Controller Design via Experimental Exploration with Robustness Guarantees

Tobias Holicki
Introductory Comments

• This talk is highly inspired by the work [1].

• Related works are, e.g., [2], [3], [4], [5].

• The aim is to extend some of the aspects of [1] while focusing on a deterministic setup.

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Setting and Goal

Let us consider the feedback interconnection

\[
\begin{pmatrix}
\dot{x}(t) \\
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e(t) \\
y(t)
\end{pmatrix} =
\begin{pmatrix}
A & B_1 & B_2 & B_3 \\
C_1 & D_{11} & D_{12} & D_{13} \\
C_2 & D_{21} & D_{22} & D_{23} \\
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\end{pmatrix}
\begin{pmatrix}
x(t) \\
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for some uncertain parameter \(\Delta_0\) contained in a known compact set \(\Delta\).
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**Goal:** We wish to find a state-feedback controller

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u(t) = F_* x(t)
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which stabilizes \(\Delta_0 \star P\) and turns the closed-loop \(H_\infty\) norm is as small as possible.

I.e., we search for a minimizer of the function

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Issue: Finding an (close-to-)optimal controller is difficult as \(\Delta_0\) is unknown.
Standard Design Approaches

Via standard $H_\infty$ design, we can compute for any fixed $\Delta \in \Delta$:

$$\gamma_{\text{nom}}(\Delta) := \inf_{F \text{ stabilizes } \Delta P} \| \Delta P F \|_\infty.$$

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**Goal:** We wish to determine $\gamma_{\text{nom}}(\Delta_0)$ and design a corresponding controller.

Via standard robust design (by exploiting knowledge of $\Delta$), we can compute upper bounds $\gamma_{\text{sep}}$ on the worst-case closed-loop $H_{\infty}$ norm:

$$\inf_{F \in F} \sup_{\Delta \in \Delta} \|\Delta \star P \star F\|_{\infty} \leq \gamma_{\text{sep}}.$$  

Here, we abbreviate the set of robustly stabilizing controllers as

$$F := \{F : F \text{ stabilizes } \Delta \star P \text{ for all } \Delta \in \Delta\}.$$
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Via standard robust design (by exploiting knowledge of $\Delta$), we can compute upper bounds $\gamma_{\text{sep}}$ on the worst-case closed-loop $H_\infty$ norm:

$$\inf_{F \in \mathcal{F}} \sup_{\Delta \in \Delta} \| \Delta \star P \star F \|_\infty \leq \gamma_{\text{sep}}.$$

Here, we abbreviate the set of robustly stabilizing controllers as

$$\mathcal{F} := \{ F : F \text{ stabilizes } \Delta \star P \text{ for all } \Delta \in \Delta \}.$$ 

Clearly, we have

$$\gamma_{\text{nom}}(\Delta_0) \leq \gamma_{\text{sep}}$$

and there might be a very large gap between both values.
Setting and Goal (Continued)

**Additional Assumption:** E.g. by running and measuring multiple closed-loop experiments, the function

\[ J : F \mapsto \| \Delta \circ P \circ F \|_{\infty} \]

can be evaluated for finitely many controllers \( F_1, \ldots, F_N \).

**New Goal:** Based on this additional information, find a controller \( F \) such that \( J(F) = \| \Delta \circ P \circ F \|_{\infty} \) is much closer to \( \gamma_{\text{nom}}(\Delta_0) \) than \( \gamma_{\text{sep}} \).
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The above assumption suggests to perform a numerical minimization of a function that interpolates the data points

\[(F_1, J(F_1)), \ldots, (F_N, J(F_N)).\]

This gives rise to the following essential questions.

- How can suitable test controllers \( F_1, \ldots, F_N \) be selected systematically?
- How can the resulting data points be interpolated?
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Robust Stability

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Robust Stability

**Issue:** Stability is a critical property as interconnecting a controller to the given system that is not stabilizing can lead to catastrophic results.

**Remedy:** Following [1], we only search for robustly stabilizing controllers in $\mathbb{F}$. It is not possible to include this safety requirement for free as we usually have

$$
\gamma_{\text{nom}}(\Delta_0) = \inf_{F \text{ stabilizes } \Delta_0 \star P} J(F) < \inf_{F \in \mathbb{F}} J(F).
$$

We show later on how to get closer to $\gamma_{\text{nom}}(\Delta_0)$ by increasing the set of admissible controllers while still being able to guarantee safe operation.

Sampling and Gridding

**Issue:** It can be difficult to find controllers in $F \subset \mathbb{R}^{n_u \times n_y}$ based on gridding or sampling especially if

- the dimension of $\mathbb{R}^{n_u \times n_y}$ is large,
- $F$ is an unbounded set or
- $F$ has measure zero in $\mathbb{R}^{n_u \times n_y}$.

**Remedy:** In contrast to [1], we propose a systematic approach to find such controllers based on gridding or sampling in the compact set $\Delta$. 
Motivation

As motivation, let us define the function (assuming it is well-defined)

\[ \mathcal{F} : \Delta \rightarrow \mathbb{F}, \quad \Delta \mapsto F \in \operatorname{arg \ min}_{F \in \mathbb{F}} \| \Delta \ast P \ast F \|_{\infty}. \]

Then \( \mathcal{F}(\Delta) \) is a robustly stabilizing controller that yields the smallest \( H_\infty \)

norm of \( \Delta \ast P \ast F \) among all robustly stabilizing controllers.
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By its definition we have

\[ \| \Delta \ast P \ast \mathcal{F}(\Delta) \|_\infty \leq \| \Delta \ast P \ast F \|_\infty \text{ for all } F \in \mathbb{F} \text{ and all } \Delta \in \Delta. \]
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For \( L := J \circ F : \Delta \mapsto \| \Delta_0 \ast P \ast F(\Delta) \|_{\infty} \) this implies

\[ \inf_{F \in \mathbb{F}} J(F) = \inf_{F \in \mathbb{F}} \| \Delta_0 \ast P \ast F \|_{\infty} = L(\Delta_0) \leq L(\Delta) \quad \text{for all} \quad \Delta \in \Delta. \]
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**Why useful?** We can minimize \( L : \Delta \rightarrow \mathbb{R} \) instead of \( J : F \rightarrow \mathbb{R} \) based on I/O samples.
Observations

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**Issue:** It is not easily possible to compute $\mathcal{F}(\Delta)$ or $\inf_{F \in \mathcal{F}} \|\Delta \ast P \ast F\|_{\infty}$ for any fixed $\Delta \in \Delta$ as we are facing a robust multi-objective problem.

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- The underlying problem is nonconvex and also nonsmooth in general.

**However**, as for robust controller design we can compute upper bounds on the optimal value and synthesize corresponding controllers!
Lemma 1. Let $\Delta \in \Delta$ be fixed. Then there is a controller $F \in \mathbb{F}$ satisfying $\|\Delta \star P \star F\|_\infty < \gamma$ if there exist a matrix $M$ and symmetric $Y, P$ satisfying

$$Y \succ 0,$$

$$P \in \mathbb{P}(\Delta), \quad (\bullet)^T \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} P \begin{pmatrix} I & 0 \\ -(AY+B_3M)^T & -(C_1Y+D_{13}M)^T \\ 0 & I \\ -B_1^T & -D_{11}^T \end{pmatrix} \succ 0, \quad \text{(RS)}$$

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} P^{-1} \begin{pmatrix} -A^\Delta Y+B_3^\Delta M)^T & -(C_2^\Delta Y+D_{23}^\Delta M)^T \\ 0 & I \\ -(B_2^\Delta)^T & -(D_{22}^\Delta)^T \end{pmatrix} \succ 0. \quad \text{(NP\Delta)}$$

If the above LMIs are feasible, a suitable controller is $F := MY^{-1}$. Moreover,

$$\inf_{F \in \mathbb{F}} \|\Delta \star P \star F\|_\infty \leq \gamma_{\text{mo}}(\Delta)$$

for $\gamma_{\text{mo}}(\Delta)$ being the infimal $\gamma$ such that the above LMIs are feasible.
Consequences and Remarks

Instead of using $F$ and for $\varepsilon > 0$, Lemma 1 suggests to employ the function $F_{mo}: \Delta \mapsto$ a corresp. close-to-optimal controller ($\gamma = (1 + \varepsilon)\gamma_{mo}(\Delta)$)

- $F_{mo}(\Delta)$ is easily determined by solving a convex semi-definite program.
- Optimal controllers might be bad conditioned or do not even exist.
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- $\mathcal{F}_{\text{mo}}(\Delta)$ is easily determined by solving a convex semi-definite program.
- Optimal controllers might be bad conditioned or do not even exist.

Finally, we obtain suitable test controllers by choosing

$$F_1 := \mathcal{F}_{\text{mo}}(\Delta_1), \ldots, F_N := \mathcal{F}_{\text{mo}}(\Delta_N) \text{ for samples } \Delta_1, \ldots, \Delta_N \in \Delta.$$
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- For $L_{mo} := J \circ F_{mo}$, $\Delta \mapsto \|\Delta_0 P F_{mo}(\Delta)\|_\infty$ we have

$$\gamma_{\text{nom}}(\Delta_0) \leq L(\Delta_0) \leq L_{mo}(\Delta_0) \leq (1 + \varepsilon) \gamma_{mo}(\Delta_0)$$
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- For $L_{mo} := J \circ F_{mo}, \Delta \mapsto \|\Delta_0 * P * F_{mo}(\Delta)\|_{\infty}$ we have

$$\gamma_{nom}(\Delta_0) \leq L(\Delta_0) \leq L_{mo}(\Delta_0) \leq (1 + \varepsilon)\gamma_{mo}(\Delta_0)$$

- A minimizer of $L$ is not necessarily a minimizer of $L_{mo}$ and, conversely, a minimizer of $L_{mo}$ is not necessarily a minimizer of $L$.
  - This is due to the conservatism in the convex design.
Example
Let us consider a slight variation of an example from COMPl_eib [6] with

\[ \Delta := \delta I, \quad \delta := [-1, 1], \quad \Delta_0 := \delta_0 I, \quad \delta_0 = 0.7. \]

We obtain

\[ \gamma_{\text{nom}}(\delta_0) = 1.20, \quad \min_{\delta \in \delta} L_{\text{mo}}(\delta) = L_{\text{mo}}(0.66) = 1.39 \quad \text{and} \quad \gamma_{\text{sep}} = 2.02. \]
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- Minimizing $L_{\text{mo}}$ leads as desired to better closed-loop $H_\infty$ performance if compared to robust design.
- Safe operation is assured as robustly stabilizing controllers are designed.
- Here $F$ is a subset of $\mathbb{R}^{4 \times 8}$ which has dimension 36 and turns sampling or gridding very tedious.
- The minimizer of $L_{\text{mo}}$ is not necessarily equal to $\delta_0$.

[6] Leibfritz. COMPl\text{eib}: COnstraint Matrix-optimization Problem library - a collection of test examples for nonlinear semidefinite programs, control system design and related problems. 2004
Interesting Bonus Feature:

- We can assure that $\delta_0$ is contained in $[0.65, 0.9]$ as we have inequality

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- This allows to repeat the procedure for $\Delta$ replaced by $\tilde{\Delta} := [0.65, 0.9]/l$.

- This yields even better controllers as easier robust problems are involved:

$$\gamma_{\text{nom}}(\delta_0) = 1.20, \quad \min_{\delta \in [0.65, 0.9]} L_{\text{mo}}(\delta) = 1.31 \quad \text{and} \quad \gamma_{\text{sep}} = 1.66.$$
“Negative” Example

- Shrinking $\Delta$ by a large amount is not always possible as the curves do not have to intersect at all.

- But it can as well be possible to iteratively apply the shrinking.
Interpolation/Approximation

**Goal:** Interpolate/Approximate $L_{mo}$ with as few evaluations as possible.

- Evaluating $L_{mo}$ requires (expensive?) closed-loop experiments.
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- As in [1], we propose a kernel based approach with customized kernels.
- Allows for an extension to the stochastic setting with Gaussian processes.
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- Evaluations of closed-loop experiments allow to design safe controllers with superior performance if compared to a standard robust design.

- Systematic selection of suitable test controllers.

Outlooks:

- (Output-feedback) synthesis based on superior analysis results.

- How to handle time-varying uncertainties?

- Systematic approaches for higher dimensions.
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Thank you!

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