Differential analysis of nonlinear systems

Rodolphe Sepulchre -- University of Cambridge, UK

Toulouse, February 2015

Joint work with Fulvio Forni

What is the steady-state response of a nonlinear system?

When/how can it be analyzed by linearization techniques?

Induction motor control

Goal: regulate the speed and reject the load torque under voltage and current constraints

A bottleneck of energy-based control

\[
\dot{\theta} = \omega \\
J\dot{\omega} = -k\omega - mgl\sin{\theta} + \tau
\]

? A Lyapunov function for \( \tau = \tau^* \neq 0 \)?
Energy is a Lyapunov function only in the absence of input

\[ \tau = 0: \quad V = \frac{1}{2} \omega^2 + mgl(1 - \cos \theta) \]
\[ \dot{V} \leq -k\omega^2 \quad \Rightarrow \quad (\theta, \omega) = (0, 0) \]

Almost global asymptotic stability

\[ \tau \neq 0: \]

the question could be a hard one...

\[ \tau = 0: \quad V = J_2 + mgl(1 - \cos \theta) \]
\[ \dot{V} \leq -k\omega^2 \quad \Rightarrow \quad (\theta, \omega) = (0, 0) \]

Differential stability (contraction)

A linear map is (Lyapunov) stable if it leaves a ball invariant.

A dynamical system is differentially stable (non expanding) if its linearization along an arbitrary trajectory is Lyapunov stable.

Alternative: first method of Lyapunov...

From the linearization

\[ \begin{pmatrix} 0 & 1 \\ -\cos(\theta) & -k \end{pmatrix} \]

we infer a local steady-state response around the fixed point \( \theta \)

provided that \( |\bar{\theta}| < \frac{\pi}{2} \)

Pitfalls of a differential approach

Kalman's conjecture (1964): stability of the “frozen” linearization implies stability of the NL feedback system.

(disproven for \( n > 3 \)).

Studying differential stability is about studying Lyapunov stability of a time-varying linear system... Possibly hard question

In systems and control: Lohmiller & Slotine (1998), and many others since then...

Terminology: contraction, convergence, incremental stability, ...
Lyapunov analysis of a time-varying linear system

Linear consensus algorithms are linear time-varying systems

\[ x(t+1) = A(t)x(t), \]

where for each \( t \), \( A(t) \) is row stochastic, i.e.

A is nonnegative: \( a_{ij} \geq 0 \)
each row sums to one: \( A(t)1 = 1 \)

Convergence analysis relies on a non-quadratic Lyapunov function proposed by Tsitsiklis (1986):

It is known that no common quadratic Lyapunov exists in general. (See Olshevsky & Tsitsiklis 08 for a discussion)

\[ V(x) = \max_{1 \leq i \leq n} x_i - \min_{1 \leq i \leq n} x_i \]

Tsitsiklis Lyapunov function follows from the projective contraction property

Consensus algorithms are positive maps.

In the positive orthant, Hilbert (projective) metric is

\[ d(x, y) = \log \frac{\max(x_i/y_i)}{\min(x_i/y_i)} \]

Lyapunov function = (projective) distance to consensus

A key lesson: linear time-varying positive systems admit a common (non-quadratic) projective contraction measure
Compared to contraction, a cone replaces the ball...

A linear map is \textit{(Lyapunov) stable} if it leaves a ball invariant.

A dynamical system is \textit{differentially stable (non expanding)} if its linearization along an arbitrary trajectory is Lyapunov stable.

In systems and control: Lohmiller & Slotine (1998), and many others since then...

Terminology: contraction, convergence, incremental stability, ...

Textbook Perron-Frobenius theory

As a geometric concept, positivity is \textit{not} antagonist to oscillations

An obstacle to think of positivity as a geometric property?
Strict differential positivity and nonlinear oscillations

Theorem 3 is a differential version of Perron-Frobenius theory.
The corollary is akin to Poincare Bendixon theorem for planar systems.
Strict differential positivity, similarly to the topology of the plane, enforces a one-dimensional asymptotic behavior.

Contents

- Preview
- Lyapunov analysis versus contraction analysis
- The integral properties of differential stability and differential positivity
- The asymptotic behavior of differentially positive systems
- Differential positivity of the nonlinear pendulum
- Outlook

Contraction and Lyapunov: (only) two ways to analyze asymptotic behavior

\[ x_+ = F(x) \]

contraction
Lyapunov

\[ d(F(x), F(y)) \leq \alpha \, d(x, y) \]
\[ \alpha < 1 \]

contraction rate

\[ V(F(x)) < V(x) \]
AND minimum at \( x = x^* \)

Distance between any two trajectories decreases
Distance to fixed point decreases

Contraction is an incremental stability property

\[ x_+ = F(x) \]

contraction
Lyapunov

\[ d(F(x), F(y)) \leq \alpha \, d(x, y) \]
\[ \alpha < 1 \]

contraction rate

\[ V(F(x)) < V(x) \]
AND minimum at \( x = x^* \)

Distance between any two trajectories decreases
Distance to fixed point decreases
Contraction can be studied \textit{differentially}

\[
x_+ = F(x)
\]

\[d(F(x), F(y)) \leq \alpha \, d(x, y)\]

For \( y = x + \delta x \), this means

\[d(F(x), F(x) + DF(x)\delta x) \leq \alpha \, d(x, x + \delta x)\]

i.e. a contraction property for the differential system

\[\delta x_+ = DF(x)\delta x\]

Contraction analysis is about building an infinitesimal contraction measure, not a distance

Contraction: a sample of the control literature


\begin{center}
\textbf{Contents}
\end{center}

- Preview
- Lyapunov analysis versus contraction analysis
- The integral properties of differential stability and differential positivity
- The asymptotic behavior of differentially positive systems
- Differential positivity of the nonlinear pendulum
- Outlook
Integral properties

The integral property of differential stability is **incremental stability**

The integral property of differential positivity is **monotonicity**

**Theorem 1**: Consider the system (1) on a smooth manifold $M$ with $f$ of class $C^1$, a connected and forward invariant set $C$, a set of isolated points $\Omega \subset M$, and a function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. Let $V$ be a candidate Finsler Lyapunov function such that

$$DV(x, \delta x)[f(t, x), Df(t, x)[0, \delta z]] = -\alpha(V(x, \delta z))$$

for each $t \geq t_0$, each $x \in C \subseteq M$, and each $\delta x \in T_xM$. Then, (1) is

(1S) incrementally stable on $C$ if $\alpha(s) = 0$ for each $s \geq 0$;
(1AS) incrementally asymptotically stable on $C$ if $\alpha$ is a K function;
(1EAS) incrementally exponentially stable on $C$ if $\alpha(s) = \lambda s > 0$ for each $s > 0$.

Incremental Lyapunov stability

Key observation (D. Angeli): incremental stability is equivalent to stability of the set

$$\{(x, z) \mid x = z\}$$

for the product dynamics

$$\dot{x} = f(x, t) \quad \dot{z} = f(z, t)$$

Example of a global Lyapunov function:

$$V(x, z) = (x - z)^T P(x - z)$$

Any other example?

Finsler Lyapunov function: a Lyapunov function in the tangent bundle

**Definition 2**: Consider a manifold $M$ and a set of isolated points $\Omega \subset M$. A function $V : M \times TM \to \mathbb{R}_{\geq 0}$ that maps every $(x, \delta x) \in M \times T_xM$ to $V(x, \delta x) \in \mathbb{R}_{\geq 0}$ is a candidate Finsler Lyapunov function for (1) if

(i) $V$ is of class $C^1$ for each $x \in M$ and $\delta x \in T_xM \setminus \{0\}$;
(ii) $V(x, \delta x) > 0$ for each $x \in M \setminus \Omega$ and each $\delta x \in T_xM \setminus \{0\}$;
(iii) for some $p \geq 1$, $V(x, \delta x) = \lambda V(x, \delta x)$ for each $\lambda > 0$, $x \in M$, and each $\delta x \in T_xM$ (homogeneity);
(iv) for some $p \geq 1$, $V(x, \delta x_1 + \delta x_2)^p < V(x, \delta x_1)^p + V(x, \delta x_2)^p$ for each $x \in M \setminus \Omega$ and each $\delta x_1, \delta x_2 \in T_xM$ such that $\delta x_1 \neq \lambda \delta x_2$ and $\lambda \in \mathbb{R}$ (strict convexity).
A glocal Lyapunov theory

The local construction provides a global distance through integration along curves:

\[ d(x_1, x_2) = \inf_{\Gamma(x_1, x_2)} \int_I V(\gamma(s), D\gamma(s)[1])^{\frac{1}{2}} ds. \]

Integral properties

The integral property of differential stability is \textit{incremental stability}

The integral property of differential positivity is \textit{monotonicity}

Monotone Control Systems

David Angeli and Eduardo D. Sontag, Fellow, IEEE

I. INTRODUCTION

One of the most important classes of dynamical systems in theoretical biology is that of monotone systems. Among the classical references in this area are the textbook by Smith [27] and the papers [14] and [15] by Hirsh and [26] by Smale.

Monotone systems are those for which trajectories preserve a partial ordering on states. They include the subclass of cooperative systems.

Definition II.1: A controlled dynamical system \( \phi : \mathbb{R}_{\geq 0} \times X \times \mathcal{U}_{\infty} \rightarrow X \) is monotone if the following implication holds for all \( t \geq 0 \):

\[ u_1 \succeq u_2, \; x_1 \succeq x_2 \implies \phi(t, x_1, u_1) \succeq \phi(t, x_2, u_2). \]

Remark VIII.3: Looking at cooperativity as a notion of “incremental positivity” one can provide an alternative proof of the infinitesimal condition for cooperativity, based on the positivity of the variational equation. Indeed, assume that each system (35) is a positive linear time-varying system, along trajectories of (1). Pick arbitrary initial conditions \( \xi_1 \geq \xi_2 \in X \) and inputs \( u_1 \geq u_2 \). Let \( \Phi(t) := \phi(t, \xi_2 + h(\xi_1 - \xi_2), u_2 + h(u_1 - u_2)) \).

We have (see, e.g., [28, Th. 1]) that \( \phi(t, \xi_1, u_1) - \phi(t, \xi_2, u_2) = \Phi(1) - \Phi(0) = \int_0^1 \Phi'(h) dh = \int_0^1 z_h(t, \xi_1 - \xi_2, u_1 - u_2) dh \), where \( z_h \) denotes the solution of (35) when \( (\partial f/\partial u)(x, u) \) and \( (\partial f/\partial u)(x, u) \) are evaluated along \( \phi(t, \xi_2 + h(\xi_1 - \xi_2), u_2 + h(u_1 - u_2)) \). Therefore, by positivity, and monotonicity of the integral, we have \( \phi(t, \xi_1, u_1) - \phi(t, \xi_2, u_2) \succeq 0 \), as claimed. \( \Box \)

We remark that monotonicity with respect to other orthants corresponds to generalized positivity properties for linearizations, as should be clear by Corollary III.3.
The importance of Monotone Dynamical Systems

M.W. Hirsch* Hal Smith †

We will see that the long-term behavior of monotone systems is severely limited. Typical conclusions, valid under mild restrictions, include the following:

- If all forward trajectories are bounded, the forward trajectory of almost every initial state converges to an equilibrium.
- There are no attracting periodic orbits other than equilibria, because every attractor contains a stable equilibrium.
- In $\mathbb{R}^3$, every compact limit set that contains no equilibrium is a periodic orbit that bounds an invariant disk containing an equilibrium.
- In $\mathbb{R}^2$, each component of any solution is eventually increasing or decreasing.

A differential geometric viewpoint on PF theory

VI. Differential Perron-Frobenius theory

A. Contraction of the Hilbert metric

Bushell [10] (after Birkhoff [7]) used the Hilbert metric on cones to show that the strict positivity of a mapping guarantees contraction among the rays of the cone, opening the way to many contraction-based results in the literature of positive operators [10], [30], [39], [8], [26],

$$dx(t)(\delta x(t), \delta y(t)) \to 0$$

B. The Perron-Frobenius vector field

The Perron-Frobenius vector of a strictly positive linear map is a fixed point of the projective space. Its existence is a consequence of the contraction of the Hilbert metric, [10]. To exploit the

and nonlinear operators on Banach spaces. The usefulness of operators that are positive in some sense stems from the theorem of Perron [154] and Frobenius [48], now almost a century old, asserting that for a linear operator on $\mathbb{R}^n$ represented by a matrix with positive entries, the spectral radius is a simple eigenvalue having a positive eigenvector, and all other eigenvalues have smaller absolute value and only nonpositive eigenvectors. In 1912 Jentzsch [84] proved the existence of a positive eigenfunction with a positive eigenvalue for a homogeneous Fredholm integral equation with a continuous positive kernel.

In 1935 the topologists Alexandroff and Hopf [2] reproved the Perron-Frobenius theorem by applying Brouwer’s fixed-point theorem to the action of a positive $n \times n$ matrix on the space of lines through the origin in $\mathbb{R}^n$. This was perhaps the first explicit use of the dynamics of operators on a cone to solve an eigenvalue problem. In 1940 Rutman [169] continued in this vein by reproving Jentzsch’s theorem by means of Schauder’s fixed-point theorem, also obtaining an infinite-dimensional analog of Perron-Frobenius, known today as the Krein-Rutman theorem [103, 213]. In 1957 G. Birkhoff [20] initiated the dynamical use of Hilbert’s projective metric for such questions.

The dynamics of cone-preserving operators continues to play an important role in functional analysis; for a survey, see Nussbaum [145, 146]. One outgrowth of this work

Contents

- Preview
- Lyapunov analysis versus contraction analysis
- The integral properties of differential stability and differential positivity
- The asymptotic behavior of differentially positive systems
- Differential positivity of the nonlinear pendulum
- Outlook
The main result: the PF vector field determines the asymptotic behavior

Suppose that the trajectories of \( \Sigma \) are bounded. Then, for every \( \xi \in \mathcal{X} \), the \( \omega \)-limit set \( \omega(\xi) \) satisfies one of the following two properties:

(i) The vector field \( f(x) \) is aligned with the Perron-Frobenius vector field \( w(x) \) for each \( x \in \omega(\xi) \), and \( \omega(\xi) \) is either a fixed point or a limit cycle or a set of fixed points and connecting arcs;

(ii) The vector field \( f(x) \) is nowhere aligned with the Perron-Frobenius vector field \( w(x) \) for each \( x \in \omega(\xi) \), and either \( \liminf_{t \to \infty} |\partial_x \psi(t, 0, x) w(x)|_{\psi(t, 0, x)} = \infty \) or \( \lim_{t \to \infty} f(\psi(t)) = 0 \).

The well-behaved situation

\[ \sigma(t) w(y(t)) = f(y(t)) \]

The pathological situation

(ii) The vector field \( f(x) \) is nowhere aligned with the Perron-Frobenius vector field \( w(x) \) for each \( x \in \omega(\xi) \), and either \( \liminf_{t \to \infty} |\partial_x \psi(t, 0, x) w(x)|_{\psi(t, 0, x)} = \infty \) or \( \lim_{t \to \infty} f(\psi(t)) = 0 \).

Corollary (rephrasing)

Take \( \omega(\xi) \) of Part (ii). Then, for any \( z \in \mathcal{X} \) such that \( \omega(z) \subseteq \omega(\xi) \), the trajectory \( \psi(\cdot, 0, z) \) satisfies \( \psi(t, 0, z) \notin \mathcal{K}(\psi(t, 0, z)) \setminus \{0\} \) for each \( t \geq 0 \). If \( \omega(z) \) is not a singleton, then \( \liminf_{t \to \infty} |\partial_x \psi(t, 0, z) w(z)|_{\psi(t, 0, z)} = \infty \).

The main corollary

Under boundedness of trajectories, consider an open, forward invariant region \( C \subseteq \mathcal{X} \) that does not contain any fixed point. If the vector field \( f(x) \in \text{int} \mathcal{K}(x) \) for any \( x \in C \), then there exists a unique attractive periodic orbit contained in \( C \).
Pendulum

$$\Sigma : \begin{cases} \dot{\vartheta} &= v \\ \dot{v} &= -\sin(\vartheta) - kv + \tau \end{cases}$$

$$\delta \Sigma : \begin{cases} \delta \dot{\vartheta} &= \delta v \\ \delta \dot{v} &= -\cos(\vartheta) \delta \vartheta - k \delta v \end{cases}$$

$$k \geq 2$$

Strict Diff+ with respect to the cone field

$$K(\vartheta, v) := \delta \vartheta \geq 0, \delta v + \delta v \geq 0$$

$$k=2 \rightarrow \text{Diff}$$

Contents

- Preview
- Lyapunov analysis versus contraction analysis
- The integral properties of differential stability and differential positivity
- The asymptotic behavior of differentially positive systems
- Differential positivity of the nonlinear pendulum
- Outlook
Contents

- Preview
- Lyapunov analysis versus contraction analysis
- The integral properties of differential stability and differential positivity
- The asymptotic behavior of differentially positive systems
- Differential positivity of the nonlinear pendulum
- Outlook

Outlook:
differential positivity
= local monotonicity
+ smooth patching of local orders

- projective contraction is cheap for positive linear maps
- for a constant cone field in a linear space, differential positivity = monotonicity
- the damped pendulum is diff positive with respect to an invariant cone on the cylinder
How weak is differential positivity?

Local ordering is a weak property. Smooth global patching is a demanding property.