Regulation of linear PDE's by P-I controller using a Lyapunov approach inspired by forwarding methods.
Theory and application to the drilling case

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Introduction: What is output regulation ?

- Problem statement
- Regulation of abstract Cauchy problem

Forwarding Lyapunov functional

- For a general abstract Cauchy problem
- For a $n \times n$ hyperbolic system

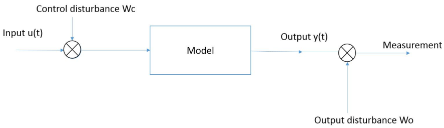
Application of this Lyapunov method

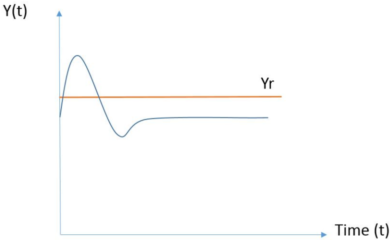
- Drilling system
- Study-case of a simple delayed ODE

Conclusion

Regulation problem

Given a system one wants to ensure that an output follows a prescribed reference despite uncertainties and disturbances.





Static error

Disturbances in real model : error of the modelisation, linearisation, sensors, $\cdots$

$\Rightarrow$ Static error between the measurement output and the reference.

Output regulation problem

An old problem with an old solution...

In 1788 James Watt : regulation of the admission of steam into an engine.

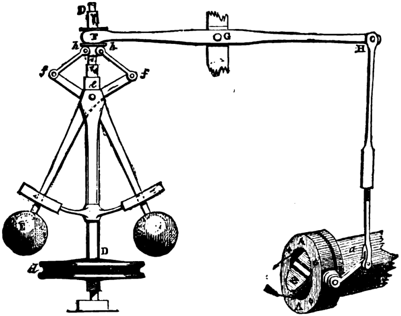


FIG. 4.—*Governor and Throttle-Valve.*

It takes time for the ball to go up $\Rightarrow$ There is a kind of "integral action".

Example of regulation by integral action

Example : A very trivial system:

$$\dot{x} = u + d$$

$$y = x$$

State $x \in \mathbb{R}$, control $u \in \mathbb{R}$, unknown constant disturbance $d \in \mathbb{R}$, measure $y \in \mathbb{R}$.

$\Rightarrow$ Given a reference y_r in $\mathbb{R}$, design u such that $y \to y_r$.

- If $u = y_r - y \Rightarrow$ equilibrium is stable but $y \nrightarrow y_r$.
- If $u = y_r - y - z$, where $\dot{z} = y - y_r \Rightarrow$ equilibrium is stable and $y \to y_r$.

Conclusion : The integral term added can reject the constant disturbance.

Abstract Input/Output Cauchy problem

By Fattorini's transformation $\Longrightarrow$ Kalman form :

$$\begin{aligned} \dot{\varphi} &= A\varphi + Bu + w, \quad \varphi \in \mathbb{X}, \quad u \in U \\ y &= C\varphi \end{aligned}$$

where $\mathbb{X}$ = Hilbert space with norm $\|\cdot\|_{\mathbb{X}}$ and $w \in \mathbb{X}$ = unknown constant vector .

$$\begin{aligned} &A : D(A) \mapsto \mathbb{X} \\ &U \subset \mathbb{R}^m, \quad B : U \mapsto D(A) \\ &y \in \mathbb{R}^m, \quad C : \mathbb{X} \mapsto \mathbb{R}^m \end{aligned}$$

$D(A)$ = the definition domain of A.

C_0-semigroups of contraction

With $u = k_i K_i \xi, \dot{\xi} = y$ the closed-loop systems is :

$$\dot{\varphi}_e = \begin{pmatrix} 0 & C \\ Bk_i K_i & A \end{pmatrix} \varphi_e + \begin{pmatrix} -y_r \\ w \end{pmatrix}$$

where $\varphi_e = \begin{pmatrix} \xi \\ \varphi \end{pmatrix} \in \mathbb{X}_e = \mathbb{R}^m \times \mathbb{X} =$ extended state space

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Open-loop stability Assumption

Let the operator A generates a C_0-semigroup exp. stable. Then there exist $k, \nu > 0$ such that, if $w = 0$, then $\forall \varphi \in \mathbb{X}$, $\forall t \in [0, +\infty)$:

$$\|T(t)\varphi\|_{\mathbb{X}} \leqslant k exp(-\nu t)\|\varphi_0\|_{\mathbb{X}}$$

This assumption can be obtain using a proportional feedback.

A brief review of existing results

Using abstract input/output Cauchy problem

- Perturbation theory for linear operator, Kato in 66'
- Pohjolainen in 82' for parabolic system.
- C.-Z. Xu and Jerby 95' for general abstract Cauchy problem.

About the assumption needed in semigroup theory

C.-Z. Xu and Jerby 95', Pohjolainen 82'

Assume assumption on Open-loop stability and :

1. Operator B is bounded;
2. Operator C is A-bounded, i.e :

$$|C\varphi| \leq c(\|x\|_X + \|A\varphi\|_X) \ , \ \forall \ \varphi \in D(A).$$

3. Rank condition : $\mathtt{rank}\{CA^{-1}B\} = m$

then there exists a positive real number k_i^* and a $m \times m$ matrix K_i such that for all $0 < k_i \leq k_i^*$ the operator

$$A_e = \begin{bmatrix} 0 & C \\ Bk_iK_i & A \end{bmatrix} \qquad (1)$$

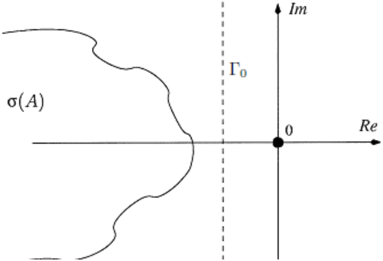
is the generator of an exponentially stable C_0-semigroup in the extended state space X_e.

Remarks

- Require B bounded
- Based on a spectral approach $\rightarrow$ difficult to extend to nonlinear systems.
- k_i^* is small and difficult to compute. For $K_i = (CA^{-1}B)^{-1}$, previous theorem impose

$$k_i^* = min_{\lambda \in \Gamma_0}\{\|B_e K_i\|^{-1}\|R(\lambda; A_e)\|^{-1}\}$$

$$B_e = \begin{pmatrix} 0 & 0 \\ Bk_iK_i & 0 \end{pmatrix}$$
$$A_e = \begin{pmatrix} 0 & C \\ k_iBK_i & A \end{pmatrix}$$
$R(\lambda; A_e)$ = resolvent of A_e.



Regulation of abstract Cauchy problem

Under assumption on the open-loop stability, $\exists$ P bounded and self-adjoint, c_1, c_2, w positive constants s.t. :

$$V(\varphi) = \langle \varphi, P\varphi \rangle, \qquad c_1 \|\varphi\|_{\mathbb{X}} \leqslant V \leqslant c_2 \|\varphi\|_{\mathbb{X}}, \qquad \dot{V} \leqslant -w \|\varphi\|_{\mathbb{X}}$$

Theorem : Forwarding Lyapunov for ACP (CDC18: ATJ-VA-VDSM-CZX)

Assume that all assumptions of Theorem Xu-Jerbi 95' are satisfied. There exists a bounded operators $M : \mathbb{X} \to \mathbb{R}^m$ and positive real numbers p and k_i^* such that for all $0 \leqslant k_i \leqslant k_i^*$, there exists $\omega_e > 0$ such that the functional :

$$W(x_e) = \langle \varphi, P\varphi \rangle + p\left(\xi - M\varphi\right)^T \left(\xi - M\varphi\right)$$

satisfies :

$$\dot{W} \leqslant -\omega_e \|\varphi_e\|_{\mathbb{X}_e}$$

Remark :
Lyapunov functional approach = same result of stability - spectral approach.

Sketch of the proof

$$\dot{W} = \dot{V} + 2p\,(\xi - M\varphi)^T \left(\dot{\xi} - M\varphi_t\right)$$

Select M and K_i as :

$$M\varphi_t = MA\varphi = C\varphi, \quad K_i = \left(CA^{-1}B\right)^{-1}$$

then $W(\varphi_e(t))$ according to time yields

$$\dot{W} \leq \left(-\omega + \frac{k_i}{a} + \frac{pk_i}{b}\right) \|\varphi\|^2_{\mathbb{X}} + k_i \left(p(-2 + b\|M\|^2) + a\|BPK_i\|^2_{\mathbb{X}}\right) |\xi|^2$$

So, one obtains a k_i max :

$$k_i^* = \frac{\nu}{\|CA^{-1}\|k^2\|B(CA^{-1}B)^{-1}\|}$$

where ν, k are selected in assumption on Open-Loop stability.

System of $n \times n$ hyperbolic PDE

Let a linear hyperbolic system (in Riemann coordinates):

$$\phi_t = \Lambda \phi_x \ , \quad \text{where } \phi : [0, \infty) \times [0,1] \to \mathbb{R}^n$$

with

- $\Lambda = diag\{\lambda_1, \ldots, \lambda_n\}, \quad \lambda_i > 0, \forall i \in \{1, \ldots \ell\}, \quad \lambda_i < 0, \forall i \in \{\ell + 1, \ldots, n\}$
- Perturbated boundary control conditions

$$\begin{bmatrix} \phi_+(t,0) \\ \phi_-(t,1) \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} \phi_+(t,1) \\ \phi_-(t,0) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) + Dw_b$$

- Perturbated output to be regulated

$$y(t) = L_1 \begin{bmatrix} \phi_+(t,0) \\ \phi_-(t,1) \end{bmatrix} + L_2 \begin{bmatrix} \phi_+(t,1) \\ \phi_-(t,0) \end{bmatrix} + w_y$$

Aim : Output regulation

$$\lim_{t \to +\infty} |y(t) - y_{ref}| = 0$$

System of $n \times n$ hyperbolic PDE

Integral control law :

$$u(t) = -k_i K_i \xi(t), \quad \dot{\xi}(t) = y(t) - y_{ref}$$

Assumptions :

1. Open-loop ISS properties: $\exists V \in \left(L^2(0,1)\right)^n$ and $\mu, c > 0$ verifying

$$\dot{V}(\phi(t)) \leq -2\mu V(\phi(t)) + c|u(t)|^2 \ . \tag{2}$$

2. Regulator equation

$$\mathtt{Im}\left(\begin{bmatrix} \mathrm{Id}_m & 0 \\ 0 & D \end{bmatrix}\right) \subset \mathtt{Im}\left(\begin{bmatrix} L_1 + L_2 & 0 \\ \mathrm{Id}_n - K & B \end{bmatrix}\right) . \tag{3}$$

3. Rank condition

$$T = (L_1 + L_2)(\mathrm{Id}_n - K)^{-1} B \tag{4}$$

is full rank.

Spaces of solution for $n \times n$ hyperbolic PDE

Let the Hilbert space:

$$\mathbb{X}_h = (L^2(0,1))^n \times \mathbb{R}^m$$

with the norm:

$$\|v\|_{\mathbb{X}_h} = \|\phi\|_{L^2(0,1)^n} + |\xi|$$

and the smoother Hilbert space :

$$\mathbb{X}_{h1} = (H^1(0,1))^n \times \mathbb{R}^m$$

From [Bastin,Coron 16']:

- $\forall v_0 \in \mathbb{X}_h$ satisfying the BC's, it exists a unique solution

$$v \in C^0([0,+\infty), \mathbb{X}_h)$$

- If $v_0 \in \mathbb{X}_{h1}$ and satisfies the C^1-compatibility condition, solution is strong and :

$$v \in C^0([0,+\infty), \mathbb{X}_{h1}) \cap C^1([0,+\infty), \mathbb{X}_h)$$

Output regulation for $n \times n$ hyperbolic PDE

Theorem : Forwarding Lyapunov for hyperbolic system (CDC18: ATJ-VA-VDSM-CZX)

Assume assumptions 1, 2 and 3 and select $K_i = T^{-1}$. Then, there exist $k_i^* > 0$ such that for all $0 < k_i < k_i^*$ the output regulation is achieved. More precisely

- There exist an equilibrium state $v_\infty = (\phi_\infty, \xi_\infty)^T$
- $v_\infty \in \mathbb{X}_h$ is a globally exponentially stable equilibrium

$$\|v(t) - v_\infty\|_{\mathbb{X}_h} \leq k \exp(-\nu t)\|v_0 - v_\infty\|_{\mathbb{X}_h}.$$

- if v_0 satisfies the C^1-compatibility condition and is in $\mathbb{X}_{h1}$, the regulation is achieved, i.e.

$$\lim_{t \to +\infty} |y(t) - y_{ref}| = 0.$$

Sketch of the proof 1/3

Assumptions 2 and 3 (regulator eq. + rank condition) $\Rightarrow \exists v_\infty$

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Suppose that

- $v_0 \in \mathbb{X}_{h1}$ and satisfies the C^1 compatibility conditions
- there exist a Lyapunov functional for the closed-loop system verifying

$$\frac{\|v_\infty - v\|^2_{\mathbb{X}_{h1}}}{L_w} \leqslant W(v) \leqslant L_w \|v_\infty - v\|^2_{\mathbb{X}_{h1}}.$$

Sketch of the proof 1/3

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Then, using Grönwall lemma

$$W(v(t)) \leq e^{-\omega t} W(v_0)\ .$$

And finally, using Sobolev embedding

$$\lim_{t\to+\infty} |y(t) - y_{ref}| = 0.$$

Thus, it only remains to build $W(v(t))$ to conclude !

Sketch of the proof 2/3

The Lyapunov candidate is

$$W(\xi, \phi) = V(\phi) + p(\xi - \mathcal{M}\phi)^T(\xi - \mathcal{M}\phi)$$

ISS and open-loop stability assumptions

$$\dot{V}(\phi, t) \leq -2\mu V(\phi, t) + c|k_i K_i \xi(t)|^2$$

Key Idea

Find out $\mathcal{M}\phi$ such that

$$\xi_t(t) - \mathcal{M}\phi_t(t, .) = -k_i \xi(t)$$

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Solution

$$\mathcal{M}\phi = \int_0^1 M\Lambda^{-1}\phi(s)ds, \quad \text{with } M = \begin{bmatrix} \mathrm{I_d}_\ell & 0 \\ 0 & -\mathrm{I_d}_{n-\ell} \end{bmatrix} (L_1 K + L_2)(\mathrm{I_d}_n - K)^{-1}$$

it yields,

$$\mathcal{M}\phi_t(t, .) = z_t(t) - Tu(t)$$

Sketch of the proof 3/3

Then

$$W_t(v(t)) \leq -2\mu V(\phi(t)) + c|u(t)|^2 - 2p\left[\xi(t) - \mathcal{M}\phi(\cdot,t)\right]^\top Tu(t)$$

using $K_i = T^{-1}$ and $u(t) = -k_i K_i \xi(t)$

$$W_t(v(t)) \leq -2\mu V(\phi(t)) + pk_i a|\mathcal{M}\phi(\cdot,t)|^2 + k_i\left(ck_i|K_i|^2 - 2p + \frac{p}{a}\right)|\xi(t)|^2$$

- Same form as for the abstract Cauchy problem
- Always possible to find $w_1, a, p \in \mathbb{R}^+$ s.t

$$W_t(v(t)) \leq -2w_1 W(v(t))$$

Drilling system

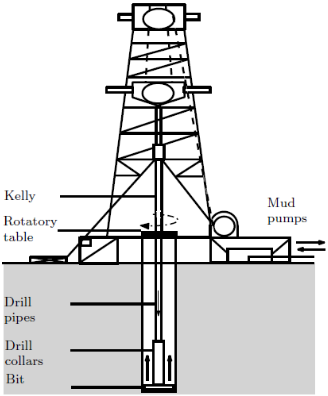

Figure: Diagram of a drilling device

Drilling system

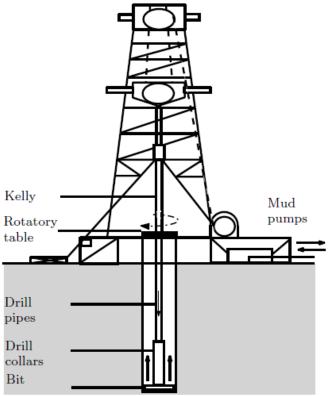

Figure: Diagram of a drilling device

Mechanical vibrations :

- Stick-Slip
- Bit bounce
- Lateral vibrations

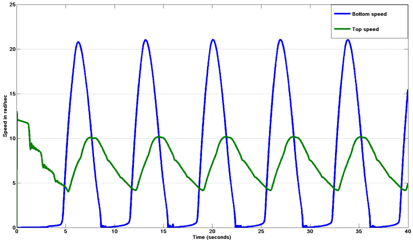

Figure: Oscillation of the radial velocities due to Stick-Slip

Drilling model

Using an equation of balance laws

Inside the pipe:

$$\theta_{tt}(x,t) = \frac{\partial_x \left(G(x)\theta_x(x,t)\right)}{\rho} - \beta(x)\theta_t(x,t)$$

Top boundary condition :

$$GJ\theta_x(0,t) = c_a\left(\theta_t(0,t) - \Omega(t)\right)$$

Bottom boundary condition:

$$I_B\theta_{tt}(L,t) = -GJ\theta_x(L,t) - T_{fr}\left(\theta_t(L,t)\right)$$

Problem statement

Rotatory table control :

$$u(t) = \Omega(t) + d_u$$

Topside velocity measurement :

$$y(t) = \theta_t(0, t)$$

Friction term + a constant perturbation :

$$I_B \theta_{tt}(L, t) = -G(L) J \theta_x(L, t) - c_b \theta_t(L, t) - T_0$$

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Objective

to regulate the downside velocity despite of constant perturbations :

$$\lim_{t \to \infty} |\theta_t(L, t) - y_{ref}| = 0$$

Riemann coordinates

The system written into Riemann coordinates:

$$\begin{aligned}
\varphi_t(x,t) &= \Lambda(x)\varphi_x(x,t) + N(x)\varphi(x,t),\ \forall x \in (0,1),\\
z_t(t) &= -(a+b)z(t) + a\varphi^-(1,t) + d_0,\\
\xi_t(t) &= \varphi^-(0,t) + \varphi^+(0,t) - \tilde{y}_{ref}
\end{aligned}$$

Boundary conditions :

$$\begin{aligned}
\varphi^-(0,t) &= \alpha_0\varphi^+(0,t) + k_i\xi(t) + d_u\\
\varphi^+(1,t) &= -\varphi^-(1,t) + 2z(t),
\end{aligned}$$

with $\varphi(x,t) = \begin{bmatrix}\varphi^-(x,t)\\ \varphi^+(x,t)\end{bmatrix}$, $\quad c(x) = \frac{G(x)}{\rho}$

$$\Lambda(x) = \begin{bmatrix}-c(x) & 0\\ 0 & c(x)\end{bmatrix},\quad N(x) = \begin{bmatrix}-(\lambda(x)+G_x(x)) & -(\lambda(x)-G_x(x))\\ -(\lambda(x)+G_x(x)) & -(\lambda(x)-G_x(x))\end{bmatrix}$$

Result for drilling system

Theorem (IEEE TAC 18 : ATJ-VA-MTF-VDSM)

Considering that all physical parameters are positives, and provided that $\forall x \in [0,1], |G_x(x)| \leqslant \overline{G_x}$, there exist real numbers k_i and positive real numbers k and ν such that $\forall(\tilde{y}_{ref}, d) \in \mathbb{R}^2$ and $\forall v(x,0) \in \mathbb{X}$, the following holds

1. $\exists$ an equilibrium state denoted v_∞ globally exponentially stable in $\mathbb{X}$ for the previous system. More precisely, we have :

$$\|v(t) - v_\infty\|_{\mathbb{X}} \leq k \exp(-\nu t) \|v_0 - v_\infty\|_{\mathbb{X}};$$

2. If moreover v_0 satisfies the C^1-compatibility condition and is in $\mathbb{X}_1$, the regulation is achieved, i.e.

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$\Longrightarrow$ we regulate the downside velocity !

Lyapunov design

1. One can find μ, p, q s.t for all parameters with $G_x(x) = 0$ the functional

$$V(z,\varphi) = qz^2 + \int_0^1 (\varphi^-(s,.))^2 e^{-\mu x} + p(\varphi^-(s,.))^2 e^{+\mu x} ds$$

Verifying

$$\dot{V}(z,\varphi) \leqslant -w_1 V(z,\varphi) + c|k_i \xi(t)|^2$$

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Verifying

$$\dot{V}(z,\varphi) \leqslant -w_1 V(z,\varphi) + c|k_i \xi(t)|^2$$

2. Using forwarding Lyapunov method, build $W(z,\xi,\varphi)$ and find k_i^*, w_2 s.t

$$\dot{W}(z,\xi,\varphi) \leqslant -w_2 W(z,\xi,\varphi)$$

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$$\dot{W}(z,\xi,\varphi) \leqslant -w_2 W(z,\xi,\varphi)$$

3. Let $G_x(x) \neq 0$. For all $0 < k_i < k_i^*$, there exist k_G s.t
$$\dot{W}(z,\xi,\varphi) \leqslant -w_2 W(z,\xi,\varphi) + k_G |G_x(x)| W(z,\xi,\varphi)$$

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$$\dot{W}(z,\xi,\varphi) \leqslant -w_2 W(z,\xi,\varphi) + k_G |G_x(x)| W(z,\xi,\varphi)$$

4. Select $\overline{G_x} = \frac{w_2}{2k_G}$ s.t

$$\dot{W}(z,\xi,\varphi) \leqslant -\frac{w_2}{2} W(z,\xi,\varphi)$$

Problem statement

We want to know the possible values of k such that

$$\dot{z}(t) = -kz(t-1)$$

converge to zero. The pole s verify

$$Re(s) < 0 \iff 0 < k < \frac{\pi}{2}$$

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Equivalence Delay/Transport PDE

$$\phi_t(x,t) = -\phi_x(x,t) \Rightarrow \phi(1,t) = \phi(0,t-1)$$

Can be formulate

$$\begin{aligned} \phi_t(x,t) &= -\phi_x(x,t) \\ \phi(0,t) &= -kz(t) \\ y(t) &= \phi(1,t), \quad \dot{z}(t) = y(t) \end{aligned}$$

Lyapunov design

Lyapunov candidate

$$W = \int_0^1 \phi(s,.)^2 e^{-\mu s} ds + p\left(z + \int_0^1 \phi(s,.)ds\right)^2$$

Note that

$$\frac{d}{dt}\left(z + \int_0^1 \phi(s,.)ds\right) = -kz(t)$$

Using Holder inequality:

$$\dot{W} \leqslant -\phi^2(1,t)e^{-\mu} - w^T(t)\mathcal{M}w(t)$$

with $\mathcal{M} = \begin{pmatrix} \frac{\mu^2}{e^{\mu}-1} & -pk \\ -pk & k(2p-k) \end{pmatrix}$ and $w(t) = \begin{pmatrix} \int_0^1 \phi(s,t)ds \\ z(t) \end{pmatrix}$

Applying Sylvester criterion

$$k < \frac{2p\mu^2}{\mu^2 + p^2(e^{\mu}-1)} = ln(2) \approx 0.69 \quad \text{avec } \mu = p = ln2$$

Using Lyapunov of [Ngoc Tu 16'] for hyperbolic PDE $\Rightarrow k_{max} \approx 0.39$

Results

Existing control methods based on 1D drilling model :

- Time-delay, backstepping, flatness, Smith predictor with $\lambda = 0$.
- Backstepping for a drilling model ($\lambda = cst, c = cst$) [Roman et al. 16', ...]

Remarks on the Forwarding Lyapunov design :

- Can be less conservative than the Lyapunov functional initiated by [Coron, Xu-Gauthier]
- Allow to deal with complex system (see drilling result)
- Give an explicit limit for the gain k_i
- Improve the stability result obtains using the Lyapunov functional initiated by [Coron, Xu-Gauthier] or [Ngoc Tu 16']

Perspectives

1. Lyapunov approach :
   - Extension to nonlinear controller (local stability)
   - Time varying perturbation $d(t)$
   - Extension to the general case of system of balance laws
   - Extension to coupled hyperbolic PDE with linear ODE

2. Drilling system :
   - Take account of rotatory table inertia.
   - Consider axial dynamics and coupled axial torsional friction function.

Thanks for your attention