \beta_H I_m(t, x))S_H(t, x) + b_H + d_1 \Delta S_H(t, x) \quad (a) \\
l'_H(t, x) &= \beta_H I_m(t, x)S_H(t, x) - (\gamma + b_H)l_H(t, x) + d_2 \Delta l_H(t, x) \quad (b) \\
l'_m(t, x) &= -(s_L u(t, x) + \beta_m l_H(t, x))l_m(t, x) + \beta_m l_H(t, x) \quad (c) \\
y_1 &= S_H, \; y_2 = l_H, \; \tilde{I}_m := \beta_H I_m.
\end{align*}
\]

Second equation: \(\tilde{I}_m\) is known almost everywhere and does not depend on \(\beta_m\) and \(\beta_H\).

Proposition

Let \(S_H\) and \(l_H\) solutions of (23) in \(C([0, \infty); H^2_N(\Omega)) \cap C^1((0, \infty); L^2(\Omega))\). Suppose that

1. \(d_1\) and \(d_2\) are identifiable

2. there exist \(t_1, t_2 \in (0, T)\) such that

\[
\begin{vmatrix}
<y_2 \tilde{I}_m>(t_1) & <y_2>(t_1) \\
<y_2 \tilde{I}_m>(t_2) & <y_2>(t_2)
\end{vmatrix} \neq 0.
\]

The model is identifiable at \(\beta_m, \beta_H\).
The spatio-temporal chikungunya model

Identifiability

\[
\begin{align*}
S'_H(t, x) &= -(b_H + \beta_H l_m(t, x))S_H(t, x) + b_H + d_1 \Delta S_H(t, x) \quad (a) \\
I'_H(t, x) &= \beta_H l_m(t, x)S_H(t, x) - (\gamma + b_H)I_H(t, x) + d_2 \Delta I_H(t, x) \quad (b) \\
l'_m(t, x) &= -(s_L u(t, x) + \beta_m I_H(t, x))l_m(t, x) + \beta_m I_H(t, x) \quad (c)
\end{align*}
\]

(23)

\[ y_1 = S_H, \ y_2 = I_H, \ \tilde{l}_m := \beta_H l_m. \]

Second equation: $\tilde{l}_m$ is known almost everywhere and does not depend on $\beta_m$ and $\beta_H$.

Proof

Third equation:

\[-\beta_m y_2(t, x)\tilde{l}_m(t, x) + \beta_H \beta_m y_2(t, x) = s_L r(t, x)\tilde{l}_m(t, x) + \partial_t \tilde{l}_m(t, x).\]
\[
\begin{aligned}
S_H'(t, x) &= -(b_H + \beta_H l_m(t, x)) S_H(t, x) + b_H + d_1 \Delta S_H(t, x) \quad (a) \\
I_H'(t, x) &= \beta_H l_m(t, x) S_H(t, x) - (\gamma + b_H) I_H(t, x) + d_2 \Delta I_H(t, x) \quad (b) \\
l_m'(t, x) &= - (s_L u(t, x) + \beta_m l_H(t, x)) l_m(t, x) + \beta_m l_H(t, x) \quad (c)
\end{aligned}
\]

Second equation: \( \tilde{l}_m \) is known almost everywhere and does not depend on \( \beta_m \) and \( \beta_H \).

**Proof**

Third equation: 
\[
-\beta_m y_2(t, x) \tilde{l}_m(t, x) + \beta_H \beta_m y_2(t, x) = s_L r(t, x) \tilde{l}_m(t, x) + \partial_t \tilde{l}_m(t, x).
\]

System:
\[
\begin{cases}
-\beta_m y_2(t, x) \tilde{l}_m(t, x) + \beta_H \beta_m y_2(t, x) = s_L r(t, x) \tilde{l}_m(t, x) + \partial_t \tilde{l}_m(t, x) \\
-\bar{\beta}_m y_2(t, x) \tilde{l}_m(t, x) + \bar{\beta}_H \bar{\beta}_m y_2(t, x) = s_L r(t, x) \tilde{l}_m(t, x) + \partial_t \tilde{l}_m(t, x)
\end{cases}
\]

\[
\Rightarrow (\beta_m - \bar{\beta}_m) (-y_2(t, x) \tilde{l}_m(t, x)) + (\beta_H \beta_m - \bar{\beta}_H \bar{\beta}_m) y_2(t, x) = 0
\]

Mean: 
\[
-(\beta_m - \bar{\beta}_m) < -y_2 \tilde{l}_m > + (\beta_H \beta_m - \bar{\beta}_H \bar{\beta}_m) < y_2 >= 0.
\]
The spatio-temporal chikungunya model

Identifiability

\[ \begin{align*}
S'_H(t, x) &= -(b_H + \beta_H l_m(t, x)) S_H(t, x) + b_H + d_1 \Delta S_H(t, x) \quad (a) \\
l'_H(t, x) &= \beta_H l_m(t, x) S_H(t, x) - (\gamma + b_H) l_H(t, x) + d_2 \Delta l_H(t, x) \quad (b) \\
l'_m(t, x) &= -(s_L u(t, x) + \beta_m l_H(t, x)) m(t, x) + \beta_m l_H(t, x) \quad (c)
\end{align*} \]

(25)

\[ y_1 = S_H, \ y_2 = l_H, \ \tilde{l}_m := \beta_H l_m. \]

{\bf Second equation:} \( \tilde{l}_m \) is known almost everywhere and does not depend on \( \beta_m \) and \( \beta_H \).

\[ \text{Proof} \]

{\bf Third equation:} \(-\beta_m y_2(t, x) \tilde{l}_m(t, x) + \beta_H \beta_m y_2(t, x) = s_L r(t, x) \tilde{l}_m(t, x) + \partial_t \tilde{l}_m(t, x) \).

{\bf System:}

\[ \begin{align*}
-\beta_m y_2(t, x) \tilde{l}_m(t, x) + \beta_H \beta_m y_2(t, x) &= s_L r(t, x) \tilde{l}_m(t, x) + \partial_t \tilde{l}_m(t, x) \\
-\bar{\beta}_m y_2(t, x) \tilde{l}_m(t, x) + \beta_H \bar{\beta}_m y_2(t, x) &= s_L r(t, x) \tilde{l}_m(t, x) + \partial_t \tilde{l}_m(t, x)
\end{align*} \]

\[ \Rightarrow (\beta_m - \bar{\beta}_m) (-y_2(t, x) \tilde{l}_m(t, x)) + (\beta_H \beta_m - \beta_H \bar{\beta}_m) y_2(t, x) = 0 \]

{\bf Mean:} \(- (\beta_m - \bar{\beta}_m) < -y_2 \tilde{l}_m > + (\beta_H \beta_m - \beta_H \bar{\beta}_m) < y_2 > = 0. \)

{\bf Assumption 2:} there exist \( t_1, t_2 \in (0, T) \) such that

\[ \begin{vmatrix}
<y_2 \tilde{l}_m > (t_1) & < y_2 > (t_1) \\
<y_2 \tilde{l}_m > (t_2) & < y_2 > (t_2)
\end{vmatrix} \neq 0, \]

hence: \( \beta_m = \bar{\beta}_m, \ \beta_H = \bar{\beta}_H. \)
Estimation of $d_1$, $d_2$ from $P_1$

\[ P_1 := d_1 \Delta y_1(t, \mathbf{x}) + d_2 \Delta y_2(t, \mathbf{x}) - \partial_t y_1(t, \mathbf{x}) - y_2(t, \mathbf{x}) - b_H(y_1(t, \mathbf{x}) - 1) - (\gamma + b_H)y_2(t, \mathbf{x}). \]

After multiplying $P_1$ by a test function and integrating:

\[
\begin{align*}
    d_1 \int_0^t \left( \int_{\Omega_1} S_H(\tau, s) \cdot \Delta \psi(s) \, ds \right) \, d\tau &+ d_2 \int_0^t \left( \int_{\Omega_1} l_H(\tau, s) \cdot \Delta \psi(s) \, ds \right) \, d\tau \\
    = \int_{\Omega_1} \left( S_H(t, s) + l_H(t, s) \right) \psi(s) \, ds - \int_{\Omega_1} \left( S_H(0, s) + l_H(0, s) \right) \psi(s) \, ds \\
    &+ (\gamma + b_H) \int_0^t \left( \int_{\Omega_1} l_H(\tau, s) \psi(s) \, ds \right) \, d\tau - b_H \int_0^t \left( \int_{\Omega_1} \left( 1 - S_H(\tau, s) \right) \psi(s) \, ds \right) \, d\tau.
\end{align*}
\]
Estimation of $d_1, d_2$ from $P_1$

$P_1 := d_1 \Delta y_1(t, x) + d_2 \Delta y_2(t, x) - \partial_t y_1(t, x) - y_2(t, x) - b_H(y_1(t, x) - 1) - (\gamma + b_H)y_2(t, x)$.

After multiplying $P_1$ by a test function and integrating:

$$d_1 \int_0^{t^+} \left( \int_{\Omega_1} S_H(\tau, s) \cdot \Delta \psi(s) \, ds \right) d\tau + d_2 \int_0^{t^+} \left( \int_{\Omega_1} I_H(\tau, s) \cdot \Delta \psi(s) \, ds \right) d\tau$$

$$= \int_{\Omega_1} \left( S_H(t, s) + I_H(t, s) \right) \psi(s) \, ds - \int_{\Omega_1} \left( S_H(0, s) + I_H(0, s) \right) \psi(s) \, ds$$

$$(\gamma + b_H) \int_0^{t^+} \left( \int_{\Omega_1} I_H(\tau, s) \psi(s) \, ds \right) d\tau - b_H \int_0^{t^+} \left( \int_{\Omega_1} \left( 1 - S_H(\tau, s) \right) \psi(s) \, ds \right) d\tau. \tag{27}$$

Given a discretized time interval of $[0, T], 0 = t_1 < t_2 < \cdots < t_{M+1} = T$, for $k = 1, 2, \ldots, M$, we denote

$$A(k, 1) = \int_0^{t_{k+1}} \left( \int_{\Omega_1} S_H(\tau, s) \cdot \Delta \psi(s) \, ds \right) d\tau; \quad A(k, 2) = \int_0^{t_{k+1}} \left( \int_{\Omega_1} I_H(\tau, s) \cdot \Delta \psi(s) \, ds \right) d\tau,$$

$$B(k) = \int_{\Omega_1} \left( S_H(t_{k+1}, s) + I_H(t_{k+1}, s) \right) \psi(s) \, ds - \int_{\Omega_1} \left( S_H(0, s) + I_H(0, s) \right) \psi(s) \, ds$$

$$(\gamma + b_H) \int_0^{t_{k+1}} \left( \int_{\Omega_1} I_H(\tau, s) \psi(s) \, ds \right) d\tau - b_H \int_0^{t_{k+1}} \left( \int_{\Omega_1} \left( 1 - S_H(\tau, s) \right) \psi(s) \, ds \right) d\tau. \tag{28}$$

Since (23) is linear with respect to $d_1$ and $d_2$, the following system is deduced:

$$Ad = B,$$

where $d = (d_1, d_2)^T$. 
Estimation of $d_1$, $d_2$ from $P_1$

$$P_1 := d_1 \Delta y_1(t, x) + d_2 \Delta y_2(t, x) - \partial_t y_1(t, x) - y_2(t, x) - b_H(y_1(t, x) - 1) - (\gamma + b_H)y_2(t, x)$$

$(y_1 = S_H, y_2 = I_H)$

Simulations

- $\Omega = (0, 1)^2$ and the measurements are supposed to be done during 20 days at discrete times $(t_i)_{i=1,...,M}$ with a sampling period equal to 0.1.
- $d_1 = 0.1$ and $d_2 = 0.01$ (susceptible populations diffuse through the domain faster than the infected populations)
- Initial conditions:

![Initial distributions in the domain $\Omega$](image)

Figure: Initial distributions in the domain $\Omega$

- The outputs $S_H$ and $I_H$ obtained with the exact parameter values are disturbed by a Gaussian noise so that the relative error $\lambda$ has a maximal value equal to 1%, 5%, 10% and 20%.
✓ Ad = B, where \( d = (d_1, d_2)^T \) is solved with the QR factorization which does not necessitate any initial guess.

✓ The integrals are estimated by the trapezoidal formula in one dimension and the composite trapezoidal formula in two-dimensional.

✓ \( d_1 = 0.1 \) and \( d_2 = 0.01 \).

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Towards discontinuous and/or spatial models...

- Extension of identifiability results of ODEs systems to discontinuous systems and more generally to PDEs systems.

- Develop numerical methods to exploit IO relations in that case.

- Work on other systems with other constraints.
Thank you for your attention!