Experimental Wind Field Estimation and Aircraft Identification

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Abstract

The presented work is focusing on the wind estimation and airframe identification based on real flight experiments as part of the SkyScanner project. The overall objective of this project is to estimate the local wind field in order to study the formation of cumulus-type clouds with a fleet of autonomous mini-UAVs involving many aspects including flight control and energy harvesting. For this purpose, a small UAV has been equipped with airspeed and angle of attack sensors. Flight data are recorded on-board at high speed for post-analyses. An approach based on Unscented Kalman Filter (UKF) is proposed for nonlinear wind estimation. As a first result, wind updraft estimation is highlighted by exploiting recorded flight test data. In addition to this, a motor test bench have been designed for this purpose, with an automatic recording sequence controlled by a computer.

In the sequel, first section presents the models and the problem formulation. Then, the theoretical basis of the estimation and identification algorithms are presented. The next section details the experimental setup and the instruments embedded on the UAV. To conclude, the final part gathers all the motor test bench and the experimental results obtained from the flights.

2 Problem Formulation

2.1 Wind field

Small UAVs are very sensitive to relative high wind gusts because of their size, hence satisfying the real-life demands becomes difficult. In general wind speed is assumed to be a spatially and temporally varying vector field s.t.

\[ \mathbf{w}(x,y,z,t) = \begin{pmatrix} w_x(x,y,z,t) \\ w_y(x,y,z,t) \\ w_z(x,y,z,t) \end{pmatrix} \]

where the superscript \( i \) denotes components expressed in the inertial frame. A small vehicle flying through this field is influenced by three components of the wind field and gradients of the wind field such that:

\[ \frac{d}{dt} \mathbf{w}(x,y,z,t) = \begin{pmatrix} w_x' \\ w_y' \\ w_z' \end{pmatrix} = \begin{pmatrix} w_{x-x} \\ w_{y-y} \\ w_{z-z} \end{pmatrix} + \nabla \mathbf{w}(x,y,z) \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \]

(1)

where the subscript \( \zeta \) denotes the time rate of change of wind velocity at the point \((x,y,z)\). Three component \( \dot{x}, \dot{y} \) and \( \dot{z} \) represent the velocity of the vehicle with respect to inertial
frame and $\nabla w(x,y,z)$ is the gradient of the wind field. Assuming a constant wind field as seen by the vehicle, the last term of Eq.1 becomes to zero, however this approximation is only applicable when vehicle velocity is large compared to the “point” rates of change of wind velocity (e.g. wing span is significantly larger than tail height, so vertical gradient of the lateral airmass velocity has negligible effect on roll rate). This paper considers the simultaneous nonlinear state estimation of aircraft body-axis velocity component and wind velocity component in the North-East-Down (NED) inertial reference frame using this assumption.

2.2 Vehicle Dynamics and Kinematics

In order to tackle a wide range of applications, various implementations of flight dynamics models, in terms of assumptions and numerical techniques, therefore exist. To overcome the difficulty for an UAV to derive a reliable representative aerodynamic model, they are commonly represented using a 6 Degrees of Freedom (DoF) kinematic model (3 DoF correspond to the translational motion $(V_N, V_E, V_D)$ and the 3 remaining DoF to the rotational motion $(\varphi, \theta, \psi)$).

![Fig. 1: Reference frames.](image)

Assuming a flat non-rotating Earth the flying rigid body motion in turbulent conditions can be located at $r$ with velocity $v_i$ having components $V_N, V_E, V_D$ in an inertial frame $I$, where $\hat{x}_i, \hat{y}_i$ and $\hat{z}_i$ define unit vector (see figure 1). Using a standard body-fixed coordinate frame with airmass-relative velocity $v_a$ having components $u, v, w$ in the body $\hat{x}_b, \hat{y}_b, \hat{z}_b$ directions, respectively, the acceleration of the aircraft in the inertial frame can be mathematically described s.t:

$$\ddot{r} = \dot{v}_i + \ddot{w}$$

Developing inertial velocity

$$\ddot{r} = \frac{d}{dt} v_a + \omega \times v_a + \frac{d}{dt} w$$

Hence

$$\frac{d}{dt} v_a = (\ddot{v}) - \omega \times v_a - \frac{d}{dt} w \quad (2)$$

Using angular velocity $\omega = (p \quad q \quad r)^T$ and aerodynamics forces $X, Y, Z$ depending on functions of trust $T$, drag $D$ and lift $L$ expressed in the body $x, y, z$ directions, the fundamental principle of dynamics now becomes:

$$m \left[ \begin{array}{c} \dot{\mathbf{v}}_a \\ \dot{w} \end{array} \right] = \left[ \begin{array}{ccc} 0 & -w & v \\ -v & 0 & -u \\ -u & w & 0 \end{array} \right] \left[ \begin{array}{c} p \\ q \\ r \end{array} \right] - \frac{d}{dt} \left[ \begin{array}{c} \dot{w}_x \\ \dot{w}_y \\ \dot{w}_z \end{array} \right] \quad (3)$$

In Eq.3, the latter quantity $\frac{d}{dt} w$ is expressed in the inertial frame and can be converted in the body frame through a Direction Cosine Matrix (DCM) $R_o^b$ which is defined by successive rotation of the roll, pitch and yaw angles of the aircraft s.t:

$$R_o^b = (R_q)^T = \begin{pmatrix} c_\theta c_\psi & c_\theta s_\psi & -s_\theta \\ s_\psi s_\theta c_\psi - c_\psi s_\theta & s_\psi s_\theta s_\psi + c_\psi c_\theta & s_\psi c_\theta \\ s_\theta s_\psi c_\psi + c_\psi s_\theta & -s_\psi c_\theta s_\psi + c_\psi s_\theta & c_\psi c_\theta \end{pmatrix}$$

In the next sections, Galilean transformations will be made by using a standard quaternional form, i.e., $R_q^b \cdot r = q^{-1} \ast r \ast q, R_q^b \cdot r = q \ast r \ast q^{-1}$. Note that symbol $\ast$ corresponds to the quaternion product. Using the aforementioned standard quaternional form provides at the same time:

- a global parametrization;
- avoids the mathematical singularities inherent to Euler angles;
- and is convenient for calculations and simulations.

For more details about formulas used on quaternion in this paper, see Appendix A. Finally, the state dynamics for the body-axis velocity states are given by:

$$\frac{\ddot{w}_x}{m} = -g \sin \theta - qw + rv - \dot{w}_x \cos \theta \cos \psi$$

$$\frac{\ddot{w}_y}{m} = -w \cos \theta \sin \psi + \dot{w}_y \sin \theta$$

$$\frac{\ddot{w}_z}{m} = \dot{w}_z \sin \theta$$
\[ \dot{v} = \frac{Y}{m} - g \sin \phi \cos \theta + pw - ru \]
\[ - \dot{w}\text{th} (\sin \phi \sin \theta \cos \psi - \cos \phi \sin \psi) \]
\[ - \dot{w}\text{th} (\sin \phi \sin \theta \sin \psi + \cos \phi \cos \psi) \]
\[ - \dot{w}\text{th} \sin \phi \cos \theta \]

\[ \dot{\psi} = \frac{Z}{m} - g \cos \phi \cos \theta + qu - pv \]
\[ - \dot{w}\text{th} (\cos \phi \sin \theta \cos \psi + \sin \phi \sin \psi) \]
\[ - \dot{w}\text{th} (\cos \phi \sin \theta \sin \psi - \sin \phi \cos \psi) \]
\[ - \dot{w}\text{th} \sin \phi \cos \theta \]

with \( \dot{w}\text{th} \) denotes the rate of change of a component of the wind velocity expressed in the inertial frame.

2.3 Aerodynamic and propulsion

Aerodynamic lift and drag forces in stability axes can be written as follow:

\[ L = \frac{1}{2} \cdot \rho \cdot V_a^2 \cdot S_{ref} \cdot C_L \]  \hspace{1cm} (4)

\[ D = \frac{1}{2} \cdot \rho \cdot V_a^2 \cdot S_{ref} \cdot C_D \]  \hspace{1cm} (5)

with,

\[
\begin{align*}
C_L &= C_{L0} + C_{LA} \cdot \alpha + C_{L\beta} \cdot \beta + \frac{1}{2V_a} \begin{pmatrix} bC_{Lp} \\ \frac{\partial C_{Lq}}{\partial \alpha} \end{pmatrix} \cdot \begin{pmatrix} p \\ q \end{pmatrix} \\
C_D &= C_{D0} + C_{DK} \cdot C_L^2 + \sum_{sfc} C_{Dsf} \cdot \delta_{sfc}
\end{align*}
\]  \hspace{1cm} (6)

where \( \alpha \) is the angle of attack, \( \beta \) the side-slip angle and \( \delta_{sfc} \) the elevator deflection. Since we are mostly interested in steady flight conditions, close to straight line without sideslip, and that the effect of the elevator on lift and drag forces is small, the equations can be reduced for performance analysis to:

\[ L = \frac{1}{2} \cdot \rho \cdot V_a^2 \cdot S_{ref} (C_{L0} + C_{LA} \cdot \alpha) \]  \hspace{1cm} (7)

\[ D = \frac{1}{2} \cdot \rho \cdot V_a^2 \cdot S_{ref} (C_{D0} + C_{ DK} \cdot C_L^2) \]  \hspace{1cm} (8)

In steady level flight, the weight is equilibrated by the lift, and the drag by the thrust. As a first approximation, the thrust, or more precisely the propulsive power \( P \) (the product of the thrust \( T \) by the airspeed \( V_a \)) can be expressed as:

\[ P = \eta \cdot P_{elec} \]  \hspace{1cm} (9)

where, \( P_{elec} \) is the electrical power drown from the batteries and \( \eta \) is an efficiency coefficient, function of the advance ratio, the propeller and motor characteristics, the Reynolds number and the electrical efficiency of the global propulsive system.

A more complex propulsion model [14], with a first order DC motor model, might be used in later studies in order to define an analytic description of \( \eta \).

3 Wind field estimation and aircraft model identification

3.1 Wind field estimation

This problem considers simultaneous estimation of aircraft body-axis velocity (\( u, v, w \)) and wind velocity components (\( w_{\text{th}}, w_{C\text{p}}, w_{D\text{c}} \)). Both process and measurement equations are not dependent on aerodynamic force described above. The estimation is performed through a fusion algorithm of low-cost inertial sensors used for UAV navigation [12]. The navigation quality is limited by inertial sensors performance specifies by the size, power and cost constraints of the UAV. To recover navigation accuracy using low-cost aided-INS (Inertial Navigation System), it is necessary to use, if possible, additional instruments (e.g. magnetometers, barometer, which are used to improve the heading and position accuracies) and/or nonlinear estimation algorithms to improve the flight handling qualities of the aerial vehicle. The nonlinear state estimation makes use of 2 triaxial sensors plus both GPS and Pitot tube sensor units which deliver a total of 10 scalar measurement signals:

- 3 gyroscopes providing a measurement of the instantaneous angular velocity vector denoted by \( \omega = [p, q, r]^T \in \mathbb{R}^3 \),

- 3 accelerometers giving a measurement of the specific acceleration denoted by \( a = [a_{mx}, a_{my}, a_{mc}]^T \in \mathbb{R}^3 \),

- 1 GPS unit measuring both position (not used) and velocity vectors denoted by \( y_v = [V_N, V_E, V_D]^T \in \mathbb{R}^3 \), vector \( \psi = [V_N, V_E, V_D]^T \) is used in the observation equations;

- 1 pitot tube sensor providing a scalar measurement of the air data denoted by \( y_{\text{Va}} \).

All the sensors embedded are low-cost, and therefore have imperfections. The major error sources in the navigation system are due to: - all of the disturbances (noises) that affect all the instruments; - the potential incorrect navigation system initialization (e.g. on magnetometers or barometric sensor); - and the inadequacy between the real local Earth’s gravity value and the one used for computation. The largest error is usually a bias instability (expressed respectively in deg/hr for gyro and mu g for the accelerometers).
All these measurements are obviously corrupted by additive noises for which it appears reasonable to assimilate their stochastic properties to the ones of Gaussian processes. Their covariances have been identified in [15] from logged sensor data using the Allan variance method [16]. Moreover, these errors correspond to the random nature of wind evolution necessary in Eq.2. Using these values, the state space representation corresponding to $\mathcal{M}_s$ can be described in a compact form such as:

$$\dot{x} = f(x,u) \quad \text{and} \quad y = h(x,u)$$

where $x = [u,v,w,w_{ix},w_{iy},w_{iz}]^T$, $u = [\omega_m^T,a_m^T]^T$ and $y = [y_V^T,y_{\alpha}]^T$ are the state, input and output vectors respectively.

$$\mathcal{M}_s\left\{\begin{align*}
\dot{y}_V &= \begin{bmatrix} u \\ v \\ w \end{bmatrix} * q^{-1} \quad \text{(measurement)} \\
\dot{y}_a &= \begin{bmatrix} u \\ v \\ w \end{bmatrix} - q^{-1} * \begin{bmatrix} w_{ix} \\ w_{iy} \\ w_{iz} \end{bmatrix} * q \\
\end{align*}\right.$$ 

Obviously, to implement these equation in a discrete-time filter, a first order discretization is used [17].

$$\begin{align*}
x_k &= x_{k-1} + T_f (x_{k-1}, u_{k-1}) + v_k \\
y_k &= h(x_k) + \mu_k
\end{align*}$$

where $f$ is the discrete-time state transition, $h$ is the nonlinear observation function which depend on DCM through quaternion and $T_f$ is the sampling time of the system. $v, \mu$ are the zero-mean Gaussian process noise vectors with covariance matrix, Q, R. Using these relationships, the angle of attack and side-slip are calculated from the body axis velocity components by

$$\alpha = \tan^{-1} \left( \frac{v}{u} \right)$$

$$\beta = \sin^{-1} \left( \frac{v}{\sqrt{u^2 + v^2 + w^2}} \right)$$

Since the measurement equation formulation contains nonlinear function, a nonlinear state estimation technique such as EKF [18] or UKF is required. The SR-UKF (Square-Root UKF) was selected for this work due to its ease of implementation and outperforms relative to EKF. Identification of both aerodynamic coefficient $C_L$ and $C_D$ can be lead by changing the process equation which is ongoing work.

This study uses flight data collected with a small UAV which is susceptible to perform some trajectories which leads to reduce maneuverability and so unobservability on two components of wind speed. Wind speed unobservability points out a particular behavior of UKF which is shown by a slow drift on state vector due to using sigmas-points for prediction and correction steps. Indeed, in case of strong nonlinearity, predictions are obtained by weighting the sigma-point predictions whose results differs from the direct calculation of the prediction from the estimated state (see figure 2). Firstly, effects can be observed in the calculation of the predicted state where the successive accumulation of these differences can lead to significant drifts, and secondly, in the correction of the state from the prediction measure. These effects can be eliminated by performing the calculation of prediction of the state and measures from the estimated average state, while retaining the sigma points for the calculation of covariance.

### 3.2 Aircraft model identification

The aircraft model identification is using the equation from section 2.3 during steady flight phases, when the airspeed is almost constant, thus the acceleration is zero. As a results, lift equals weight and thrust equals drag. Since the propulsion model was still under investigation at the time of the first flight experiments, the methodology have been adapted in order to estimate the drag. The procedure described in [19] have been used. It consists of performing several gliding phases at different airspeed and angle of attack in order to estimate the drag from the glide flight path $\gamma$.

$$\tan \gamma = -\frac{D}{L} = -\frac{C_D}{C_L} \quad (10)$$

The identification of the lift coefficient is done using Eq. 7. For each flight phase, the airspeed $V_a$ and the angle of attack $\alpha$ are averaged. Then a linear regression is used to estimate the two parameters $C_{L0}$ and $C_{L\alpha}$.

In order to identify the drag coefficient, only the gliding phases are considered as stated above and equations 8 and 10 are used. Three parameters are then needed, the lift coefficient and the angle of attack that are already computed, and the flight path angle $\gamma$. Since this angle can’t be directly measured, two methods have been evaluated. The first method is using an angle of attack installed on the UAV and the pitch
angle $\theta$ estimated using the IMU sensor. With the kinematic relation $\theta = \alpha + \gamma$, the path angle is estimated by averaging over the complete phase. The second method is using the relation that the lift over drag ratio is also the ratio between the horizontal distance and altitude lost during a glide. The main constraint is that the experiment needs to be done with almost no wind so that the ground and aerodynamic flight paths are the same. After estimating the parameter $\gamma$, the drag coefficient is computed with Eq. 10, and second order polynomial regression is done between $C_D$ and $\alpha$ in order to estimate $C_{D0}$, $C_{Dk}$.

Experimental results are presented in section 5.1 and they are showing a good correlation between the two methods.

For a future work, the UKF estimation algorithm presented at the previous section will be applied for these parameters identification.

4 EXPERIMENTAL SETUP

4.1 UAV instrumentation

As mention above, a mini UAV has been equipped with several sensors in order to make in-flight measurements. The frame itself is a commercial foam plane Solius from Multiplex\(^2\), a 2.16 meter wingspan motorized glider. The autopilot is an Apogee\(^3\) board using the Paparazzi system [20, 21], which includes a SD card slot for high speed logging.

It would have been possible to connect all the required sensors to the main autopilot, but due to electrical problems with long cables, it has been decided to split the Data Acquisition System (referred as DAQ board) from the flight control (referred as AP board). Hence, a second Apogee board was used to record the sensors, which is possible since there is no feedback of the wind estimation to the flight control at this stage of the project. The DAQ board is already equipped with 3-axis gyroscopes and accelerometers, and a low resolution barometer. The INS filter [13] used to estimate the position and orientation of the plane also requires a magnetometer and a GPS, that have been externally mounted.

The SkyScanner project aims at studying the formation of clouds. The meteorological parameters will be measured using a dedicated board Meteo-Stick. This board has high resolution absolute pressure sensor, differential pressure sensor, temperature sensor and humidity sensor. For this study, only the differential pressure sensor was used connected to a Pitot tube in order to measure the airspeed of the plane.

The figure 3 is showing the final integration of the measurement system, with the DAQ and Meteo-Stick stacked, the GPS and the magnetometer at the back.

The main sensor addition is an angle of attack sensor, mounted on the wing close to the Pitot tube. This sensor is made of a US DIGITAL MA3-P12-125-B\(^4\) angular sensor. It is an absolute position sensor using hall effect with 12-bit internal converter giving less than 0.09° of resolution with a very low noise. The vane has been 3D-printed and mounted directly on the shaft of the sensor. The figure 4 shows the final integration of this two sensors on the wing. The piece holding them has also been made with a 3D printer.

Fig. 3: Integration of the data acquisition system in the nose of the UAV.

Fig. 4: Integration of the Pitot tube and the angle of attack sensor on the leading edge of the wing.

The final setup for the SkyScanner project will also include a current sensor in order to measure the electrical power drawn by the motor that will be used in the aerodynamic and propulsion models, and some additional scientific sensors dedicated to the atmospheric research part, such as Liquid Water Content sensors, not directly related to the wind estimation.

4.2 Motor test bench

In order to integrate the propulsion model to the estimation process, it is necessary to establish the relation between the electrical current consumed by the motor, the rotation speed, the flight speed and the resulting propulsion forces and torque generated. Since it is not possible to embedded the necessary sensors to estimate this late parameters in flight, a motor test bench provides the propulsion model based on wind tunnel experiments.

The figure 5 shows the bench. Two force sensors are used to measure the propulsive force and the motor torque. The bench itself is an assembly of 3D-printed pieces and aluminum bars.

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\(^2\) http://www.multiplex-rc.de

\(^3\) http://wiki.paparazziuav.org/wiki/Apogee/v1.00

\(^4\) http://www.usdigital.com/products/encoders/absolute/rotary/shaft/ma3
In addition to the mechanical mounting of the motor and its propeller, an electronic board is required for the sensors’ signal conditioning. Finally, a myRIO data acquisition board from National Instruments connects them to the lab computer. A graphical interface developed using LabView allows to control the motor PWM command and synchronize all the measurements, making the process almost fully automated (the wind tunnel speed is currently control by hand). The figure 6 presents the global architecture and wiring of the motor test bench.

Fig. 6: Overview of the motor test bench experimental setup.

5 EXPERIMENTAL RESULTS

5.1 Flight tests
A few flight tests using the experimental setup described in the previous section have already been conducted (see figure 7).

Some preliminary results have been analyzed in order to assess the quality of the measurements, especially the angle of attack sensor since the Meteo-Sick sensors have already been validated in a previous project. The figure 8 shows the good correlation between the variation of the airspeed and the angle of attack (one increasing when the other decrease). Note that study uses angle of attack data collected with a constant offset which can be removed from raw data.

Flight plans In order to perform a correct estimation of the wind or the aerodynamic model, it is necessary to perform appropriate flight patterns. Concerning the wind estimation, observability is achieved by changing the flight direction. Hence, the chosen flight patterns are circles or small variation along a given segment. Figure 9 shows the horizontal wind field estimation along the aircraft trajectory. Further flight data analysis will be conducted in order to correctly tune the algorithm. The extraction of the vertical component of the wind will be possible by including the angle of attack measurements and not only the airspeed norm as it is currently the case.

An other type of flight pattern have been used in order to estimate the lift and drag coefficients as a function of the angle of attack. Several gliding phases are done at different
Fig. 9: Estimation of a wind updraft (red) during a gliding phase with confidence bounds (green).

airspeed following the same protocol than [19]. The figure 10 is showing four of these flight phases extracted from the complete experimental flight.

From the complete flight, four gliding phases and three cruise level flight phases have been selected for their stable airspeed and away from stall point. The figure 11 is a plot of the lift coefficient $C_L$ over the angle of attack $\alpha$, using both cruise and gliding phases. Figure 12 shows the drag coefficient $C_D$ over $\alpha$ computed with the two methods as described section 3.2. Both methods are giving very similar results, which is validating the identification methodology.

The table 1 summarizes the estimated aerodynamic coefficients (with $\alpha$ in degree):

<table>
<thead>
<tr>
<th>Method</th>
<th>$C_{D0}$</th>
<th>$C_{DK}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>method 1</td>
<td>0.01848</td>
<td>0.2034</td>
</tr>
<tr>
<td>method 2</td>
<td>0.01839</td>
<td>0.1912</td>
</tr>
</tbody>
</table>

Tab. 1: Aerodynamic coefficients identification for lift and drag.

Fig. 10: Different gliding flight path.

Fig. 11: Lift coefficient versus angle of attack.

The motor test bench have been placed in a wind tunnel and the parameters have been recorded at different airspeed from 0 (static thrust) up to 22 m/s. The resulting thrust versus RPM is shown on figure 13. We can see that the motor is generating a fair amount of static thrust (up to 10 N for a 1.5 kg plane) allowing easy take-off. But at higher speed, the motor is not generating positive thrust at low RPM (at 22 m/s it needs at least 80% of throttle) since the glider was not originally designed for high speed.

The main interest for improving the wind estimation and aircraft identification is to find a simple relation between the propulsive power and a measurable parameter. The figure 14 is showing this propulsive power versus the electrical power drown from the battery. This later value can be measured onboard from voltage and current sensors. As we can see, for the useful flight speeds from 10 to 15 m/s, there is a simple linear dependency of these two parameters.
Fig. 12: Drag coefficient versus squared lift coefficient, estimated with two different methods.

Fig. 13: Thrust versus RPM at different airspeed from wind tunnel experiment.

6 Conclusion

This article has presented the theoretical basis of a wind estimation algorithm based on Unscented Kalman Filter. Experimental flights have already been conducted in order to get real data. A foam glider has been equipped with airspeed and angle of attack sensors in addition to the traditional GPS+IMU units needed for the autonomous flight. The propulsion model has been identified using a motor test bench in a wind tunnel. Further developments will integrate this model to the aircraft aerodynamic identification process and to the wind estimation.

This work is only a first step in a larger project aiming at conducting atmospheric research using a fleet of cooperative UAVs. The estimation of the local wind field is therefore an important part of the project, but can be reused in many other applications such as long endurance flights based on autonomous soaring.

Acknowledgements

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References


APPENDIX A: QUATERNIONS AND ROTATIONS

An unit quaternion provide a convenient mathematical notation and a global parametrization for representing orientation and rotation of a rigid body in three dimensions. Indeed, for any unit quaternion

\[ q = q_0 + q = \cos \frac{\theta}{2} + u \sin \frac{\theta}{2} \]

and for any vector \( \mathbf{p} \in \mathbb{R}^3 \) the action of the operator

\[ q^{-1} * \mathbf{p} * q = R_q \cdot \mathbf{p} \]

is associated to a rotation matrix \( R_q \in SO(3) \) which is a rotation of the coordinate frame about axis \( \mathbf{u} = \frac{q}{|q|} \) through an angle \( \theta \).

Note that a vector \( \mathbf{p} \in \mathbb{R}^3 \) can be viewed as a pure quaternion whose real part is zero \( \begin{pmatrix} 0 \\ \mathbf{p} \end{pmatrix} \). Thus, when the real part is a scalar denoted \( p_0 \in \mathbb{R} \) the quaternion \( p \) is given as :

\[ p = \begin{pmatrix} p_0 \\ \mathbf{p} \end{pmatrix} \]