Nonlinear Observers Robust to Measurement Errors and their Applications in Control and Synchronization

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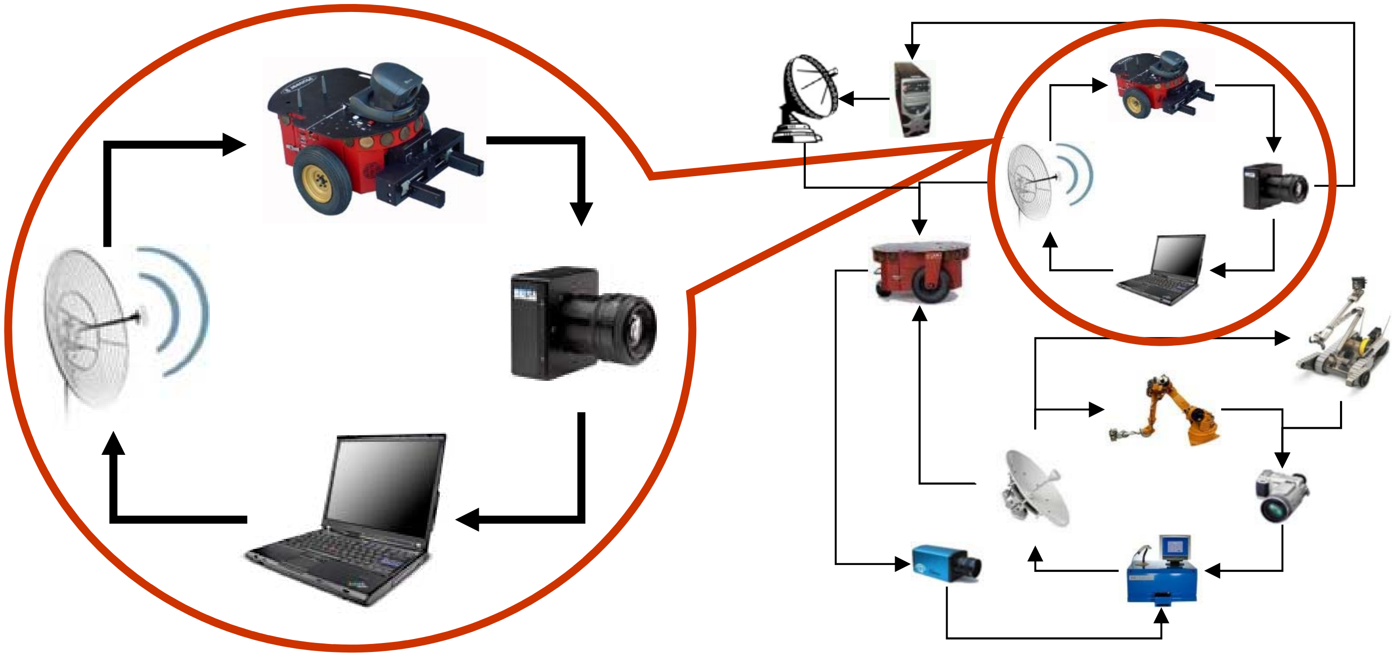
LAAS-CNRS Toulouse, 28 June 2017

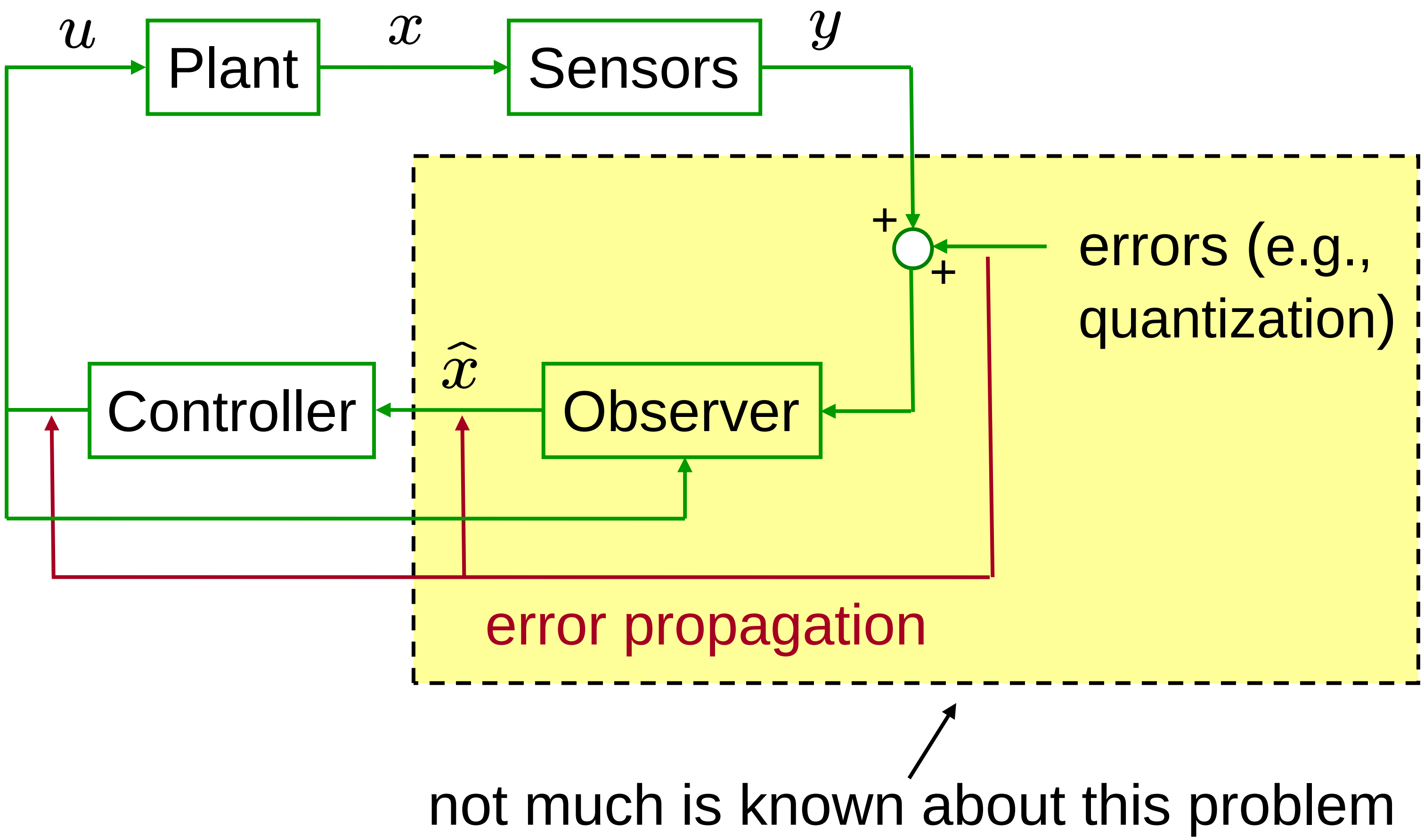
INFORMATION FLOW in CONTROL SYSTEMS



- Coarse sensing
- Limited communication capacity
- Security considerations
- Event-driven actuators
- Theoretical interest

Limited information $\Rightarrow$ errors $\Rightarrow$ need robust algorithms

OBSERVER–BASED OUTPUT FEEDBACK CONTROL



Input-to-state stability (ISS) provides a framework for quantifying robustness (graceful error propagation)

TALK OUTLINE

- Fresh look at input-to-state stability (ISS): asymptotic ratio
  [L-Shim, An asymptotic ratio characterization of ISS, TAC, 2015]
- Observers robust to measurement disturbances: formulation and Lyapunov condition

Hyungbo Shim

- Application to output feedback control design
  [Shim-L, Nonlinear observers robust to measurement disturbances in an ISS sense, TAC, 2016], see also [Shim-L-Kim, CDC, 2009]
- Applications to robust synchronization
  - electric power generators [with A. Domínguez-Garcia]
  - Lorenz chaotic system [with B. Andrievsky and A. L. Fradkov]

TALK OUTLINE

- Fresh look at input-to-state stability (ISS): asymptotic ratio

- Observers robust to measurement disturbances: formulation and Lyapunov condition

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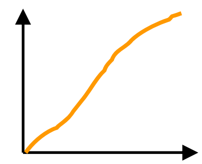
- Applications to robust synchronization

INPUT–to–STATE STABILITY (ISS) [Sontag '89]

System $\dot{x} = f(x, d)$ is ISS if its solutions satisfy

$$\boxed{|x(t)| \le \beta(|x(0)|, t) + \gamma\left(\|d\|_{[0,t]}\right)}$$



class-$\mathcal{K}_\infty$ fcn

where $\gamma \in \mathcal{K}_\infty$, $\beta(\cdot, t) \in \mathcal{K}_\infty$, $\beta(r, \cdot) \searrow 0$

ISS $\Leftrightarrow$ existence of ISS Lyapunov function:
pos. def., rad. unbdd, C^1 function V satisfying

$$|x| \ge \rho(|d|) \;\Rightarrow\; \dot{V} < 0 \qquad (\rho \in \mathcal{K}_\infty)$$

or equivalently

$$\dot{V} \le -\alpha(|x|) + \chi(|d|) \qquad (\alpha, \chi \in \mathcal{K}_\infty)$$

$\dot{x} = f(x, 0)$ GAS $\not\Rightarrow$ $\dot{x} = f(x, d)$ ISS, e.g.:

$\dot{x} = -x + xd$ (x unbdd for $d \equiv 2$)

$\dot{x} = -x + x^2 d$ (may have $x \uparrow \infty$ even if $d \to 0$)

ASYMPTOTIC–RATIO ISS LYAPUNOV FUNCTIONS

$$|x| \geq \rho(|d|) \;\Rightarrow\; \dot V < 0 \qquad (\rho \in \mathcal{K}_\infty) \qquad (1)$$

$$\dot V \leq -\alpha(|x|) + \chi(|d|) \qquad (\alpha, \chi \in \mathcal{K}_\infty) \qquad (2)$$

Definition: pos. def., rad. unbdd, C^1 function V is an asymptotic-ratio ISS Lyapunov function if

$$\dot V \leq -\alpha_3(|x|) + g(|x|, |d|)$$

where $\alpha_3 \in \mathcal{K}_\infty$, g is continuous non-negative, $g(r, \cdot)$ is non-decreasing for each r, with $g(r,0) = 0$, and

$$\limsup_{r\to\infty} \frac{g(r,s)}{\alpha_3(r)} < 1 \qquad \forall\, s \geq 0$$

Theorem: ISS $\Leftrightarrow \exists$ asymptotic-ratio ISS Lyapunov function

Proof of $\Rightarrow$ follows from characterization of ISS via (2)

Proof of $\Leftarrow$ proceeds by constructing ρ as in (1)

ASYMPTOTIC–RATIO ISS LYAPUNOV FUNCTIONS

$$|x| \geq \rho(|d|) \;\Rightarrow\; \dot{V} < 0 \qquad (\rho \in \mathcal{K}_\infty) \tag{1}$$

$$\dot{V} \leq -\alpha(|x|) + \chi(|d|) \qquad (\alpha, \chi \in \mathcal{K}_\infty) \tag{2}$$

Definition: pos. def., rad. unbdd, C^1 function V is an *asymptotic-ratio ISS Lyapunov function* if

$$\dot{V} \leq -\alpha_3(|x|) + g(|x|, |d|)$$

where $\alpha_3 \in \mathcal{K}_\infty$, g is continuous non-negative,

$g(r, \cdot)$ is non-decreasing for each r, with $g(r, 0) = 0$, and

$$\limsup_{r \to \infty} \frac{g(r,s)}{\alpha_3(r)} < 1 \quad \forall\, s \geq 0$$

Example (scalar): $\dot{x} = -\frac{1}{1+d^2}x + d, \;\; V(x) := \frac{1}{2}x^2$

$$\dot{V} = -\frac{x^2}{1+d^2} + xd = \underbrace{-x^2}_{\alpha_3(|x|)} + \underbrace{x^2\frac{d^2}{1+d^2} + xd}_{g(|x|,|d|)} \quad \checkmark$$

No info about ISS gain

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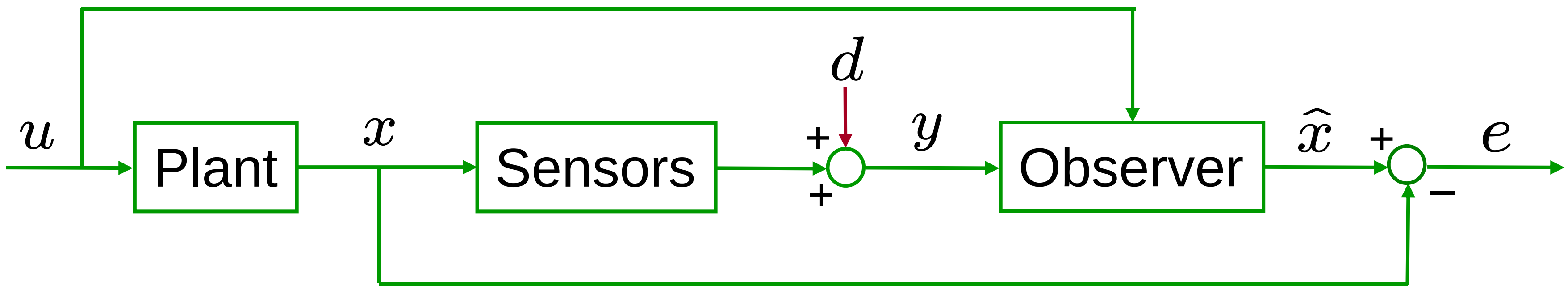
ROBUST OBSERVER DESIGN PROBLEM



Plant: $\dot{x} = f(x,u), \quad y = h(x,d) \quad (x \in \mathbb{R}^n)$

Observer: $\dot{z} = F(z,y,u), \quad \widehat{x} = H(z,y) \quad (z \in \mathbb{R}^m)$

Full-order observer: $\widehat{x} = z,\ m = n$; reduced-order: $m < n$

State estimation error: $e := \widehat{x} - x = H(z, h(x,d)) - x$

Robustness issue: can have $e \to 0$ when $d \equiv 0$
yet $e \nearrow \infty$ for arbitrarily small $d \neq 0$

DISTURBANCE–to–ERROR STABILITY (DES)

Plant: $\dot{x} = f(x,u), \quad y = h(x,d)$

Observer: $\dot{z} = F(z,y,u), \quad \hat{x} = H(z,y)$

Estimation error: $e := \hat{x} - x$

ISS-like robustness notion: call observer DES if

$$|e(t)| \le \beta(|e(0)|,t) + \gamma\left(\|d\|_{[0,t]}\right) \qquad \begin{array}{l}\beta \in \mathcal{KL},\\ \gamma \in \mathcal{K}_\infty\end{array}$$

Known conditions for this [Sontag-Wang '97, Angeli '02] are very strong

Also, DES is coordinate dependent as global error convergence is coordinate dependent: $z \to x \not\Rightarrow \Phi(z) \to \Phi(x)$

(e.g. $x(t) = e^{2t}$, $z(t) = e^{2t} + e^{-t}$, $\Phi(z) := z^2 = \underbrace{e^{4t}}_{x^2} + \underbrace{2e^{t}}_{\to\infty} + e^{-2t}$)

Path toward less restrictive, coordinate-invariant robustness property: impose DES only as long as x, u are bounded

QUASI–DISTURBANCE–to–ERROR STABILITY (qDES)

$$\dot{x} = f(x,u), \quad y = h(x,d)$$
$$\dot{z} = F(z,y,u), \quad \hat{x} = H(z,y)$$
$$e = \hat{x} - x$$

Definition: observer is quasi-Disturbance-to-Error Stable (qDES) if $\forall\, K > 0\ \exists\, \beta_K \in \mathcal{KL},\ \gamma_K \in \mathcal{K}_\infty$ such that

$$\boxed{|e(t)| \le \beta_K(|e(0)|, t) + \gamma_K(\|d\|_{[0,t]})}$$

whenever $\|u\|_{[0,t]}, \|x\|_{[0,t]} \le K$

Example: $\dot{x} = -x + x^2 u,\ y = x + d,\ \dot{z} = -z + y^2 u$

$\dot{e} = -e + 2xud + ud^2$ qDES but not DES

The qDES property is invariant to coordinate changes

LYAPUNOV CONDITION for qDES OBSERVER

$$\dot{x} = f(x,u), \quad y = h(x,d)$$
$$\dot{z} = F(z,y,u), \quad \hat{x} = H(z,y)$$
$$e = \hat{x} - x$$

Theorem: Observer is qDES if $\exists\, V(z,x)$ satisfying:

- $\underset{\mathcal{K}_\infty}{\underset{\cap}{\alpha_1}}(|e|) \le V(z,x) \le \underset{\text{positive non-decreasing}}{\lambda}(|x|)\underset{\mathcal{K}_\infty}{\underset{\cap}{\alpha_2}}(|e|)$

- $\dot{V}(z,x,u,d) \le -W(z,x,u,d) + g(z,x,u,d)$ s.t.

  $W \ge \alpha_3(|e|, |x| \vee |u|) \in \mathcal{KL}$ $\qquad$ $g(z,x,u,0) \le 0$

$\forall\, K > 0$, $\dfrac{g}{W} \le \theta_K(|e|,|d|)$ when $|u|,|x| \le K$

$\limsup_{r\to\infty} \theta_K(r,s) < 1$, $\theta_K(r,\cdot)$ non-decreasing

- Some mild regularity assumptions

LYAPUNOV CONDITION for qDES OBSERVER

$$\dot{x} = f(x,u), \quad y = h(x,d)$$
$$\dot{z} = F(z,y,u), \quad \hat{x} = H(z,y)$$
$$e = \hat{x} - x$$

Theorem: Observer is qDES if $\exists\, V(z,x)$ satisfying:

- $\underbrace{\alpha_1}_{\in\, \mathcal{K}_\infty}(|e|) \le V(z,x) \le \underbrace{\lambda}_{\text{positive non-decreasing}}(|x|)\underbrace{\alpha_2}_{\in\, \mathcal{K}_\infty}(|e|)$
- $\boxed{\dot{V}(z,x,u,d) \le -W(z,x,u,d) + g(z,x,u,d)}$ s.t.

$$W \ge \alpha_3(|e|, |x| \vee |u|) \in \mathcal{KL} \qquad g(z,x,u,0) \le 0$$

$\forall\, K > 0,$ $\boxed{\dfrac{g}{W} \le \theta_K(|e|,|d|)}$ when $|u|,|x| \le K$

$\boxed{\limsup\limits_{r\to\infty} \theta_K(r,s) < 1}$ $\quad \theta_K(r,\cdot)$ non-decreasing

Recall asymptotic-ratio ISS Lyapunov functions

Proof – construct $\rho \in \mathcal{K}_\infty$ s.t. $V \ge \rho(|d|) \;\Rightarrow\; \dot{V} < 0$

EXAMPLE: LINEARIZED ERROR DYNAMICS

Plant: $\dot{x} = Ax + f(Cx, u), \quad y = Cx + d$

with (A, C) detectable pair, so $\exists\, L$ s.t. $A - LC$ is Hurwitz

Observer: $\dot{z} = Az + f(y, u) + L(y - Cz), \quad \hat{x} = z$

[Krener-Isidori '83]

Error dynamics:

$$\dot{e} = \dot{z} - \dot{x} = (A - LC)e + Ld + f(Cx + d, u) - f(Cx, u)$$

$V := e^\top P e$ where $P(A-LC) + (A-LC)^\top P = -I$

$$\dot{V} = \underbrace{-|e|^2}_{\alpha_3(|e|)} + \underbrace{2e^\top PLd + 2e^\top P(f(Cx + d, u) - f(Cx, u))}_{g(z,x,u,d)}$$

$$|u|, |x| \le K \;\Rightarrow\; \frac{g}{\alpha_3} \le \frac{2\|PL\||d| + 2\|P\|\phi_K(|d|)}{|e|} \xrightarrow[|e|\to\infty]{} 0$$

Hence Krener-Isidori observer is already qDES

Also qDES are high-gain obs [Gauthier], circle-criterion obs [Arcak]

REDUCED–ORDER qDES OBSERVERS

Plant (after a coordinate change):

$$\dot{x}_1 = f_1(x_1, x_2, u)$$
$$\dot{x}_2 = f_2(x_1, x_2, u)$$
$$y = x_1 + d$$

Observer:

$$\dot{z} = f_2(y, z, u)$$
$$\hat{x}_1 = y$$
$$\hat{x}_2 = z$$

$$e := z - x_2, \quad V = V(e)$$

$$\dot{V} = \frac{\partial V}{\partial e}\Big[f_2(y, x_2 + e, u) - f_2(x_1, x_2, u)\Big]$$

$$= \underbrace{\frac{\partial V}{\partial e}\Big[f_2(y,x_2+e,u) - f_2(y,x_2,u)\Big]}_{\text{assume this is } \leq -\alpha_3(|e|)} + \underbrace{\frac{\partial V}{\partial e}}_{\text{assume this has norm } \leq \alpha_4(|e|)}\underbrace{\Big[f_2(y,x_2,u) - f_2(x_1,x_2,u)\Big]}_{\text{upper-bounded by } \phi_K(|d|)}$$

If $\boxed{\limsup_{r\to\infty} \frac{\alpha_4(r)}{\alpha_3(r)} = 0}$ then by our theorem observer is qDES

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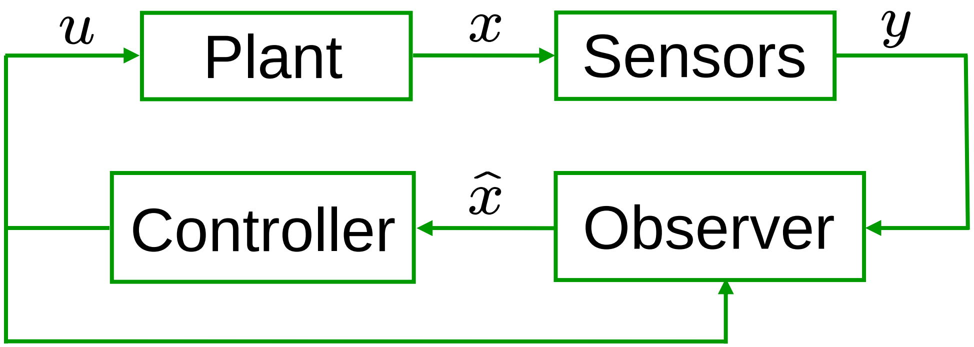
OBSERVER–BASED OUTPUT FEEDBACK REVISITED



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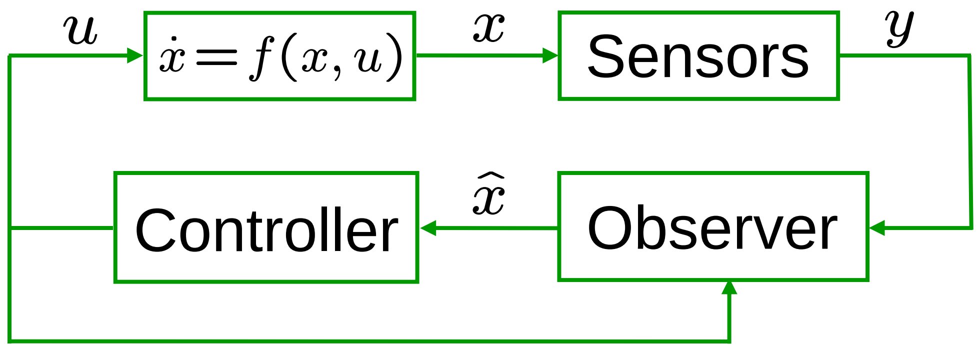

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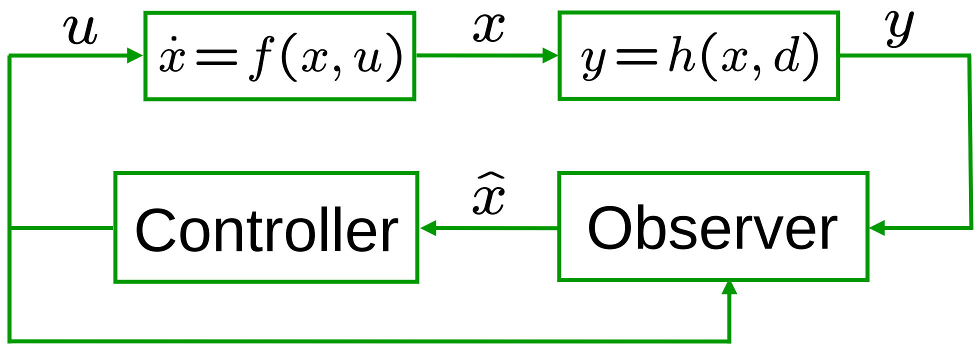

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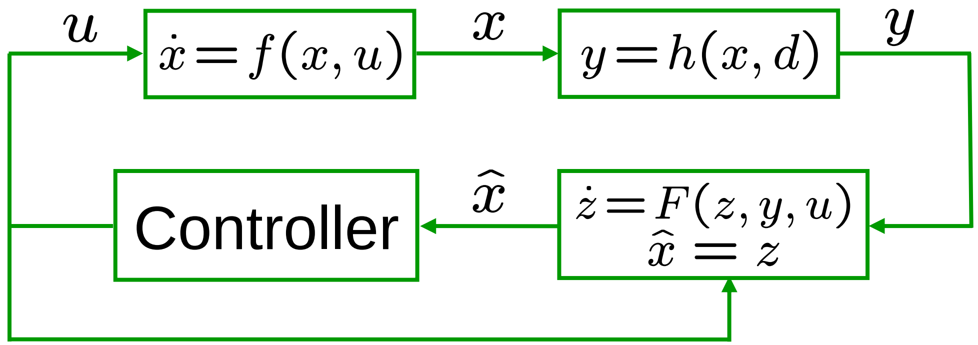
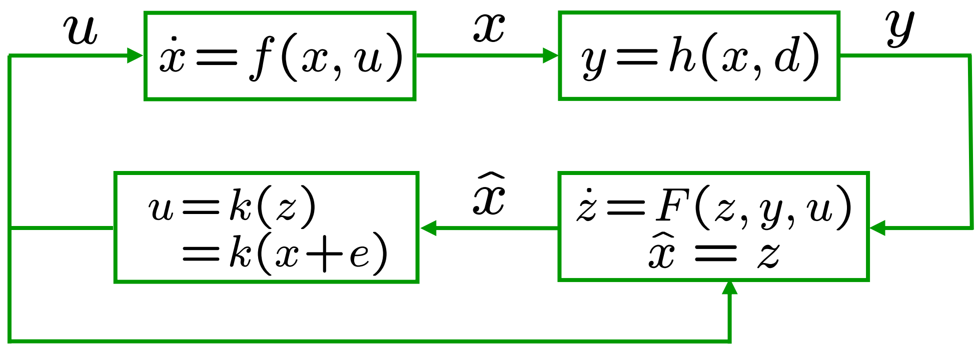


OBSERVER–BASED OUTPUT FEEDBACK REVISITED



- Assume observer is qDES w.r.t. d:

$$|e(t)| \le \beta_K(|e(0)|,t) + \gamma_K(\|d\|_{[0,t]}) \qquad \|u\|, \|x\| \le K$$

- Assume controller is ISS w.r.t. e:

$$|x(t)| \le \widehat{\beta}(|x(0)|,t) + \widehat{\gamma}\left(\|e\|_{[0,t]}\right)$$

[Freeman, Fah, Jiang et al., Sanfelice-Teel, Ebenbauer et al.]

Cascade argument: closed-loop system is quasi-ISS

$$\left|\begin{pmatrix} x(t) \\ z(t) \end{pmatrix}\right| \le \bar{\beta}_K\left(\left|\begin{pmatrix} x(0) \\ z(0) \end{pmatrix}\right|, t\right) + \bar{\gamma}_K\left(\|d\|_{[0,t]}\right) \qquad \|u\|, \|x\| \le K$$

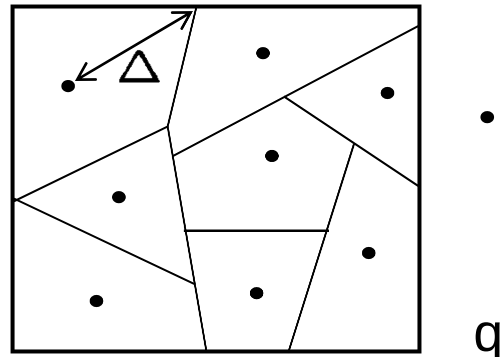
APPLICATION to QUANTIZED OUTPUT FEEDBACK

$$\left|\begin{pmatrix} x(t) \\ z(t) \end{pmatrix}\right| \le \bar\beta_K\left(\left|\begin{pmatrix} x(0) \\ z(0) \end{pmatrix}\right|, t\right) + \bar\gamma_K\left(\|d\|_{[0,t]}\right) \qquad \|u\|, \|x\| \le K$$

$y = q(h(x)) = h(x) + d$

d – quantization error

$|h(x)| \le M \;\Rightarrow\; |d| \le \Delta$



quantizer

- $\exists\, K = K(M)$ & upper bounds on Δ, $|x(0)|$, $|z(0)|$ s.t.
  - $|x(t)|$, $|u(t)|$ remain $\le K$
  - $\limsup_{t\to\infty} \left|\begin{pmatrix} x(t) \\ z(t) \end{pmatrix}\right| \le \bar\gamma_K(\Delta)$
- Contraction is guaranteed if quantization is fine enough
- Can achieve asymptotic stability by dynamic “zooming”

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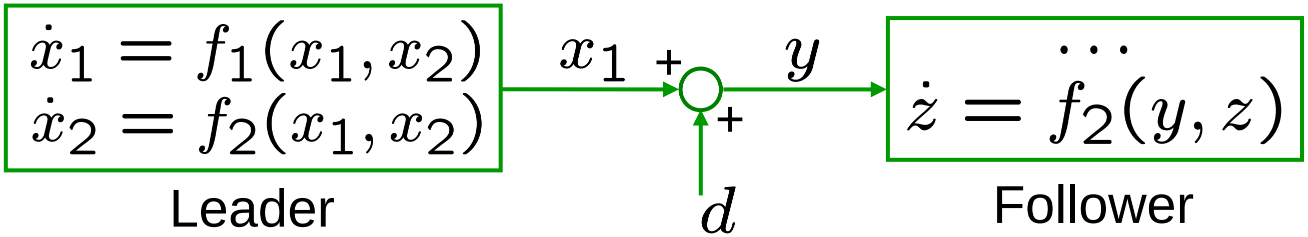
ROBUST SYNCHRONIZATION and qDES OBSERVERS



$e := z - x_2$

Robust synchronization: $\forall\, K > 0 \;\exists\, \beta_K \in \mathcal{KL},\; \gamma_K \in \mathcal{K}_\infty$ s.t.

$$\boxed{|e(t)| \le \beta_K(|e(0)|, t) + \gamma_K(\|d\|_{[0,t]})}$$

whenever $\|x\|_{[0,t]} \le K$

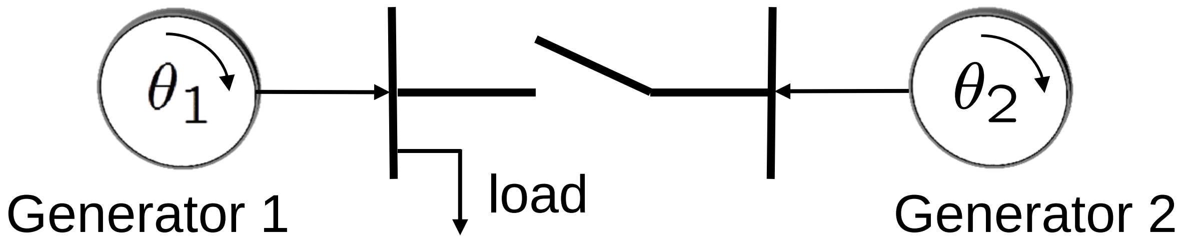
Equivalently: follower is a reduced-order qDES observer for leader

Sufficient condition from before: $\exists\, V = V(e)$ s.t. $\left|\frac{\partial V}{\partial e}\right| \le \alpha_4(|e|)$,

$\frac{\partial V}{\partial e}(e)\Big(f_2(x_1, z) - f_2(x_1, x_2)\Big) \le -\alpha_3(|e|)$, and

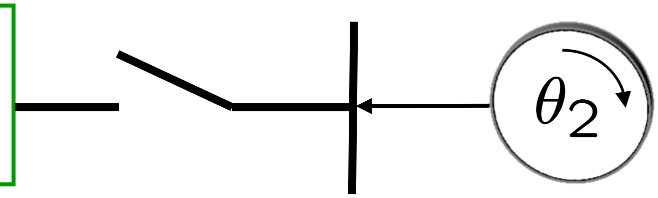
$\limsup\limits_{r \to \infty} \frac{\alpha_4(r)}{\alpha_3(r)} = 0$ (asymptotic ratio condition)

APPLICATION EXAMPLE #1



APPLICATION EXAMPLE #1

$$\boxed{\begin{aligned} \dot{\theta}_1 &= \omega_1 \\ \dot{\omega}_1 &= -D_1\omega_1 + u_1 \end{aligned}}$$



Generator 1 Generator 2

$u_1(t, \theta_1) =$ mechanical power (integral control) **+** electric load (const), makes $\omega_1 \to$ desired value $\Rightarrow$ state is bounded

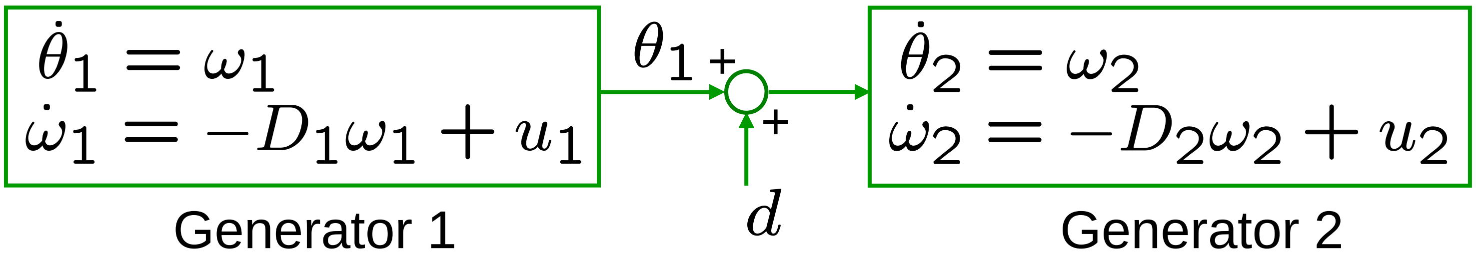
APPLICATION EXAMPLE #1

$$\dot{\theta}_1 = \omega_1 \qquad\qquad \dot{\theta}_2 = \omega_2$$
$$\dot{\omega}_1 = -D_1\omega_1 + u_1 \qquad \dot{\omega}_2 = -D_2\omega_2 + u_2$$

Generator 1 Generator 2

$u_1(t, \theta_1) =$ mechanical power (integral control) **+** electric load (const), makes $\omega_1 \to$ desired value $\Longrightarrow$ state is bounded

APPLICATION EXAMPLE #1



$u_1(t,\theta_1) =$ mechanical power (integral control) $+$ electric load (const), makes $\omega_1 \to$ desired value $\Rightarrow$ state is bounded

Measurements: PMU corrupted by disturbance

$$u_2 = (D_2 - D_1)\omega_2 + u_1(t, \theta_1 + d)$$

Objective: connect 2nd generator when $\theta_1 \approx \theta_2,\ \omega_1 \approx \omega_2$

$V = e^2$ gives DES (ISS) from d to $e := \omega_2 - \omega_1$

- ω_1 and ω_2 will synchronize with error $\frac{k}{D_1}\|d\|$
- due to phase drift, will have $\theta_1 \approx \theta_2$ at some time

$\Rightarrow$ can connect 2nd generator

APPLICATION EXAMPLE #1

$$\dot\theta_1 = \omega_1$$
$$\dot\omega_1 = -D_1\omega_1 + u_1$$

Generator 1

θ_1 + + d

$$\dot\theta_2 = \omega_2$$
$$\dot\omega_2 = -D_2\omega_2 + u_2$$

Generator 2

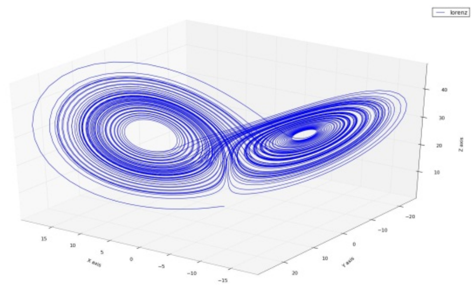
More detailed model:

$$\left.\begin{array}{ll} \bullet\ \text{damping} & D_1 = D_1(\theta_1) \\ \bullet\ \text{electrical load} & \ell = \ell(t) \end{array}\right\}\ \text{slowly varying}$$

Analysis more challenging, but can still show state boundedness and qDES from d to e

Future work: network case

APPLICATION EXAMPLE #2

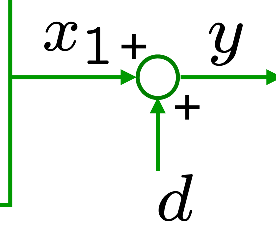


Lorenz system

APPLICATION EXAMPLE #2

$$\begin{aligned}\dot{x}_1 &= \sigma x_2 - \sigma x_1\\ \dot{x}_2 &= \theta x_1 - x_2 - x_1 x_3\\ \dot{x}_3 &= -\beta x_3 + x_1 x_2\end{aligned}$$

Lorenz system



$$\begin{aligned}\dot{z}_2 &= \theta y - z_2 - y z_3\\ \dot{z}_3 &= -\beta z_3 + y z_2\end{aligned}$$

Observer

Can show x is bounded using $V(x) = x_1^2 + x_2^2 + (x_3 - \sigma - \theta)^2$

Can show qDES from d to $e := \binom{z_2 - x_2}{z_3 - x_3}$ using $V(e) = e_2^2 + e_3^2$

For d arising from time sampling and quantization, we have an explicit bound on synchronization error which is inversely proportional to data rate

FUTURE WORK

Nonlinear qDES observer design:

- Identify system classes to which Lyapunov conditions apply
- Develop more constructive procedures for observer design

Quantized output feedback control:

- Relax ISS controller assumption

Robust synchronization:

- Study other coupled oscillator network models
- Look for examples in other areas (e.g., vehicle formations)

Papers and preprints available at *liberzon.csl.illinois.edu*