Event-based control for dynamical systems of finite or infinite dimensions.

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Outline

Introduction

Context

Control theory deals fundamentally with problems of **controllability** and **stability** of physical system modeled by ODE or PDE. For the problem of stabilization a **control feedback law** is designed in other to guarantee some desired property (decrease of the energy for example) of the resulting **closed-loop system**.

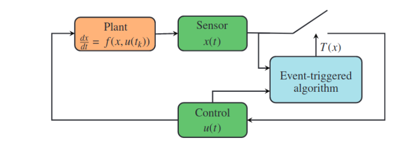
Event-based control

We are interested in the **implementation of this feedback law on digital platforms**. A choice is the **periodic control** but one can design a **triggering strategy**, which determines the time instants when the control needs to be updated: it is the Event-Triggering Mechanism (ETM).

Finite dimensional ETM

Framework and Control scheme with an ETM

Let us consider the control system : $\dot{x} = f(x, u)$



Assumption: $u(t) = kx(t)$ with k designed such that the closed-loop system $\dot{x} = f(x, k(x + e))$ is ISS with respect to measurement errors $e \in \mathbb{R}^n$.That means that, there exist a Lyapunov functional V and class $\mathcal{K}_\infty$ functions $\underline{\alpha}, \overline{\alpha}, \alpha$ and γ satisfying : $\underline{\alpha}(\|x\|) \leq V(x) \leq \overline{\alpha}(\|x\|)$ and $\dot{V}(x) \leq -\alpha(\|x\|) + \gamma(\|e\|) \, \forall x, e \in \mathbb{R}^n$.

Control objective: Implement $u(t) = k(x(t_k))$ for all $t \in [t_k, t_{k+1})$ by constructing the sequence (t_k)

Static and dynamic ETM

Static ETM [Tabuada(2007)]

Define $e(t) := x(t_k) - x(t)$ and

$$\begin{cases} t_0 & = 0, \\ t_{k+1} & = \inf\{t \in \mathbb{R},\ t > t_k \text{ and } \sigma\alpha(\|x(t)\|) - \gamma(\|e(t^-)\|) \leq 0\} \end{cases}$$

with $\sigma \in (0,1)$ a parameter and $e(t^-)$ is the limit of $e(s)$ when s approches t from the left.

Dynamic ETM [Girard(2015)]

$$\begin{cases} t_0 & = 0, \\ t_{k+1} & = \inf\{t \in \mathbb{R},\ t > t_k \text{ and } \eta(t) + \theta\left(\sigma\alpha(\|x(t)\|) - \gamma(\|e(t^-)\|)\right) \leq 0\} \end{cases}$$

where η is solution to $\begin{cases} \dot{\eta} = -\beta(\eta) + \sigma\alpha(\|x\|) - \gamma(\|e\|) \\ \eta(0) = \eta_0. \end{cases}$ with β Lipschitz continuous function and $\theta \in \mathbb{R}_0^+$ is an additional design parameter.

Asymptotical uniform stability under the ETM

Theorem (Tabuada(2007),Girard(2015))

1. *If we assume that* $-\alpha \circ \overline{\alpha}^{-1}$ *is locally Lipschitz continuous, then* ***under the static and the dynamic ETM*** *respectively, the origin is uniformly asymptotically stable for the closed-loop system* $\dot{x} = f(x, k(x+e))$.
2. *Moreover if* α^{-1} *and* γ *are Lipschitz continuous on compacts then* ***the inter-execution times*** $\{t_{k+1} - t_k\}_{i\in\mathbb{N}}$ ***are bounded from below by*** τ*, that is* $t_{k+1} - t_k > \tau$ *for any* $k \in \mathbb{N}$*.*

Linear case : $\dot{x} = Ax + Bu,\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$

With $u = Kx$, the closed-loop system $\dot{x} = Ax + BKx$ is G.A.S $\Longrightarrow$ $\exists$ Lyapunov functional $V(x) = x^\top Px$ where $P \succ 0$ is such that $(A+BK)^\top P + P(A+BK) = -Q$ with $Q \succ 0$.

Static ETM [Tabuada(2007)]

$$\begin{cases} t_0 & = 0, \\ t_{k+1} & = \inf\{t \in \mathbb{R},\ t > t_k \text{ and } \sigma x(t)^\top Qx(t) - 2x(t)^\top PBKe(t^-) \leq 0\} \end{cases}$$

with $\sigma \in (0,1)$ a parameter.

Dynamic ETM [Girard(2015)]

$$\begin{cases} t_0 & = 0, \\ t_{k+1} & = \inf\{t \in \mathbb{R},\ t > t_k \text{ and } \eta(t) + \sigma x(t)^\top Qx(t) - 2x(t)^\top PBKe(t^-) \leq 0\} \end{cases}$$

where η is solution to $\begin{cases} \dot{\eta} = -\lambda\eta + \sigma x^\top Qx - 2x^\top PBKe \\ \eta(0) = \eta_0. \end{cases}$ with $\lambda > 0$.

Fact: The previous Theorem holds.

Numerical Simulation : $A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

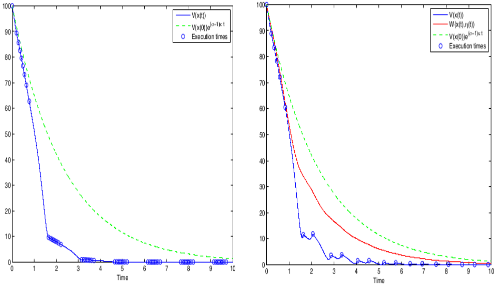
A stabilizing controller is given by the control gain $K = (1 \ -4)$. The associated Lyapunov functional is $V(x) = x^\top P x$, with $P = \begin{pmatrix} 1 & 0.25 \\ 0.25 & 1 \end{pmatrix}$ verifying

$$(A+BK)^\top P + P(A+BK) = -Q \text{ for } Q = \begin{pmatrix} 0.5 & 0.25 \\ 0.25 & 1.5 \end{pmatrix}.$$

For the initial condition $x(0) = \begin{pmatrix} 10 & 0 \end{pmatrix}^\top$, one represents the evolution of $V(x(t))$ and $W(x(t), \eta(t))$ using the static ETM, the dynamic ETM and the ETM defined by

$$\begin{cases} t_0 = 0 \\ t_{k+1} = \inf\{t \in \mathbb{R} \text{ such that } t > t_k \text{ and } V(x(t)) \geq e^{(\sigma-1)\kappa(t-t_k)} V(x(t_k))\} \end{cases}$$

Numerical Simulation



(a) Static ETM (b) Dynamic ETM

Figure 1: Evolution of the functions $V(x(t))$ and $W(x(t), \eta(t))$ using static and dynamic ETM

Event-based control for the wave equation

Consider the wave equation in a bounded domain Ω :

$$(1) \qquad \begin{cases} \partial_t^2 z(x,t) - \Delta z(x,t) = F(x,t) & \forall\ (x,t) \in \Omega \times (0,T), \\ z(x,t) = 0 & \forall\ (x,t) \in \partial\Omega \times (0,T), \\ z(x,0) = z_0(x) & \forall\ (x,t) \in \Omega, \\ \partial_t z(x,0) = z_1(x) & \forall\ x \in \Omega. \end{cases}$$

1. Taking $\boxed{F(x,t) = -\alpha \partial_t z(x,t)}$ with $\alpha > 0$, and for every $(z_0, z_1) \in H_0^1(\Omega) \times L^2(\Omega)$
   - $\rightsquigarrow$ $\exists!\, z \in C^0([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ solution to (1)
   - $\rightsquigarrow$ Exponential stability : $\exists C > 0;\ E(t) \leq CE(0)e^{-\gamma t}$ with $E(t) = \frac{1}{2}(\|\partial_t z(t)\|_{L^2(\Omega)} + \|\nabla z(t)\|_{L^2(\Omega)})$
2. **Control objective :** Can we construct a ETM and guarantee both stability and the well-posedness of the corresponding closed-loop system?

Main result

Let us consider $e_k(x,t) = \partial_t z(x,t) - \partial_t z(x,t_k)\, \forall (x,t) \in \Omega \times [t_k, t_{k+1})$, and define the ETM by

$$\begin{cases} t_0 & = 0, \\ t_{k+1} & = \inf \left\{ t \geq t_k \text{ such that } \|e_k(t)\|^2_{L^2(\Omega)} \geq \gamma E(t) + \delta_0 E(0) e^{-2\theta t} \right\} \end{cases}$$

where $\gamma > 0$, $\delta_0 > 0$ and $\theta > 0$ are design parameters.

Theorem (Exponential stability and avoidance of Zeno phenomenon)

1. $\forall (z_0, z_1) \in H_0^1(\Omega) \times L^2(\Omega)$, *there* ***exists a unique weak solution to the closed-loop system under the designed ETM*** *such that* $z \in C^0([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ *and* ***there exists a minimal dwel-time ensuring the avoidance of Zeno behavior***
2. ***Under some LMI condition***, *the closed-loop system (system* (1) *with* $F(x,t) = -\alpha \partial_t z(x,t_k)$*) is exponentially stable*

Remark: A similar result is obtained in [Baudouin,Marx, Tarbouriech(2019)]

Strategy to prove the main result

Idea of the proof

1. Well-posedness
   - $\rightsquigarrow$ Based on Induction on every sample interval $[t_k, t_{k+1}]$,
   - $\rightsquigarrow$ and well-posedness of the damping wave equation
2. Avoidance of Zeno behavior
   - The continuity of $\partial_t z(\cdot, t)$ leads to uniform continuity,
   - Since $\delta_0 E(0) e^{-\theta T} > 0 \rightsquigarrow$ there exists $\tau > 0$, such that one has $t_{k+1} - t_k \geq \tau$,

Exponential stability

Let us consider the following **Lyapunov functional**:

$$V(t) = E(t) + \frac{\alpha\varepsilon}{2}\|z(t)\|^2_{L^2(\Omega)} + \varepsilon \int_\Omega z(x,t)\partial_t z(x,t)dx$$

1. **Step 1**:Equivalence of the energy and a designed Lyapunov functional

$$(1 - \varepsilon C_\Omega)\, E(t) \leq V(t) \leq \left(1 + \varepsilon C_\Omega + \varepsilon\alpha C_\Omega^2\right) E(t)$$

2. **Step 2**: Find the conditions on which for a desired decay rate δ

$$\dot{V}(t) + 2\delta V(t) \leq \delta_0 E(0) e^{-2\theta t}$$

LMI condition to estimate $\dot{V}(t) + 2\delta V(t)$

- $\dot{V}(t) + 2\delta V(t) = \int_\Omega \psi^\top(x,t) M_1 \psi(x,t) dx$
  with $\psi = \begin{pmatrix} z \\ \partial_t z \\ \nabla z \\ e_k(t) \end{pmatrix}$ and $M_1 = \begin{pmatrix} \alpha\varepsilon\delta & \delta\varepsilon & 0 & \frac{\alpha\varepsilon}{2} \\ \star & \varepsilon - \alpha + \delta & 0 & \frac{\alpha}{2} \\ \star & \star & \delta - \varepsilon & 0 \\ \star & \star & \star & 0 \end{pmatrix}$.
- $\int_\Omega |z(t)|^2 dx \leq C_\Omega^2 \int_\Omega |\nabla z(t)|^2 dx \implies \int_\Omega \psi^\top(x,t) M_2 \psi(x,t) dx \geq 0$ with
  $$M_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & C_\Omega^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$
- ETM : $\|e_k(t)\|^2_{L^2(\Omega)} \leq \gamma E(t) + \delta_0 E(0) e^{-2\theta t} \implies \int_\Omega \psi^\top M_3 \psi dx \geq 0$ with
  $$M_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\gamma}{2} & 0 & 0 \\ 0 & 0 & \frac{\gamma}{2} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

As a consequence

$$\dot{V}(t) + 2\delta V(t) \leq \int_\Omega \psi^\top M_1 \psi dx + \delta_0 E(0) e^{-\theta t}$$

where $\int_\Omega \psi^\top(t) M_1 \psi(t) \leq 0$ is subject to

$$\int_\Omega \psi^\top(t) M_2 \psi(t) \geq 0 \text{ and } \int_\Omega \psi^\top(t) M_3 \psi(t) \geq 0.$$

S-procedure

Ensures the existence of $\lambda_1 > 0$ and $\lambda_2 \geq 0$ such that

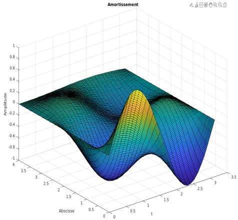
$$\int_\Omega \psi^\top (\underbrace{M_1 + \lambda_1 M_2 + \lambda_2 M_3}_{G}) \psi dx \leq 0$$

Condition on which $G \prec 0$

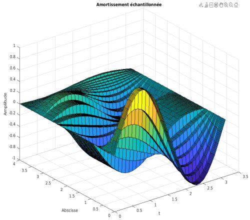
1. Use Shur complement and
2. Finsler lemma

End of the proof!

Numerical Simulation : 1D damping wave equation

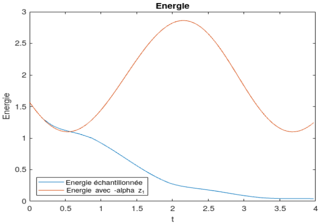


(a) Solution to continuous-time closed-loop system

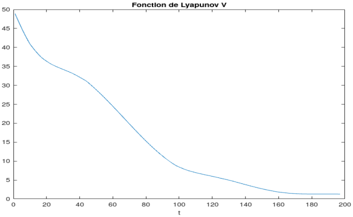


(b) Solution to the closed-loop system under the ETM

Decreasing of the Lyapunov functional



(a) Evolution of $E(t)$



(b) Decreasing of $V(t)$

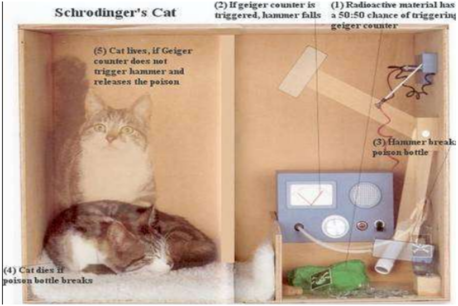
Conclusion and Future works

We studied the static and dynamic event-triggering mechanism and proposed an event-based control rule for the linear wave equation.

Future work

1. We would like to get rid of with the term $\delta_0 E(0)e^{-2\theta t}$ and prove that the zeno phenomenon does not occur.

   From the ETM law we have $\frac{\|e_k(t)\|^2_{L^2(\Omega)}}{\gamma E(t)} \leq 1, \forall t \in [t_k, t_{k+1}]$. It follows that a lower bound on the inter-execution time is actually the time taken by the function $\varphi : t \mapsto \varphi(t) = \frac{\|e_k(t)\|^2_{L^2(\Omega)}}{\gamma E(t)}$ to go from 0 to 1. We then compute the time-derivative of $\varphi(t)$ and use Comparison Lemma...


Schrodinger's Cat
(2) If geiger counter is triggered, hammer falls
(1) Radioactive material has a 50:50 chance of triggering geiger counter
(5) Cat lives, if Geiger counter does not trigger hammer and releases the poison
(3) Hammer breaks poison bottle
(4) Cat dies if poison bottle breaks

Event-triggered damping stabilization for the linear Schrödinger equation

Consider the linear Schrödinger equation in a bounded domain Ω :

$$(2)\qquad \begin{cases} i\partial_t z(x,t) + \Delta z(x,t) = f(x,t) & \forall\ (x,t) \in \Omega \times (0,T), \\ z(x,t) = 0 & \forall\ (x,t) \in \partial\Omega \times (0,T), \\ z(x,0) = z_0(x) & \forall\ (x,t) \in \Omega. \end{cases}$$

1. Taking $\boxed{f(x,t) = -iaz(x,t)}$ with $a > 0$, and for every $z_0 \in L^2(\Omega)$
   - $\rightsquigarrow$ $\exists!\, z \in C^0([0,T];L^2(\Omega)) \cap C^1([0,T];(H^2(\Omega) \cap H^1(\Omega)))$ solution to (2)
   - $\rightsquigarrow$ Exponential decay: $\exists C, \gamma > 0;\ F(t) = \frac{1}{2}\|z(t)\|_{L^2(\Omega)} \leq CF(0)e^{-\gamma t}$ [Machtyngier, Zuazua(1992)]
2. **Control objective :** For $f(x,t) = -iaz(x,t_k), t \in\ [t_k, t_{k+1})$, what (t_k) looks like in order to maintain this exponential decay result and avoid Zeno behavior?

Event-triggered **boundary control** of linear Schrödinger equation via **Backstepping approach**

Linear Schrödinger in $(0,1)$

$$\begin{cases} \partial_t z(x,t) + i\partial^2_{xx} z(x,t) = 0 \\ \partial_x z(0,t) = 0 \\ z(1,t) = Z(t). \end{cases}$$

$$\xrightarrow{\Pi_\lambda}$$

Target system

$$\begin{cases} \partial_t u(x,t) + i\partial^2_{xx} u(x,t) + \lambda u(x,t) = 0, \\ \partial_x u(0,t) = 0 \\ u(1,t) = 0. \end{cases}$$

Exponentially stable thanks to spectral theory

- Assume that Π_λ is continous with continuous inverse. Then:
  $\|z(t)\| = \|\Pi_\lambda^{-1} u(t)\| \leq C_{-1}\|u(t)\| \leq C_{-1}e^{-\lambda t}\|u(0)\| \leq C_{-1}Ce^{-\lambda t}\|z(0)\|$
- Proposition:**Voltera operator**[Krstic et all (2007)]
  $u(x,t) = \Pi_\lambda(z) := z(x,t) - \int_0^x k(x,x')z(x',t)dx'$
  $\rightsquigarrow Z(t) = z(1,t) = \int_0^1 k(1,x')z(x',t)dx'$

Event-triggered **boundary control** of linear Schrödinger equation via Backstepping approach

The futur work

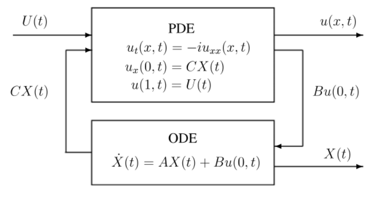
Construct (t_k) in such a way that updating the feedback law $Z(t) := \int_0^1 k(1,x')z(x',t_k)dx'$ the exponential decay result and avoidance Zeno behavior of the closed-loop system are still maintained.

Idea

We propose considering the **Schrödinger equation formally as a heat equation with the imaginary diffusion coefficient** and solving the ETM stabilization problem using the method presented in [Espitia, Karafyllis,Krstic(2020)] for reaction-diffusion equation.

ETM for ODE-Schrödinger cascade system

Block diagram for the coupled ODE-PDE system



ODE-Schrödinger equation cascade

$$\begin{cases} \dot{X}(t) = AX(t) + Bu(0,t), t > 0 \\ \partial_t u(x,t) = -i\partial_{xx}^2 u(x,t), x \in (0,1), t > 0 \\ \partial_x u(0,t) = CX(t) \\ u(1,t) = U(t). \end{cases}$$

Previous result: In [Ren,Wang,Krstic(2013)] a two step backstepping transformation, $U(t) = \int_0^1 [k(1,y) + q(1,y) - \int_0^1 k(1,l)q(l,y)dl]u(y,t)dy + [\gamma(t) - \int_0^1 k(1,y)\gamma(t))dy]X(t)$ exponentially stabilizes the system.

Goal: Get the same result of stability by constructing an ETM $U(t) := U(t_k)$.

The nonlinear case

1. ETM for the nonlinear Schrödinger equation
   $i\partial_t u(x,t) = -\partial_{xx} u(x,t) - \mu |u(x,t)|^2 u(x,t)$
2. ETM for the Schrödinger equation with saturating distributed input
3. Static and dynamic ETM for non-autonomous systems with saturating input

pour votre aimable attention.