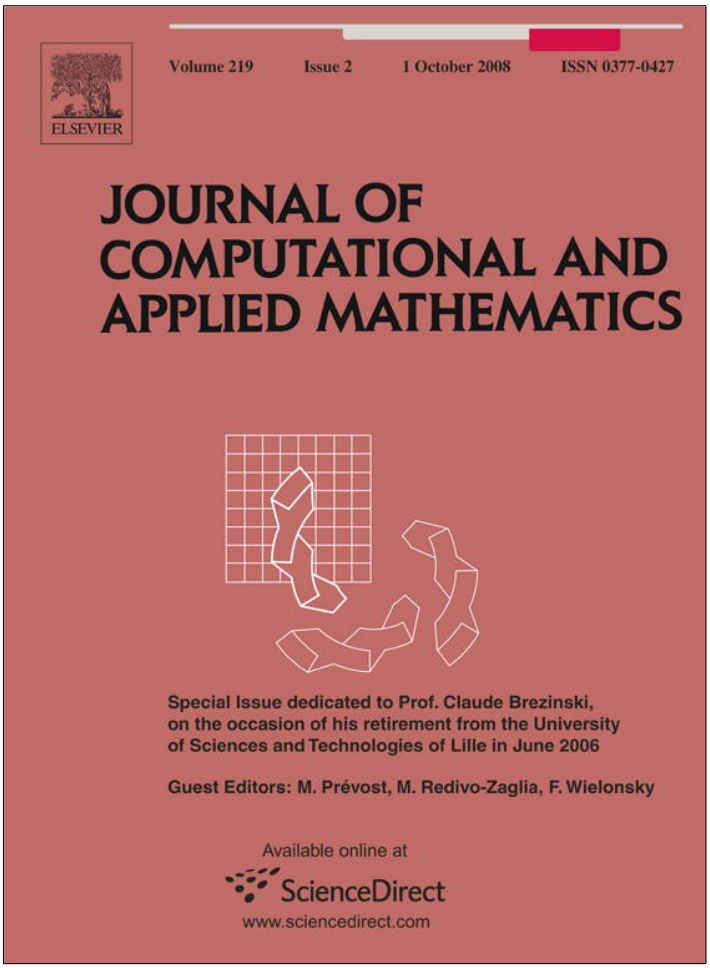

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A new stopping criterion for linear perturbed asynchronous iterations

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Abstract

A new stopping criterion is proposed for asynchronous linear fixed point methods in finite precision. The case of absolute error is considered. The originality of this stopping criterion relies on the fact that all tests are made within the same macroiteration.

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1. Introduction

This paper deals with parallel asynchronous iterative algorithms. This issue is gaining considerable attention with the advent of peer to peer and grid computing (see [1,14]). The paper concentrates on stopping criterion for linear parallel asynchronous iterations in the case where successive approximations are perturbed by round off errors. This topic is particularly important since for nonlinear problems one has to detect convergence of auxiliary linearized problems. Stopping asynchronous iterations is a difficult topic, either with respect to computer science or to numerical analysis. As opposed to the special situation of parallel synchronous scheme, the development of stopping criteria for asynchronous iterations is particularly nontrivial since there is no global clock nor synchronization points between processors, and processors go at their own pace.

Several approaches for the convergence detection of parallel asynchronous iterations have been proposed in the literature. On what concerns approaches related to pure computer sciences aspects we can quote [2,7,18]. For mixed approaches combining computer sciences and numerical analysis we refer to the following works [5,6,9–11,23]. We have proposed several results in the perturbed case (See [12,21]). The perturbation of parallel asynchronous fixed point methods in finite precision has been studied in [20], see also [24] in the linear case and [22] in the sequential linear case. In [21] some predictive results have been given in relationship with backward and forward errors; in particular,

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stopping criteria have been proposed. In this paper, we present an original stopping criterion with respect to absolute error whereby decision is taken on the base of data collected during the same macroiteration. More precisely, the originality of our study consists in making the convergence test within the same macroiteration which ends up at the given current update p, i.e., starting from the update p, we build a macroiteration going back in the sequence of events which are related to previous updates.

Recall that when the successive approximation method is applied to the computation of the fixed point problem associated with a contractive mapping $\widetilde{T} : u \in R^n \to \widetilde{T}(u) \in R^n$, with constant of contraction l and fixed point $u^\star$, then stopping the iterations when $\|u^p - \widetilde{T}(u^p)\| = \|u^p - u^{p+1}\| \leqslant \eta$, implies $\|u^p - u^\star\| \leqslant \eta/(1-l)$; namely, such a stopping test is simply based on the implication $\|v - \widetilde{T}(v)\| \leqslant \eta \Rightarrow \|u^\star - v\| \leqslant \eta/(1-l)$.

An analogous estimate can be obtained for asynchronous iterations and then used with respect to various kinds of perturbations issued both from the intrinsic nature of asynchronous iterations and from round-off errors. Namely, our stopping criterion, with respect to round-off errors, will be based on the following extension of the previous estimate

$$\begin{cases} \max\limits_{i=1,\dots,n} \left(\frac{|y_i - T_i(z^i)|}{e_i} \right) \leqslant \delta, \\ \|z^i - y\|_{\varepsilon,\infty} \leqslant \eta, \quad i = 1, \dots, n, \end{cases} \tag{1}$$

where $e_i, i = 1, \dots, n$ are the components of the Perron–Frobenius eigenvectors, $u \to \|u\|_{\varepsilon,\infty}$ is an appropriate weighted uniform norm (see [13]), y is the iterate vector as will be shown in the sequel and $\{z^i\}$ represents a family of iterate vectors which belong to a macroiteration. In this framework, we can derive a stopping criterion distinct to our previous works (see [12,21,24]).

The estimates in perturbed situations are presented in the following sections which include also our main result concerning a stopping criterion for asynchronous iterations. We need mainly three kinds of informations

- the first one concerns the delays and allows to know what are the iterates included in the macroiteration, this information permits one to have previous computed values,
- the second one is the stabilization of the algorithm with respect to a small distance between these iterates, which then includes naturally the estimate (1), and leads to practical stopping test developed in forthcoming sections,
- the last one is the numerical value of a constant of contraction and also basic data related to round-off errors.

This paper is organized as follows. Section 2 deals with linear asynchronous iterations in the context of nonperturbed case and also in the case of perturbation by round-off errors. A new stopping criterion for perturbed linear asynchronous iterations is proposed in Section 3. Finally, very simple examples are presented in Section 4, in order to illustrate our approach.

2. Mathematical background

2.1. Classical asynchronous iterations

Let n be an integer and consider two matrices $A \in \mathscr{L}(\mathbb{R}^n)$ and $B \in \mathscr{L}(\mathbb{R}^n)$ such that $A = I - B$. In the sequel, $b_{ij}, i, j = 1, \dots, n$ denote the entries of matrix B and $|B|$ the matrix with entries $|b_{ij}|, i, j = 1, \dots, n$. Assume that the spectral radius of the matrix $|B|$ satisfies

$$\rho(|B|) < 1. \tag{2}$$

In [15,24] it was shown that, for all real given positive numbers ε, there exist a strictly positive vectors $e \equiv e_\varepsilon$ and a positive scalar $\lambda \equiv \lambda_\varepsilon$, satisfying

$$|B|.e \leqslant \lambda.e \quad \text{where } \lambda \in [\rho(|B|), \rho(|B|) + \varepsilon]. \tag{3}$$

The space $\mathbb{R}^n$ being normed by the Perron–Frobenius weighted uniform norm defined by

$$\|u\|_{\varepsilon,\infty} = \max_{1 \leqslant j \leqslant n} \left(\frac{|u_j|}{e_j} \right) \tag{4}$$

then, we have the following estimation of the norm of the matrix B given by $\|B\|_{\varepsilon,\infty} \leqslant \lambda$, where $\|\cdot\|_{\varepsilon,\infty}$ is derived from the scalar norm (4) (see [24]).

Remark 1. The situation where ε is equal to zero corresponds to the case where the matrix B is irreducible. In this case, e is a suitable Perron–Frobenius eigenvector of matrix $|B|$, associated with the spectral radius.

Consider the following linear system $Au^\star = c$, where $c \in \mathbb{R}^n$, to which we associate the fixed point mapping $\widetilde{T}$ defined by $\tilde{u} \in \mathbb{R}^n \to \tilde{v} = \widetilde{T}(\tilde{u}) = B\tilde{u} + c$. Clearly, $u^\star$ is the unique solution of the linear system $Au^\star = c$, if and only if $u^\star$ is the unique fixed point of the mapping $\widetilde{T}$. In order to approximate the fixed point $u^\star$, we use parallel asynchronous iterative method which generate a sequence of vectors $\{\tilde{u}^p\}_{p\in\mathbb{N}}$ defined recursively as follows (see [3]), for all $p \geqslant 0$ and $i \in \{1, \ldots, n\}$,

$$\tilde{u}_i^{p+1} = \begin{cases} \widetilde{T}_i(\ldots, \tilde{u}_j^{s_j(p)}, \ldots), & \forall i \in J(p), \\ \tilde{u}_i^p, & \forall i \notin J(p), \end{cases} \tag{5}$$

where $\tilde{u}^0$ is the initial guess and

$$\mathbf{J} = \{J(p)\}_{p\in\mathbb{N}} \text{ is a sequence of nonempty subsets of } \{1, 2, \ldots, n\}, \tag{6}$$

$$\mathbf{S} = \{(s_1(p), s_2(p), \ldots, s_n(p))\}_{p\in\mathbb{N}}, \text{ is a sequence of elements of } \mathbb{N}^n. \tag{7}$$

Moreover $\mathbf{J}$ and $\mathbf{S}$, satisfy

$$\forall i \in \{1, 2, \ldots, n\}, \quad \mathrm{Card}\{p \in \mathbb{N} | i \in J(p)\} = +\infty, \tag{8}$$

$$\forall j \in \{1, 2, \ldots, n\}, \quad \forall p \in \mathbb{N}, \quad s_j(p) \leqslant p, \tag{9}$$

$$\forall j \in \{1, 2, \ldots, n\}, \quad \lim_{p\to\infty} s_j(p) = +\infty. \tag{10}$$

Remark 2. The reader is referred to [3,8] for some convergence results concerning asynchronous iterations (5); see also [16] for a survey, [13,19] for convergence results using weighted uniform norm. Reference is also made to [4] for convergence study using sequences of embedded subsets.

2.2. *Perturbed asynchronous iterations by round-off errors*

Let us consider now the general case of perturbed asynchronous iterations. In the framework of round-off error perturbations and floating point computations, we must replace in (5) the exact mapping $\widetilde{T}$ by an approximate mapping denoted by T; similarly the exact value of the iterate vector $\tilde{u}$ is replaced by the perturbed one denoted now by u. In order to deal with this perturbation, let us introduce now the following notations (see [24]). We define the real number $\tau = \mu(t+1)\upsilon$, where t is the maximum number of nonzero elements in a row of the matrix B, μ is a positive constant, $\mu = 1.0101\ldots$ (see [17, p. 63]) and υ is such that for any floating point number ξ we have $\mathrm{fl}(\xi) = \xi(1+\vartheta)$, $|\vartheta| \leqslant \upsilon$. If β denotes the base and s the precision, then υ is defined by $\upsilon = \frac{1}{2}\beta^{1-s}$ in the case of rounding and $\upsilon = \beta^{1-s}$ in the case of chopping (see [17]). Let us also consider a strengthened form of assumption (2) in which the spectral radius of the matrix $|B|$ satisfies

$$\rho(|B|) < \frac{1}{1+\tau}. \tag{11}$$

According to (11), ε can be chosen sufficiently small so that

$$l = (1+\tau)\lambda < 1. \tag{12}$$

Let us associate to $u \in \mathbb{R}^n$ the vectorial norm $|u|$ with components $|u_i|$, $\forall i \in \{1, \ldots, n\}$. If we make the additional assumption that the error of approximation $|Tu - \widetilde{T}u|$ is proportional to $|\widetilde{T}u|$, we have (see [24])

$$|Tu - \widetilde{T}u| \leqslant \tau(|B||u| + |c|), \quad \forall u \in \mathbb{R}^n. \tag{13}$$

We present now an important result related to approximate contraction (see [22] in the sequential case, [20,24] in the parallel context); for the proof the reader is referred to [24].

Lemma 1. *Assume that* ε *is sufficiently small so that* (12) *is valid. Then, the perturbed fixed point mapping satisfies*

$$|u^\star - Tv| \leqslant (1+\tau)|B||v-u^\star| + \tau(I-|B|)^{-1}|c|, \quad \forall v \in \mathbb{R}^n$$

and we have

$$\|u^\star - Tv\|_{\varepsilon,\infty} \leqslant (1+\tau)\lambda\|u^\star - v\|_{\varepsilon,\infty} + \frac{\tau}{1-\lambda}\|c\|_{\varepsilon,\infty}, \quad \forall v \in \mathbb{R}^n.$$

Thus, the mapping T satisfies the approximate contraction property

$$\|u^\star - Tv\|_{\varepsilon,\infty} \leqslant l.\|u^\star - v\|_{\varepsilon,\infty} + \theta^\star, \quad \forall v \in \mathbb{R}^n,$$

where $\theta^\star = \tau/(1-\lambda)\|c\|_{\varepsilon,\infty}$.

In the perturbed case, an asynchronous iterative algorithm produces a sequence of iterates, denoted by $\{u^p\}_{p\in\mathbb{N}}$, initialized by $u^0 = \tilde{u}^0$, and defined recursively as follows for all $p \geqslant 0$ and $i \in \{1,\ldots,n\}$,

$$u_i^{p+1} = \begin{cases} T_i(\ldots, u_j^{s_j(p)}, \ldots), & \forall\, i \in J(p), \\ u_j^p, & \forall\, i \notin J(p), \end{cases} \tag{14}$$

where the sequences $\mathbf{J}$ and $\mathbf{S}$, respectively, are defined by (6) and (7), respectively, and the conditions (8)–(10), are satisfied. We point out that in the case of parallel asynchronous iterative methods, whereby computations are performed in parallel without any order nor synchronizations, it is particularly important and challenging to derive stopping criteria so that certain bounds on the error hold. These aspects will be developed in the sequel. In this context, the concept of macroiteration will play a major role; let us recall now the classical notion of macroiteration (see [19]) which gather several relaxations; in a macroiteration all components of the iterate vector are updated at least one time using available values of the components associated to the previous macroiteration. Consider now

$$s(p) = \min_{r \geqslant p} \min_{1 \leqslant j \leqslant n} (s_j(r)) \tag{15}$$

then, the macroiteration sequence $\{p_k\}_{k\in\mathbb{N}}$ is defined as follows:

$$\begin{cases} p_0 = 0, \\ p_{k+1} = \min\left\{ p \,\middle|\, \bigcup_{p_k \leqslant s(r) \leqslant r \leqslant p} J(r) = \{1,\ldots,n\} \right\}. \end{cases} \tag{16}$$

In several works dealing with asynchronous iterations (see in particular [5,20]), the convergence of the successive iterates is based on the relations $u^p \in \mathbf{E}^k, \forall p \geqslant p_k$, where $\{\mathbf{E}^k\}_{k\in N}$ is a sequence of nested Cartesian products sets E_j^k, verifying on the one hand the nested inclusion property $\mathbf{E}^{k+1} \subset \mathbf{E}^k, \forall k \in \mathbb{N}$ and the convergence condition $T(\mathbf{E}^k) \subset \mathbf{E}^{k+1}, \forall k \in \mathbb{N}$, and on the other hand the box condition given in [5, p. 431], $\mathbf{E}^k = \prod_{j=1}^n E_j^k, \forall k \in \mathbb{N}$.

The above approach is well suited to the study of perturbed asynchronous iterations which converge to a limit set denoted in the sequel $\mathbf{E}^\infty$ (see [20,21,24]); this last approach will be also used in the present study. Note that the only norm compatible with the so called box condition is the weighted uniform norm.

In the context of nonperturbed asynchronous iterations, weighted uniform norm (see [13]) is also used. In such a framework (see [13,19]), when considering the convergence of asynchronous iterations (5) associated with a contracting fixed point mapping $\widetilde{T}$, the nested sets can be defined as follows:

$$\mathbf{E}^k = \{u \in \mathbb{R}^n \,|\, \|u^\star - u\|_{\varepsilon,\infty} \leqslant l^k.\|u^\star - u^0\|_{\varepsilon,\infty}\} \quad \text{with } l < 1. \tag{17}$$

In the case of asynchronous iterations (14) associated with approximate contracting perturbed fixed point mapping T, the nested sets are given as follows (see [20,24]):

$$\mathbf{E}^k = \left\{ u \in \mathbb{R}^n \,\middle|\, \|u^\star - u\|_{\varepsilon,\infty} \leqslant l^k.\|u^\star - u^0\|_{\varepsilon,\infty} + \frac{1-l^k}{1-l}\theta^\star \right\}, \tag{18}$$

where $\theta^\star = \tau/(1-\lambda)\|c\|_{\varepsilon,\infty}$ and with limit set

$$\mathbf{E}^\infty = \bigcap_{k\in\mathbb{N}} \mathbf{E}^k = \left\{ u \in \mathbb{R}^n \,|\, \|u^\star - u\|_{\varepsilon,\infty} \leqslant \frac{\theta^\star}{1-l} \right\}. \tag{19}$$

3. Stopping criterion with respect to absolute error

In the case of linear asynchronous iterations in finite precision, we propose an original stopping criterion with respect to a given bound of the error.

3.1. Preliminary results

Let us state first some technical results.

Lemma 2. *Assume that* (11) *holds and that* ε *is sufficiently small so that* (12) *is valid*; *assume also that* (3) *is valid. Then*, *using the previous notations*, *for any two positive real numbers* δ *and* η, *let us consider a n-dimensional vector* y *and a set of n-dimensional vectors* $\{z^i\}$, $i = 1, 2, \ldots, n$, *such that*,

$$\max_{i=1,\ldots,n} \frac{|y_i - T_i(z^i)|}{e_i} \leqslant \quad \delta \quad \text{and} \quad \|y - z^i\|_{\varepsilon,\infty} \leqslant \eta, \quad \forall i = 1, \ldots, n. \tag{20}$$

Then,

$$\|y - u^\star\|_{\varepsilon,\infty} \leqslant \frac{1}{1-\lambda}(\delta + l.\eta + \tau(\lambda\|y\|_{\varepsilon,\infty} + \|c\|_{\varepsilon,\infty})) \tag{21}$$

and

$$\|y - u^\star\|_{\varepsilon,\infty} \leqslant \frac{1}{1-l}\left(\delta + l.\eta + \frac{\tau}{1-\lambda}\|c\|_{\varepsilon,\infty}\right). \tag{22}$$

Proof. For all $i = 1, \ldots, n$, let us consider the following relation:

$$\frac{|y_i - T_i(z^i)|}{e_i} = \frac{|y_i - \widetilde{T}_i(z^i) + \widetilde{T}_i(z^i) - T_i(z^i)|}{e_i}$$

then

$$\frac{|y_i - T_i(z^i)|}{e_i} \geqslant \frac{|y_i - \widetilde{T}_i(z^i)|}{e_i} - \frac{|\widetilde{T}_i(z^i) - T_i(z^i)|}{e_i}. \tag{23}$$

According to (13) we have

$$\frac{|\widetilde{T}_i(z^i) - T_i(z^i)|}{e_i} \leqslant \||\widetilde{T}_i(z^i) - T_i(z^i)|\|_{\varepsilon,\infty} \leqslant \tau(\||B|\|_{\varepsilon,\infty}\||z^i|\|_{\varepsilon,\infty} + \||c|\|_{\varepsilon,\infty})$$

thus, for every $i = 1, \ldots, n$

$$\frac{|\widetilde{T}_i(z^i) - T_i(z^i)|}{e_i} \leqslant \tau(\lambda\||z^i|\|_{\varepsilon,\infty} + \||c|\|_{\varepsilon,\infty}). \tag{24}$$

Therefore (23) and (24) imply that, for all $i = 1, \ldots, n$

$$\frac{|y_i - T_i(z^i)|}{e_i} \geqslant \frac{|y_i - \widetilde{T}_i(z^i)|}{e_i} - \tau(\lambda\||z^i|\|_{\varepsilon,\infty} + \||c|\|_{\varepsilon,\infty})$$

since $\||z^i|\|_{\varepsilon,\infty} = \|z^i\|_{\varepsilon,\infty}$ and $\||c|\|_{\varepsilon,\infty} = \|c\|_{\varepsilon,\infty}$, according to (20), we obtain for all $i = 1, \ldots, n$

$$\frac{|y_i - T_i(z^i)|}{e_i} \geqslant \frac{|y_i - \widetilde{T}_i(z^i)|}{e_i} - \tau\lambda(\|y\|_{\varepsilon,\infty} + \eta) - \tau\|c\|_{\varepsilon,\infty}. \tag{25}$$

Moreover

$$\frac{|y_i - \widetilde{T}_i(z^i)|}{e_i} = \frac{|y_i - \widetilde{T}_i(y) + \widetilde{T}_i(y) - \widetilde{T}_i(z^i)|}{e_i} \geqslant \frac{|y_i - \widetilde{T}_i(y)|}{e_i} - \frac{|\widetilde{T}_i(y) - \widetilde{T}_i(z^i)|}{e_i}$$

note that for all $i = 1, \ldots, n$

$$\frac{|\widetilde{T}_i(y) - \widetilde{T}_i(z^i)|}{e_i} \leqslant \|\widetilde{T}(y) - \widetilde{T}(z^i)\|_{\varepsilon,\infty} = \|B(y - z^i)\|_{\varepsilon,\infty} \leqslant \lambda \|y - z^i\|_{\varepsilon,\infty}.$$

Thus, due to (20), we have for all $i = 1, \ldots, n$

$$\frac{|y_i - \widetilde{T}_i(z^i)|}{e_i} \geqslant \frac{|y_i - \widetilde{T}_i(y)|}{e_i} - \lambda\eta. \tag{26}$$

Then, (25) and (26) imply for all $i = 1, \ldots, n$

$$\frac{|y_i - T_i(z^i)|}{e_i} \geqslant \frac{|y_i - \widetilde{T}_i(y)|}{e_i} - \lambda\eta - \tau\lambda(\|y\|_{\varepsilon,\infty} + \eta) - \tau\|c\|_{\varepsilon,\infty}$$

therefore, for all $i = 1, \ldots, n$

$$\frac{|y_i - T_i(z^i)|}{e_i} \geqslant \frac{|y_i - u_i^\star|}{e_i} - \frac{|\widetilde{T}_i(y) - \widetilde{T}_i(u^\star)|}{e_i} - \lambda\eta - \tau(\lambda(\|y\|_{\varepsilon,\infty} + \eta) + \|c\|_{\varepsilon,\infty})$$

the previous inequality can be written

$$\frac{|y_i - T_i(z^i)|}{e_i} \geqslant \frac{|y_i - u_i^\star|}{e_i} - (1 + \tau)\lambda\eta - \tau(\lambda\|y\|_{\varepsilon,\infty} + \|c\|_{\varepsilon,\infty}). \tag{27}$$

Let us denote by i_m, $1 \leqslant i_m \leqslant n$, an index such that $|y_{i_m} - u_{i_m}^\star|/e_{im} = \|y - u^\star\|_{\varepsilon,\infty}$; then, for $i = i_m$, (27) reads

$$(1 - \lambda)\|y - u^\star\|_{\varepsilon,\infty} \leqslant \frac{|y_{i_m} - T_{i_m}(z^i)|}{e_{i_m}} + \lambda\eta + \tau(\lambda(\|y\|_{\varepsilon,\infty} + \eta) + \|c\|_{\varepsilon,\infty})$$

by taking into account (20), we obtain (21). In order to prove (22), let us start from (21); since $\|y\|_{\varepsilon,\infty} \leqslant \|y - u^\star\|_{\varepsilon,\infty} + \|u^\star\|_{\varepsilon,\infty}$, then

$$\|y - u^\star\|_{\varepsilon,\infty} \leqslant \frac{1}{1 - \lambda}(\delta + \lambda\eta + \tau(\lambda(\|y - u^\star\|_{\varepsilon,\infty} + \|u^\star\|_{\varepsilon,\infty} + \eta) + \|c\|_{\varepsilon,\infty})),$$

which can be also written as follows:

$$(1 - \lambda(1 + \tau))\|y - u^\star\|_{\varepsilon,\infty} \leqslant (\delta + \lambda\eta + \tau\lambda(\|u^\star\|_{\varepsilon,\infty} + \eta) + \tau\|c\|_{\varepsilon,\infty})$$

since $\|u^\star\|_{\varepsilon,\infty} = \|(I - B)^{-1}c\|_{\varepsilon,\infty} \leqslant \|(I - B)^{-1}\|_{\varepsilon,\infty}\|c\|_{\varepsilon,\infty} \leqslant 1/(1 - \lambda)\|c\|_{\varepsilon,\infty}$ and according to (12), we obtain (22) and the proof is complete. □

In the sequel, we will use the concepts of order intervals and order interval hull. Consider two vectors $\underline{w}, \overline{w} \in \mathbb{R}^n$ such that $\underline{w} \leqslant \overline{w}$ and the associated order interval $\langle \underline{w}, \overline{w} \rangle = \{w \in \mathbb{R}^n | \underline{w} \leqslant w \leqslant \overline{w}\}$, which can also be written as follows $\langle \underline{w}, \overline{w} \rangle = \{w | \underline{w}_k \leqslant w_k \leqslant \overline{w}_k, k = 1, \ldots, n\}$, where w_k denotes a component of w.

Let us also denote, in the sequel, $\mathrm{Diam}_{\varepsilon,\infty}(\langle \underline{w}, \overline{w} \rangle) = \|\overline{w} - \underline{w}\|_{\varepsilon,\infty}$ the diameter, with respect to the weighted uniform norm (4), of the order interval $\langle \underline{w}, \overline{w} \rangle$. Let us also consider m vectors of $\mathbb{R}^n$, denoted by $\{w^1, \ldots, w^m\}$; in the sequel w_i^j, $i = 1, \ldots, n$, denotes the ith component of the vector w^j, $j = 1, \ldots, m$. We introduce now the following vectors $\underline{w} = (\underline{w}_i)$ and $\overline{w} = (\overline{w}_i)$ defined by $\underline{w}_i = \min(w_i^1, \ldots, w_i^m)$ and $\overline{w}_i = \max(w_i^1, \ldots, w_i^m)$, $1 \leqslant i \leqslant n$.

Definition 1. Consider a set of vectors $\{w^1, \ldots, w^m\}$; then the order interval hull associated with the set of vectors $\{w^i\}_{1 \leqslant i \leqslant m}$ is defined by $\langle \underline{w}, \overline{w} \rangle = \{w \in \mathbb{R}^n | \underline{w} \leqslant w \leqslant \overline{w}\}$.

Lemma 3. *Assume that assumptions of Lemma 2 hold. Consider a positive real number η, and an order interval $\langle \underline{\underline{w}}, \overline{\overline{w}} \rangle$,such that $Diam_{\varepsilon,\infty}(\langle \underline{\underline{w}}, \overline{\overline{w}} \rangle) \leqslant \eta$. Let us consider a set of n-dimensional vectors $\{z^i\}$, $i = 1, 2, \ldots, n$, such that,*

$$z^i \in \langle \underline{\underline{w}}, \overline{\overline{w}} \rangle, \quad \forall i = 1, \ldots, n \tag{28}$$

and a vector $\bar{y} \in \langle \underline{\underline{w}}, \overline{\overline{w}} \rangle$ such that

$$\bar{y} = \{T_1(z^1), \ldots, T_i(z^i), \ldots, T_n(z^n)\}^{\mathrm{t}}. \tag{29}$$

Then $\forall z \in \langle \underline{\underline{w}}, \overline{\overline{w}} \rangle$, we obtain

$$\|z - u^\star\|_{\varepsilon,\infty} \leqslant \frac{1}{1-\lambda}((1+l)\eta + \tau(\lambda.\|z\|_{\varepsilon,\infty} + \|c\|_{\varepsilon,\infty})) \tag{30}$$

and

$$\|z - u^\star\|_{\varepsilon,\infty} \leqslant \frac{1}{1-l}((1+l)\eta + \theta^\star). \tag{31}$$

Proof. The proof follows from Lemma 2 by choosing $y = z$, $y \in \langle \underline{w}, \overline{w} \rangle$ and $\delta = \eta$; then it follows from the above assumptions, that the first inequality of (20) is satisfied and (21) and (22), respectively, imply (30) and (31), respectively, which achieves the proof. □

3.2. Stopping criterion

In order to define a practical stopping criterion we adapt the concept of macroiteration presented in (15,16) and we introduce now the paradigm of sliding macroiteration defined by the following notations:

$$\underline{\sigma}(p) = \max\left(\left\{\underline{r} \,\middle|\, \bigcup_{\underline{r} \leqslant s(r) \leqslant r \leqslant p} J(r) = \{1, \ldots, n\}\right\}\right), \quad \forall p \geqslant p_1$$

and

$$\overline{\sigma}(p) = \min(\{r | \underline{\sigma}(p) \leqslant s(r) \leqslant r \leqslant p\}), \quad \forall p \geqslant p_1,$$

which permits one to take into account all necessary information used to produce the pth update; $\underline{\sigma}(p)$ traduces the fact that all components of the pth update are computed using values which belong to the same sliding macroiteration. In this framework, updates labelled by an index smaller than the value $\underline{r}$, quoted in the definition of $\underline{\sigma}(p)$, are not used anymore in the computation of the pth update.

The set of iteration numbers related to the updating of the ith component that are between $\overline{\sigma}(p)$ and p is defined as follows:

$$\forall i \in \{1, \ldots, n\} \to \psi_p(i) = \{r \in \mathbb{N} | \overline{\sigma}(p) \leqslant r \leqslant p \wedge i \in J(r)\} \neq \emptyset \quad \text{if } p_1 \leqslant p,$$

where $a \wedge b$ means a and b. Therefore, for all $i \in \{1, \ldots, n\}$, we can choose some indices $\phi_p(i) \in \{\overline{\sigma}(p), \ldots, p\}$ such that $\phi_p(i) = \max(r \in \psi_p(i))$, $\forall p \geqslant p_1$, $\phi_p(i)$, corresponding to the iteration number related to the last update of the ith component. Note that $\bigcup_{i \in \{1,\ldots,n\}} J(\phi_p(i)) = \{1, \ldots, n\}$. Then the set of n-dimensional vectors $\{z^i\}$, $i = 1, 2, \ldots, n$, is defined such that, $z^i = \{\ldots, u_j^{s_j(\phi_p(i))}, \ldots\}$ and $\bar{y} \equiv u^{p+1}$ defined by

$$u^{p+1} = \{T_1(z^1), \ldots, T_i(z^i), \ldots, T_n(z^n)\}^t = \{u_1^{\phi_p(1)+1}, \ldots, u_i^{\phi_p(i)+1}, \ldots, u_n^{\phi_p(n)+1}\}$$

let us define w^i as follows $w^i = z^i$, $i = 1, \ldots, n$. We label by an index p the order interval hull $\langle \underline{w}, \overline{w} \rangle$, which is then denoted by $\langle \underline{w}, \overline{w} \rangle_p$. In order to establish the main result of this section, we introduce an abstract framework related to the sequence of embedded sets: we consider a sequence of order intervals denoted by $\langle \underline{\underline{v}}, \overline{\overline{v}} \rangle_p$ such that

$\langle \underline{w}, \overline{w}\rangle_p \subseteq \langle \underline{v}, \overline{\overline{v}}\rangle_p, \forall p \in \mathbb{N}$. Consider a positive real number η such that the diameter of the limit set $\mathbf{E}^\infty$ defined by (19) satisfies $\mathrm{Diam}_{\varepsilon,\infty}(\mathbf{E}^\infty) = 2\theta^\star/(1-l) < \eta$. Then we can define the stopping criterion at the $(p+1)$th sliding macroiteration as follows:

$$\mathrm{Diam}_{\varepsilon,\infty}(\langle \underline{v}, \overline{\overline{v}}\rangle_p) \leqslant \eta. \tag{32}$$

We present the main result of this section

Proposition 1. *Assume that* (11) *holds and that* ε *is sufficiently small so that* (12) *is valid*; *assume also that* (3) *is valid. The real number* η *satisfying* $\mathrm{Diam}_{\varepsilon,\infty}(\mathbf{E}^\infty) < \eta$, *there exists a sequence of order intervals* $\langle \underline{v}, \overline{\overline{v}}\rangle_p$, $p \in \mathbb{N}$ *satisfying* $\langle \underline{w}, \overline{w}\rangle_p \subseteq \langle \underline{v}, \overline{\overline{v}}\rangle_p, \forall p \in \mathbb{N}$, *such that* $u^{p+1} \in \langle \underline{w}, \overline{w}\rangle_p \forall p \in \mathbb{N}$ *and such that, for* $p \in \mathbb{N}$ *sufficiently large, the condition* (32) *is satisfied. If the condition* (32) *is satisfied, then* $\forall u^q \in \langle \underline{v}, \overline{\overline{v}}\rangle_p$ *we have the estimations*

$$\|u^q - u^\star\|_{\varepsilon,\infty} \leqslant \frac{1}{1-\lambda}((1+l)\eta + \tau(\lambda.\|u^q\|_{\varepsilon,\infty} + \|c\|_{\varepsilon,\infty})) \tag{33}$$

and

$$\|u^q - u^\star\|_{\varepsilon,\infty} \leqslant \frac{1}{1-l}((1+l)\eta + \theta^\star). \tag{34}$$

Proof. Since $\phi_p(i) \in \psi_p(i), \forall p \geqslant p_1$ and $u^{p+1} = \{T_1(z^1), \ldots, T_i(z^i), \ldots, T_n(z^n)\}^{\mathrm{t}}$, note that necessarily $u^{p+1} \in \langle \underline{w}, \overline{w}\rangle_p, \forall p \in \mathbb{N}$. Let us choose η such that the condition $\mathrm{Diam}_{\varepsilon,\infty}(\mathbf{E}^\infty) < \eta$ is satisfied and consider also $\underline{\underline{\sigma}}(p)$ satisfying $\underline{\underline{\sigma}}(p) \leqslant \underline{\sigma}(p), \forall p \in \mathbb{N}$ and $\lim_{p\to\infty}(\underline{\underline{\sigma}}(p)) = +\infty$. Then, taking into account (18) and (19), there exists an integer $\bar{p}$ such that

$$\|u^q - u^r\|_{\varepsilon,\infty} \leqslant \eta, \quad \forall q, r \geqslant \bar{p}. \tag{35}$$

So there exists an integer $\hat{p}$ such that

$$\|u^q - u^r\|_{\varepsilon,\infty} \leqslant \eta, \quad \forall q, r \geqslant \bar{p}, \quad \forall p \geqslant \hat{p}, \quad \bar{p} \leqslant \underline{\underline{\sigma}}(p) \leqslant q, \quad r \leqslant p+1. \tag{36}$$

Let us now denote $v^i = u^q$, $q = \underline{\underline{\sigma}}(p), \ldots, p+1$, $i = q - \underline{\underline{\sigma}}(p) + 1 \equiv 1, \ldots, n$, to which corresponds the order interval hull $\langle \underline{v}, \overline{v}\rangle_p$; then $\langle \underline{w}, \overline{w}\rangle_p \subseteq \langle \underline{v}, \overline{\overline{v}}\rangle_p = \langle \underline{v}, \overline{v}\rangle_p$ and (35) implies $\mathrm{Diam}_{\varepsilon,\infty}(\langle \underline{v}, \overline{\overline{v}}\rangle_p) = \mathrm{Diam}_{\varepsilon,\infty}(\langle \underline{v}, \overline{v}\rangle_p) \leqslant \eta, \forall p \geqslant \hat{p}$. Thus (32) is satisfied. It follows from Lemma 3 that if (32) is satisfied, then (33) and (34) hold and the proof is achieved. □

Remark 3. Inequalities (35) and (36) give another opportunity to derive a stopping criterion in situations where a suitable overestimation of $\varrho(p) = p - \underline{\underline{\sigma}}(p)$ is available.

Corollary 1. *Assume that assumptions of Proposition* 1 *hold. Then, there exists an integer* $\hat{p}$, *such that for any* $p \geqslant \hat{p}$, *the condition* (36) *is true*; *furthermore, when* (36) *is satisfied, for all* q, *such that* $\underline{\underline{\sigma}}(p) \leqslant \underline{\sigma}(p) \leqslant q \leqslant p+1$ *with* $\lim_{p\to\infty}(\underline{\underline{\sigma}}(p)) = +\infty, \forall p \in \mathbb{N}$, *the inequalities* (33) *and* (34) *are satisfied.*

Proof. The previous result follows in a straightforward way from the proof and the statement of Proposition 1. □

Fig. 1 illustrates what may happen if different components of the iterate vector are updated using values of the same component which are very dissimilar: one may have a bad convergence detection that delivers a very poor approximation to the fixed point. In Fig. 1, computations are made by three processors. The end of each iteration is represented by a vertical bar. Moreover each processor communicates only with adjacent processors. Communication frequency relatively to both directions is high between processor 0 and 1 and processor 1 communicates very slowly with processor 2. For simplicity of presentation, we have only represented the most significant communications using arrows. It results from the important differences in communication frequencies that processors 0 and 1 converge to a solution of a subproblem, updating the components which are assigned to them. If different components of the iterate

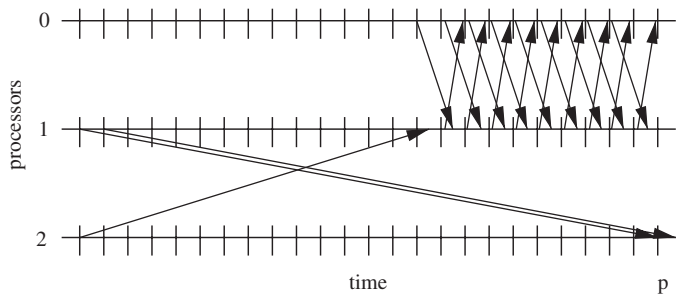

Fig. 1. Pathological situation in the convergence detection of asynchronous iterative algorithms, time diagram.

vector are updated using values very dissimilar of the same component then, we note that global convergence may be detected at time p, though processor 2 has repeated the same updating phase from time 0 to p using the same data.

Remark 4. The originality of our method relies on the fact that all convergence tests are made within the same macroiteration which ends up at the given current update p. In other words, starting from p, we build a macroiteration going back in the sequence of events which are related to previous updates. We note that the stopping criterion ensures that some kind of data coherency is satisfied. Note that the different components of the iterate vector can be computed in the same macroiteration according to the parallel asynchronous iterative model (14) using values of the same component that can be very dissimilar (see Fig. 1). The stopping criterion which must be verified using the infinite norm guarantees part of iterate vector values coherency, i.e., data relative to a given component which are used in the same macroiteration during the different updating phases must be close to each other. Inequality (36) achieves in some way computed iteration values coherency by guaranteeing that convergence conditions are satisfied for all components at the same macroiteration.

Remark 5. In the present theoretical context, the results of Proposition 1 means that the asynchronous iterations (14) lead to a stabilization asserted by the considered stopping criterion, to which corresponds the error bounds (35) and (36). Note also that in the nonperturbed case corresponding to $\tau = 0$, then the condition $\mathrm{Diam}_{\varepsilon,\infty}(\mathbf{E}^{\infty}) < \eta$ reduces to the usual choice of $\eta > 0$. Moreover (35) and (36) reduce both to $\|u^q - u^\star\|_{\varepsilon,\infty} \leqslant (1+\lambda)/(1-\lambda)\eta$, for all $q \in \mathbb{N}$.

3.3. Stopping criterion derived from other norms

All norms of the n-dimensional space being equivalent, the stopping criterion with respect to absolute error can be considered with respect to another norm of $\mathbb{R}^n$. Let us denote in the sequel such a norm as follows $v \to]|v|[, \forall v \in \mathbb{R}^n$ and by α, β the corresponding constants of equivalence between the considered norm $]|.|[$ and the uniform weighted norm (4); thus

$$0 < \alpha \leqslant \frac{]|v|[}{\|v\|_{\varepsilon,\infty}} \leqslant \beta, \quad \forall v \in \mathbb{R}^n. \tag{37}$$

In the sequel, we shall use the following notation $\mathrm{Diam}_{]|.|[}(\langle \underline{w}, \overline{w} \rangle) = \max(]|u-v|[)$, $u, v \in \langle \underline{w}, \overline{w} \rangle$. We introduce also a positive real number η' such that

$$\eta' > \frac{2\beta\tau}{\alpha(1-l)(1-\lambda)}]|c|[\geqslant \frac{2\beta\tau}{(1-l)(1-\lambda)} \|c\|_{\varepsilon,\infty} = \frac{2\beta}{1-l}\theta^\star$$

and denote $\eta = \eta'/\beta$; $\eta^\star = \eta'/\alpha$. Consider a sequence of order intervals $\langle \underline{v}, \overline{v} \rangle_p$ such that the condition $\mathrm{Diam}_{\varepsilon,\infty}(\mathbf{E}^{\infty}) = 2\theta^\star/(1-l) \leqslant \eta$ holds. In the sequel we will consider the following stopping criterion with respect to absolute error

$$\mathrm{Diam}_{]|.|[}(\langle \underline{v}, \overline{v} \rangle_p) \leqslant \eta'. \tag{38}$$

Proposition 2. *Assume that assumptions of Proposition* 1 *hold. Then*

(1) *there exists a sequence of order intervals* $\langle \underline{\underline{v}}, \overline{\overline{v}} \rangle_p$, $p \in \mathbb{N}$ *satisfying* $\langle \underline{w}, \overline{w} \rangle_p \subseteq \langle \underline{\underline{v}}, \overline{\overline{v}} \rangle_p$, $\forall p \in \mathbb{N}$, *and such that* $u^{p+1} \in \langle \underline{w}, \overline{w} \rangle_p$, $\forall p \in \mathbb{N}$, *for p sufficiently large, such that the condition* (38) *is satisfied*;

(2) *if the condition* (38) *is satisfied, then* $\forall u^q \in \langle \underline{\underline{v}}, \overline{\overline{v}} \rangle_p$ *we have the estimations*

$$]|u^q - u^\star|[\leqslant \frac{\beta}{1-\lambda}((1+l)\eta^\star + \frac{\tau}{\alpha}(\lambda]|u^q|[+]|c|[)), \quad \forall u^q \in \langle \underline{\underline{v}}, \overline{\overline{v}} \rangle_p, \tag{39}$$

and

$$]|u^q - u^\star|[\leqslant \frac{\beta}{1-l}\left((1+l)\eta^\star + \frac{\tau]|c|[}{\alpha(1-\lambda)}\right). \tag{40}$$

Proof. The proof follows by adapting the results of Proposition 1 to the actual situation. In particular, since $\eta = \eta'/\beta > 2\theta^\star/(1-l) = \mathrm{Diam}_{\varepsilon,\infty}(\mathbf{E}^\infty)$, the condition $\langle \underline{w}, \overline{w} \rangle_p \subseteq \langle \underline{\underline{v}}, \overline{\overline{v}} \rangle_p$, $\forall p \in \mathbb{N}$, holds and, for p sufficiently large, by the first point of Proposition 1 we have $\mathrm{Diam}_{]|.|[}(\langle \underline{\underline{v}}, \overline{\overline{v}} \rangle_p) \leqslant \beta\, \mathrm{Diam}_{\varepsilon,\infty}(\langle \underline{\underline{v}}, \overline{\overline{v}} \rangle_p) \leqslant \beta\eta$, which implies $\mathrm{Diam}_{]|.|[}(\langle \underline{\underline{v}}, \overline{\overline{v}} \rangle_p) \leqslant \eta'$, and the first part of Proposition 2 is proved. Assume now that (38) holds; it follows from Proposition 1, (37) and by replacing η by $\eta^\star$ that (33) and (34), respectively, imply (39) and (40) respectively, and the proof is achieved. □

Consider now the following absolute error criterion derived from the norm $v \to]|v|[$, introduced in this section

$$]|u^q - u^r|[\leqslant \eta', \quad \forall q, r \text{ such that } \underline{\underline{\sigma}}(p) \leqslant \underline{\sigma}(p) \leqslant q, r \leqslant p+1. \tag{41}$$

Corollary 2. *Assume that assumptions of Proposition* 2 *hold. Then*,

(1) *the condition* (41) *is satisfied for p sufficiently large*,
(2) *when* (41) *is satisfied, for all q such that* $\underline{\underline{\sigma}}(p) \leqslant \underline{\sigma}(p) \leqslant q \leqslant p+1$ *with* $\lim_{p\to\infty}(\underline{\underline{\sigma}}(p)) = +\infty$, $\forall p \in \mathbb{N}$, *the estimations* (39) *and* (40) *are satisfied*.

Proof. The previous result follows by an argument similar to the one developed in Corollary 1. □

4. Examples

In order to illustrate simply the concept presented in Section 3, we consider first the classical Jacobi and Gauss–Seidel methods.

4.1. The Jacobi and Gauss–Seidel methods

Consider the solution of the linear algebraic system $Au^\star = c$, by the Jacobi method; classically (see [19]), this algorithm can also be viewed as a particular case of asynchronous iterations modelled by (5)–(10), where $J(p) = \{1, \ldots, n\}$ and $s_j(p) = p$, $\forall j \in \{1, \ldots, n\}$, $\forall p \in \mathbb{N}$. In this context, we have $\underline{\underline{\sigma}}(p) = \underline{\sigma}(p) = p$, $\forall p \in \mathbb{N}$, $\phi_p(i) = p$, $i = 1, \ldots, n$, $z^i = u^p$, $i = 1, \ldots, n$, and $\bar{y} = \{T_1(z^1), \ldots, T_i(z^i), \ldots, T_n(z^n)\}^{\mathrm{t}} = \{u_1^{p+1}, \ldots, u_i^{p+1}, \ldots, u_n^{p+1}\}^{\mathrm{t}}$. Then, in the considered method, $\langle \underline{w}, \overline{w} \rangle_p$, is defined by $\underline{w} = \min(u^p, u^{p+1})$ and $\overline{w} = \max(u^p, u^{p+1})$; therefore $\mathrm{Diam}_{\varepsilon,\infty}(\langle \underline{w}, \overline{w} \rangle_p) = \|u^{p+1} - u^p\|_{\varepsilon,\infty}$. Thus, in this case, the results of Propositions 1 and 2 are valid.

Using the same notations as those considered in the previous example, the usual Gauss–Seidel method can also be viewed as an asynchronous iteration defined by (5)–(10), where $J(p) = \{j_p\}$, $j_p = 1 + p(\mathrm{mod}\, n)$ and $s_i(p) = p$, $i = 1, \ldots, n$, and $p = 0, 1, \ldots$. If $\lfloor p/n \rfloor$ denotes the integer part of p/n, satisfying $p = qn + r$, where p, q and r are integers with $r < n$, then, we can also write $j_p = 1 + (qn + r)(\mathrm{mod}\ n) = 1 + r$. Then $\underline{\sigma}(p) = \overline{\sigma}(p) = p - n + 1$, $\forall p \geqslant p_1$ and

$$\phi_p(i) = \begin{cases} p - j_p + i & \text{if } i \leqslant j_p, \\ p - n + i - j_p & \text{if } i > j_p. \end{cases}$$

Then $z^i = u^{\phi_p(i)}$; therefore

$$\bar{y} = \{T_1(z^1), \ldots, T_i(z^i), \ldots, T_n(z^n)\}^{\mathrm{t}} = \{u_1^{\phi_p(1)+1}, \ldots, u_i^{\phi_p(i)+1}, \ldots, u_n^{\phi_p(n)+1}\}^{\mathrm{t}},$$

which can also be written as follows $\bar{y} = \{u_1^{p+1}, \ldots, u_n^{p+1}\}^{\mathrm{t}}$.

Lemma 4. *Consider $n+1$ vectors $\{w^1, \ldots, w^i, \ldots, w^n, w^{n+1}\}$, such that $w^i - w^{i+1} = \{0, \ldots, w_i^i - w_{i+1}^{i+1}, 0, \ldots, 0\}$, $i = 1, \ldots, n$. Then, the order interval hull associated with $\{w^1, \ldots, w^{n+1}\}$ is $\langle \underline{w}, \overline{w} \rangle = \langle \underline{\underline{\min(w^{n+1}, w^1)}}, \overline{\overline{\max(w^{n+1}, w^1)}} \rangle$.*

Proof. Indeed, any two vectors w^i and w^{i+1}, $i = 1, \ldots, n$, differ only by the value of the ith component; thus, only vectors w^1 and w^{n+1} can differ by the value of all components. □

Consider now the vectors $w^i = z^i \equiv u^{\phi_p(i)}$, $i = 1, \ldots, n$, and $w^{n+1} = \bar{y}$; then, by a simple permutation of the index of the components of the previous vectors, we can apply Lemma 4; and we obtain

$$\langle \underline{w}, \overline{w} \rangle_p = \langle \underline{\underline{\min(u^{p+1}, w^{p+1-n})}}, \overline{\overline{\max(u^{p+1}, w^{p+1-n})}} \rangle$$

and $\mathrm{Diam}_{\varepsilon,\infty}(\langle \underline{w}, \overline{w} \rangle_p) = \|u^{p+1-n} - u^{p+1}\|_{\varepsilon,\infty}$. Then, once again, the results of Proposition 1 and 2 are still valid.

Remark 6. The result of Proposition 1 does not formally lead to distinct results for Jacobi and Gauss–Seidel methods. Note that, in the common framework, the same stopping criterion does not lead nevertheless to the same numerical value for the bounds of the two studied methods, even if these bounds have the same algebraic expressions. The Stein Rosenberg theorem can be applied to the Gauss–Seidel method if the corresponding iteration matrix is explicitly known with corresponding spectral radius smaller than the one of the Jacobi method.

4.2. An iterative method with delays

For the sake of simplicity, let us consider a very simple example, where B is defined and splitted as follows:

$$B = \begin{vmatrix} 0 & b_{12} & 0 & 0 \\ b_{21} & 0 & b_{23} & 0 \\ 0 & b_{32} & 0 & b_{34} \\ 0 & 0 & b_{43} & 0 \end{vmatrix} = \begin{vmatrix} 0 & b_{12} & 0 & 0 \\ b_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{34} \\ 0 & 0 & b_{43} & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & b_{23} & 0 \\ 0 & b_{32} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

denote $\mathscr{I}_1(i) = \{j \in \{1, \ldots, 4\} | b_{ij}^1 \neq 0\}$ and $\mathscr{I}_2(i) = \{j \in \{1, \ldots, 4\} | b_{ij}^2 \neq 0\}$, $\forall i \in \{1, \ldots, 4\}$, respectively; in this case we have $\mathscr{I}_1(i) \bigcap \mathscr{I}_2(i) = \emptyset$, $\forall i \in \{1, \ldots, 4\}$. Consider first the nonperturbed iteration with delays $\tilde{v}^{p+1} = B^1 \tilde{v}^p + B^2 \tilde{v}^{p-\varrho(p)} + c$, where $p \to \varrho(p)$ denotes the delay. Then, using the formulation (14), the corresponding perturbed iteration with delays satisfies $J(p) = \{1, \ldots, 4\}$, $p = 1, 2, \ldots, s_j(p) = p, \forall j \in \mathscr{I}_1(i)$ and $s_j(p) = p - \varrho(p), \forall j \in \mathscr{I}_2(i)$, $i \in \{1, \ldots, 4\}, \forall p \in \mathbb{N}$. We concentrate on the situation where, for a real positive given number $\bar{\mu}$ and an integer$\bar{\rho}$, if

$$\|u^{p+1} - u^p\|_{\varepsilon,\infty} > \bar{\mu} \quad \text{and} \quad \varrho(p) < \bar{\rho} \tag{42}$$

then we replace $(p+1)$ by p and $\varrho(p+1) = \varrho(p) + 1$; else we replace $(p+1)$ by p and $\varrho(p+1) = 0$. Then according to Remark 3, we can write $\varrho(p) = \bar{\rho}$ and $\underline{\underline{\sigma}}(p) = p - \bar{\rho}$. Therefore, if $\mathrm{Diam}_{\varepsilon,\infty}(\mathbf{E}^\infty) \leqslant \eta$, then we can use the stopping criterion (36) in the context of Remark 3 and use Corollary 1 in order to obtain the bounds (33) and (34).

Note that the direct use of a simpler stopping criterion, joined to (42), can lead to stop the iterations with an iterate vector which is in the neighborhood of the fixed point of the scheme $\tilde{v} \to B^1 \tilde{v}^p + B^2 \tilde{v}^{p-\varrho(p)} + c$.

4.3. A simple illustration of multiprocessing asynchronous iterations

For the sake of simplicity, let us consider a simple illustration of computational aspects related to asynchronous iterations and concentrate on communications between two processes $(\mathscr{P}_1)$ and $(\mathscr{P}_2)$ carried out on two separate

processors. Let us consider the following tridiagonal matrix $B = B_1 + B_2 + B_3$, sum of three matrices B_i, $i = 1, 2, 3$, defined as follows:

$$B = \begin{vmatrix} 0 & b_{12} & 0 & 0 \\ b_{21} & 0 & b_{23} & 0 \\ 0 & b_{32} & 0 & b_{34} \\ 0 & 0 & b_{43} & 0 \end{vmatrix} = \begin{vmatrix} 0 & b_{12} & 0 & 0 \\ b_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{34} \\ 0 & 0 & b_{43} & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & b_{23} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & b_{32} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

then, consider the following asynchronous iteration $u^{p+1} = B_1 u^p + B_2 u^{p-\varrho_1(p)} + B_3 u^{p-\varrho_2(p)} + c$, where components (u_1, u_2), (u_3, u_4), respectively, are updated by processes $(\mathscr{P}_1)$ and $(\mathscr{P}_2)$, respectively, as follows:

$$(\mathscr{P}_1) \begin{vmatrix} u_1^{p+1} \\ u_2^{p+1} \end{vmatrix} = \begin{vmatrix} 0 & b_{12} \\ b_{21} & 0 \end{vmatrix} \begin{vmatrix} u_1^p \\ u_2^p \end{vmatrix} + \begin{vmatrix} c_1 \\ c_2 \end{vmatrix} + \begin{vmatrix} 0 \\ b_{23} u_3^{p-\varrho_1(p)} \end{vmatrix},$$

$$(\mathscr{P}_2) \begin{vmatrix} u_3^{p+1} \\ u_4^{p+1} \end{vmatrix} = \begin{vmatrix} 0 & b_{34} \\ b_{43} & 0 \end{vmatrix} \begin{vmatrix} u_3^p \\ u_4^p \end{vmatrix} + \begin{vmatrix} c_3 \\ c_4 \end{vmatrix} + \begin{vmatrix} b_{32} u_2^{p-\varrho_2(p)} \\ 0 \end{vmatrix},$$

where $p \to s_3(p) = p - \varrho_1(p)$ and $p \to s_2(p) = p - \varrho_2(p)$ satisfy (9)–(10); the algorithm corresponds to a situation where the two processors perform synchronously their computational tasks while the exchanges between them are completely asynchronous. In this case we have

$$z^1 = \{u_1^p, \mathbf{u}_2^p, u_3^p, u_4^p\}^{\mathrm{t}}, z^2 = \{\mathbf{u}_1^p, u_2^p, \mathbf{u}_3^{p-\varrho_1(p)}, u_4^p\}^{\mathrm{t}},$$
$$z^3 = \{u_1^p, \mathbf{u}_2^{p-\varrho_2(p)}, u_3^p, \mathbf{u}_4^p\}^{\mathrm{t}}, z^4 = \{u_1^p, u_2^p, \mathbf{u}_3^p, u_4^p\}^{\mathrm{t}},$$

where the bold-faced letters correspond to the components of the vectors z^i, $i = 1, \ldots, 4$, which are actually used during the computation and the others are unused; in this situation the order interval hull is defined componentwise by

$$\underline{w_1} = \min(u_1^{p+1}, u_1^p); \quad \overline{w_1} = \max(u_1^{p+1}, u_1^p),$$

$$\underline{w_2} = \min(u_2^{p+1}, u_2^p, u_2^{p-\varrho_2(p)}); \quad \overline{w_2} = \max(u_2^{p+1}, u_2^p, u_2^{p-\varrho_2(p)}),$$

$$\underline{w_3} = \min(u_3^{p+1}, u_3^p, u_3^{p-\varrho_1(p)}); \quad \overline{w_3} = \max(u_3^{p+1}, u_3^p, u_3^{p-\varrho_3(p)}),$$

$$\underline{w_4} = \min(u_4^{p+1}, u_4^p); \quad \overline{w_4} = \max(u_4^{p+1}, u_4^p)$$

and the corresponding order interval hull is

$$\langle \underline{w}, \overline{w} \rangle_p = \{v = (v_1, v_2, v_3, v_4) | \underline{w}_i \leqslant v_i \leqslant \overline{w}_i, i = 1, 2, 3, 4\}$$

which can be splitted as follows:

$$\langle \underline{w}, \overline{w} \rangle_p = \begin{cases} \langle \underline{w}, \overline{w} \rangle_p^1 = \{(v_1, v_2) | \underline{w}_i \leqslant v_i \leqslant \overline{w}_i, i = 1, 2\}, \\ \langle \underline{w}, \overline{w} \rangle_p^2 = \{(v_3, v_4) | \underline{w}_i \leqslant v_i \leqslant \overline{w}_i, i = 3, 4\} \end{cases}$$

where the suborder intervals $\langle \underline{w}, \overline{w} \rangle_p^1$, and $\langle \underline{w}, \overline{w} \rangle_p^2$, respectively, are managed by the first and the second processor, respectively. In this case the stopping criterion (32), i.e., $\mathrm{Diam}_{\varepsilon,\infty}(\langle \underline{w}, \overline{w} \rangle_p) \leqslant \eta$, corresponds to

$$\max\left(\frac{\overline{w}_1 - \underline{w}_1}{e_1}, \frac{\overline{w}_2 - \underline{w}_2}{e_2}\right) \leqslant \eta \quad \text{and} \quad \max\left(\frac{\overline{w}_3 - \underline{w}_3}{e_3}, \frac{\overline{w}_4 - \underline{w}_4}{e_4}\right) \leqslant \eta$$

on the first and the second processor, respectively.

Remark 7. Note that each processor is in charge of its own suborder interval, and is unaware of the state of the sub order interval managed by the other processors. As a consequence, additional data exchanges must be carried out in order to evaluate global criterion. Additional data exchanges can be included in the same message as numerical updates.

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