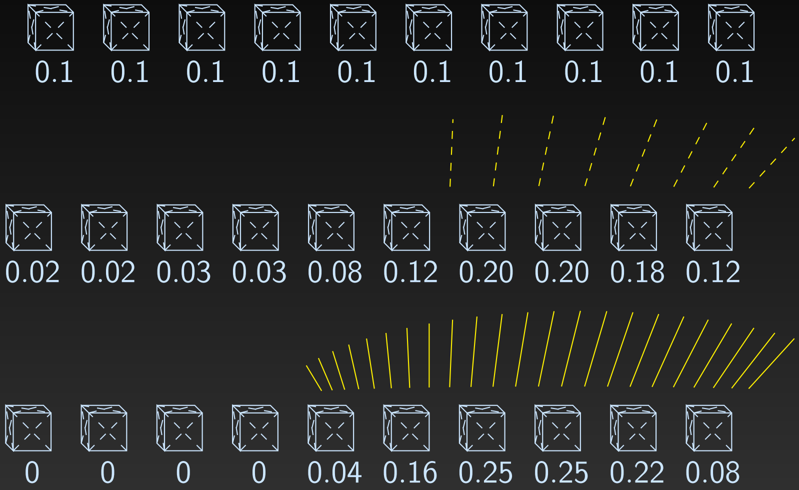
A First Contact with borel measures

7 amazing facts about borel measures that you almost surely didn't know

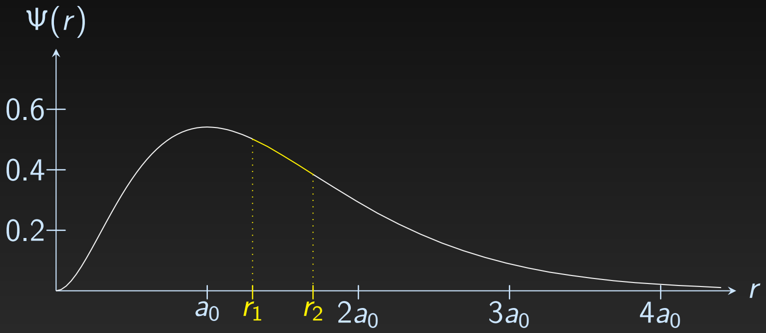
2020, October, the 16th

I) Examples and definition



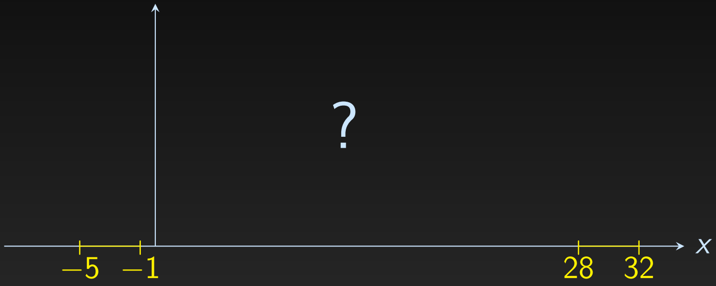
Ψ is the *density of probability of presence* of the electron:

$$p([r_1, r_2]) = \langle p, \mathbb{1}_{[r_1,r_2]} \rangle = \int_0^{+\infty} \mathbb{1}_{[r_1,r_2]}(r)\,\Psi(r)dr = \int_{r_1}^{r_2} \Psi(r)dr.$$



$$p(\{a_0\}) = \langle p, \mathbb{1}_{\{a_0\}} \rangle = \int_{a_0}^{a_0} \Psi(r)dr = 0, \quad p(\mathbb{R}_+ \setminus \{a_0\}) = 1.$$

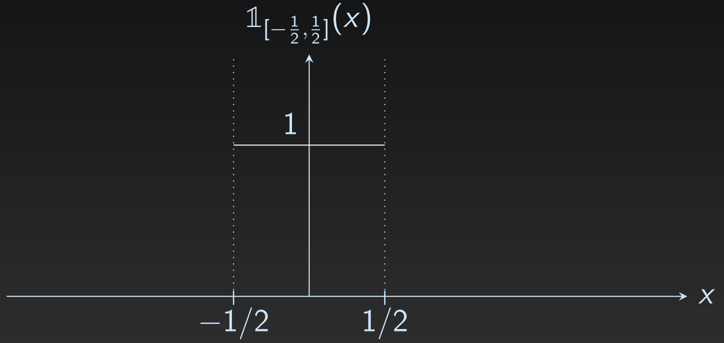
$$\langle p, \mathbb{1}_{[-5,-1]} \rangle = \langle p, \mathbb{1}_{[28,32]} \rangle = \langle p, \mathbb{1}_{[a,a+4]} \rangle \quad \forall a \in \mathbb{R}.$$



$$\text{Leb}_{[-\frac{1}{2},\frac{1}{2}]}([a,b]) = \int_a^b \mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}(x)dx = \int_{-1/2}^{1/2} \mathbb{1}_{[a,b]}(x)dx.$$

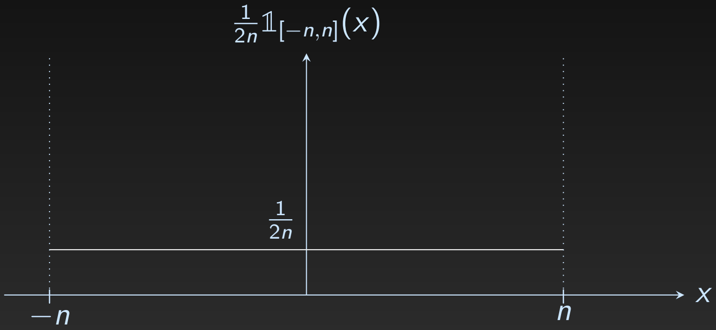
$$\text{Leb}_{[-\frac{1}{2},\frac{1}{2}]}([a,b]) = \text{Leb}_{[-\frac{1}{2},\frac{1}{2}]}([a+c,b+c])$$

$$\forall a,b,c / \ \frac{-1}{2} \leqslant a, a+c, b, b+c \leqslant \frac{1}{2}.$$



$$\text{Leb}_{[-\frac{1}{2},\frac{1}{2}]}([-\tfrac{1}{2},\tfrac{1}{2}]) = \int_{-1/2}^{1/2} 1 dx = 1.$$

$$p_n([a,b]) = \langle \tfrac{1}{2n}\mathrm{Leb}_{[-n,n]}, \mathbb{1}_{[a,b]} \rangle = \langle \tfrac{1}{2n}\mathbb{1}_{[-n,n]}, \mathbb{1}_{[a,b]} \rangle$$
$$= \int_{\mathbb{R}} \mathbb{1}_{[a,b]}(x)\, \frac{1}{2n}\mathbb{1}_{[-n,n]}(x)dx.$$



$$p_n([a,b]) = p_n([a+c, b+c]) \qquad \forall a,b,c / \; -n \leqslant a, a+c, b, b+c \leqslant n.$$

Take $\mathscr{B}$ a σ-algebra on Ω (the elements of $\mathscr{B}$ are subsets of Ω).

definition

$p : \mathscr{B} \to [0,1]$ is a *probability* on $(\Omega, \mathscr{B})$ if

- $p(\Omega) = 1$
- $p\left(\bigsqcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} p(A_i)$ for $(A_i)_{i\in\mathbb{N}} \in \mathscr{B}^{\mathbb{N}}$ mutually incompatible events.

definition

$\mu : \mathscr{B} \to [0,+\infty]$ is a (positive) *measure* on $(\Omega, \mathscr{B})$ if

- $\mu(\emptyset) = 0$
- $\mu\left(\bigsqcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} \mu(A_i)$ for $(A_i)_{i\in\mathbb{N}} \in \mathscr{B}^{\mathbb{N}}$ mutually disjoint subsets of Ω.

Examples of borel measures on $\mathbb{R}$:

- $\text{Leb}_{[0,1]}$, measure of probability with density $\mathbb{1}_{[0,1]}$. $\int_{\mathbb{R}} \mathbb{1}_{[0,1]}(x)dx = 1$.
- δ_a, measure of probability with no density. $\delta_a(\mathbb{R}) = \delta_a(\{a\}) = 1$, the real a is an *atom* for δ_a.
- $\mathcal{N}(m, \sigma^2)$, measure of probability with density $g : x \mapsto \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right)$. $\int_{\mathbb{R}} g(x)dx = 1$.
- $\text{Leb}_{\mathbb{R}}$, measure with density 1. $\text{Leb}_{\mathbb{R}}(\mathbb{R}) = \int_{\mathbb{R}} 1dx = +\infty$.

Why can we not build a uniform probability on $\mathbb{R}$?
- This is a property of the Lebesgue measure.

Why can we not build a uniform probability on $\mathbb{R}$?
This is due to a property of the Lebesgue measure.

property

There is only one translation-invariant measure on $\mathbb{R}$ such that the image of $[0,1]$ is 1. It is the Lebesgue measure $\mathrm{Leb}_{\mathbb{R}}$.

Take μ a measure s.t. $\forall A \in \mathscr{B}, \forall c \in \mathbb{R}, \mu(A) = \mu(A+c)$.

- If $\mu([0,1]) = 0$ then $\mu = 0$ and $\mu(\mathbb{R}) = 0 < 1$.
- If $\mu([0,1]) = +\infty$ then $\mu(\mathbb{R}) \geqslant \mu([0,1]) = +\infty > 1$.
- If $0 < \mu([0,1]) < +\infty$ then $\frac{1}{\mu([0,1])}\mu = \mathrm{Leb}_{\mathbb{R}}$ so $\mu(\mathbb{R}) = \mu([0,1])\mathrm{Leb}_{\mathbb{R}}(\mathbb{R}) = +\infty > 1$.

II) The Cabinet of curiosities of Pr. Lebesgue

1. The Borelians
2. The Derivative of a measure
3. The Bochner Integral of a measure-valued map
4. Limit of processes through different asymptotic scales
5. The Fat Cantor
6. The Devil's Staircase
7. The Infinite-Dimensional Lebesgue measure

II)1) The Borelians

definition

$\mathscr{B}$ is a σ-algebra on Ω if

- $\emptyset \in \mathscr{B}$,
- $A \in \mathscr{B} \implies \Omega \setminus A \in \mathscr{B}$,
- $(A_i)_{i\in\mathbb{N}} \in \mathscr{B}^{\mathbb{N}} \implies \bigcup_{i\in\mathbb{N}} A_i \in \mathscr{B}$.

The Borel σ-algebra $\mathscr{B}(\Omega)$ is the is the most little that contains all the open sets of Ω.
On $\mathbb{R}$, it is equivalent to say that $\mathscr{B}(\mathbb{R})$ is generated by all the intervals.
Typically, a union of intersection of union of intersection of... of open sets is a borelian.

II)1) The Borelians: the Vitali counterexample

Take V a $\mathbb{Q}$-base on $\mathbb{R}$ in $[0,1]$ ie. a set of representatives in $[0,1]$ of the equivalence relation $x \sim y \Leftrightarrow x - y \in \mathbb{Q}$.

So $\mathbb{R} = \bigsqcup_{q\in\mathbb{Q}} (V+q)$.

- Suppose $\mathrm{Leb}_{\mathbb{R}}(V) = 0$. Then

$$2 = \mathrm{Leb}_{\mathbb{R}}([0,2]) \leqslant \mathrm{Leb}_{\mathbb{R}}\left(\bigsqcup_{q\in\mathbb{Q}} (V+q)\right) = \sum_{q\in\mathbb{Q}} \mathrm{Leb}_{\mathbb{R}}(V) = 0.$$

- Suppose $\mathrm{Leb}_{\mathbb{R}}(V) > 0$. Then

$$2 = \mathrm{Leb}_{\mathbb{R}}([0,2]) \geqslant \mathrm{Leb}_{\mathbb{R}}\left(\bigsqcup_{q\in\mathbb{Q}\cap[0,1]} (V+q)\right) = \sum_{q\in\mathbb{Q}\cap[0,1]} \mathrm{Leb}_{\mathbb{R}}(V) = +\infty.$$

II)2) The Derivative of a measure

$\Omega \subset \mathbb{R}^d$, $\mathscr{D}'(\Omega) = \mathscr{C}_c^\infty(\Omega)' = \mathscr{L}(\mathscr{C}_c^\infty(\Omega), \mathbb{R})$ is the set of distributions.

- $f \in \mathscr{C}^0(\Omega) \;\Rightarrow\; T_f$ is a distribution

$$\langle T_f, \phi \rangle = \int_\Omega f(x)\phi(x)\,dx \quad \forall \phi \in \mathscr{C}_c^\infty(\Omega).$$

- μ s.t. $\mu(K) < +\infty \quad \forall K \subset \Omega$ compact $\Rightarrow$ μ is a distribution: Take $\phi \in \mathscr{C}_c^\infty(\Omega)$, take $A = (A_i)_{0\leqslant i\leqslant N} \in \mathscr{B}(\Omega)^N$ and $\alpha = (\alpha_i)_{1\leqslant i\leqslant N} \in \mathbb{R}^N$ s.t. $s_{A,\alpha} = \sum_i \alpha_i \mathbb{1}_{A_i} \geqslant \phi$. Define

$$\langle \mu, \phi \rangle = \int_\Omega \phi(x)\,d\mu(x) := \inf_{A,\alpha} \langle \mu, s_{A,\alpha} \rangle = \inf_{A,\alpha} \int_\Omega s_{A,\alpha}\,d\mu = \sum_{0\leqslant i\leqslant N} \alpha_i \mu(A_i).$$

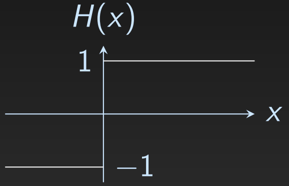
II)2) The Derivative of a measure: some derivatives

- If μ has density $g \in \mathscr{C}^1(\mathbb{R})$ then μ' has density g'

$$\langle \mu', \phi \rangle := -\langle \mu, \phi' \rangle = -\int_{\mathbb{R}} \phi' g = \int_{\mathbb{R}} \phi g' = \int_{\mathbb{R}} \phi(x)\,(g'(x)dx).$$

- ν with density $x \mapsto |x|$ has its derivative with density

$$H(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}.$$



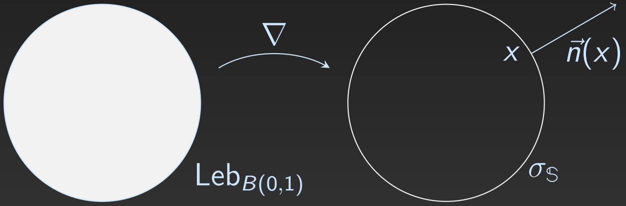
$\nu'' = 2\delta_0 : \phi \longmapsto 2\phi(0), \quad \nu^{(3)} = 2\delta_0' : \phi \longmapsto -2\phi'(0),$
and $\nu^{(k)} = 2\delta_0^{(k-2)} : \phi \longmapsto (-1)^{k-2}.2\phi^{(k-2)}(0) \quad \forall k > 2.$

II)2) The Derivative of a measure: some derivatives

- Name $B(0,1)$ the unit ball in $\mathbb{R}^3$. $\mathrm{Leb}_{B(0,1)}$ has for gradient $-\vec{n}\sigma_{\mathbb{S}}$ where $\sigma_{\mathbb{S}}$ the uniform measure on the sphere $\mathbb{S} = \partial B(0,1)$ and $\vec{n}$ is the unitary vector pointing outward from the sphere:

$$\langle \nabla \mathrm{Leb}_{B(0,1)}, \phi \rangle := -\langle \mathrm{Leb}_{B(0,1)}, \nabla.\phi \rangle = -\int_{B(0,1)} \nabla.\phi(x)\, dx$$

$$= -\int_{\mathbb{S}} \phi(x).\vec{n}(x)\, d\sigma_{\mathbb{S}}(x) = \langle -\vec{n}\sigma_{\mathbb{S}}, \phi \rangle.$$



II)3) The Bochner Integral

$\Omega \subset \mathbb{R}^d$.
$\langle \mu, \phi \rangle = \int_\Omega \phi$ has a sense when ϕ is nice enough: $0 \leqslant \int_\Omega |\phi| < +\infty$.
For B a Banach space,
$\langle \mu, \lambda \rangle$ has a sense when $\lambda : \Omega \to B$ is nice enough.
On $(\Gamma, \mathscr{B}(\Gamma))$
define $\mathscr{M}_+(\Gamma) = \{$finite (positive) measures on $\Gamma\}$ and
$\mathscr{M}(\Gamma) = \{$finite signed measures on $\Gamma\} = \{\mu_+ - \mu_-,\ \mu_+, \mu_- \in \mathscr{M}_+(\Gamma)\}$.
$(\mathscr{M}(\Gamma), \|\cdot\|)$ is a Banach for some norm $\|\cdot\|$
so if $\mu(K) < +\infty \quad \forall K \subset \Omega$ compact and $\lambda : \Omega \to \mathscr{M}(\Gamma)$ s.t.
$\int_\Omega \|\lambda(x)\| d\mu(x) < +\infty$ then $\nu = \langle \mu, \lambda \rangle = \int_\Omega \lambda(x) d\mu(x) \in \mathscr{M}(\Gamma)$

$$\forall A \in \mathscr{B}(\Gamma) \quad \nu(A) = \langle \mu, \lambda(\cdot)(A) \rangle = \int_\Omega \lambda(x)(A)\, d\mu(x).$$

II)3) The Bochner Integral: equations on measure-valued maps

The measure-valued mass evolution problem

$$\partial_t \mu(t) + \partial_x (v(x)\mu(t)) = f(x)\mu(t) \quad \text{on } [0,1]\ \forall t \in \mathbb{R}_+$$

where $\mu : \mathbb{R}_+ \to \mathscr{M}([0,1])$ and $\partial_x(v\mu(t))$ is in the sense of distributions
is equivalent to

$$\mu(t) = \mu(0)_* \Phi_t + \int_0^t (f\mu(s))_* \Phi_{(t-s)}\, ds \quad \text{on } [0,1]\ \forall t \in \mathbb{R}_+$$

where Φ is a (sort of) flow encoding the spatial evolution relative to x.

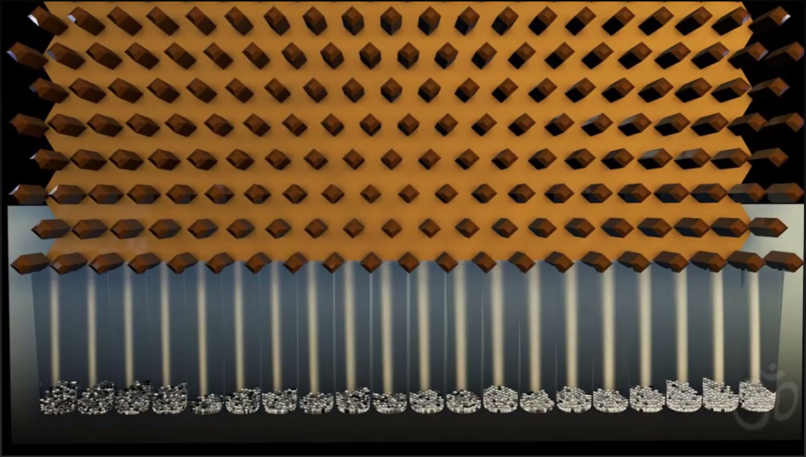
II)4) Limit processes through asymptotic scales

video made by PhysicsFun
(seen on Facebook)

X_n the position of a particle after n possible collisions at the times $t_n = \frac{1}{n}$, $2t_n$, ..., $nt_n = 1$.
$p(X_n = k) = \binom{n}{k} 2^{-n}$

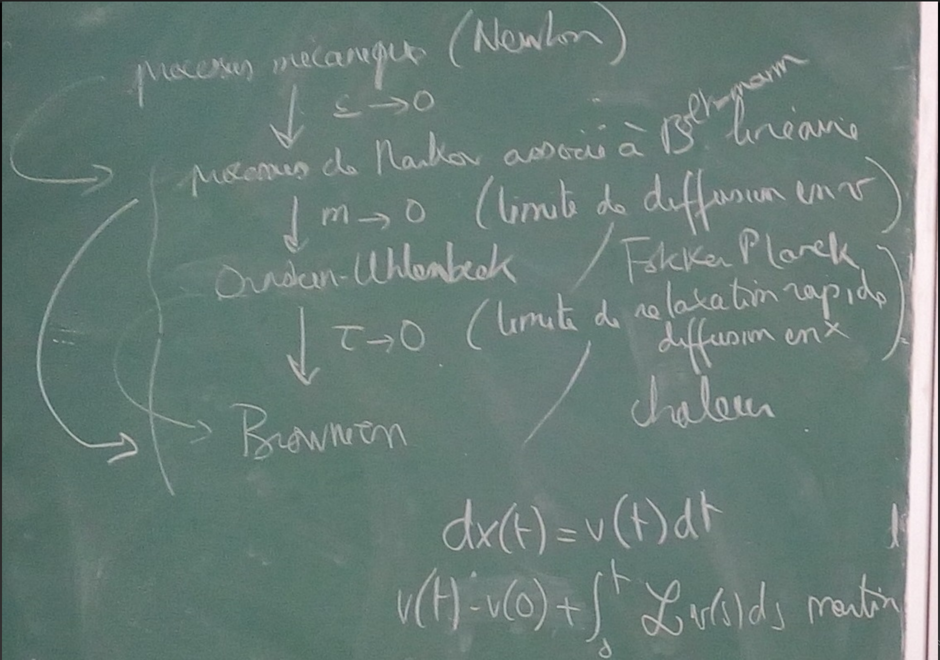
$$\frac{X_n - \mathbb{E}(X_n)}{\sqrt{\operatorname{Var}(X_n)}} \xrightarrow[n\to+\infty]{} \mathcal{N}(0,1)$$

considering $\frac{r_n}{t_n} = 1/4$ and $p_n = \frac{1}{2}$ constant.
What if $\frac{r_n}{t_n} \xrightarrow[n\to+\infty]{} 0$?



video made by 3D-PHASE (seen on Youtube)

p_n the probability of collision at each time t_n, ..., $nt_n = 1$ converges to 0 as n grows. There is no more convergence towards the normal law.


(Newton)
ε → 0
m → 0
Ornstein-Uhlenbeck
Fokker Planck
τ → 0
chaleur
Brownien
dx(t) = v(t) dt

Newton's laws: rectilinear movement or elastic chocs.
$t \mapsto (x_i(t), v_i(t))_{0\leqslant i\leqslant N}$ deterministic process.

$$\downarrow \quad Nr^2 \sim 1 \text{ as } N \to +\infty \text{ (Boltzmann-Grad)}$$

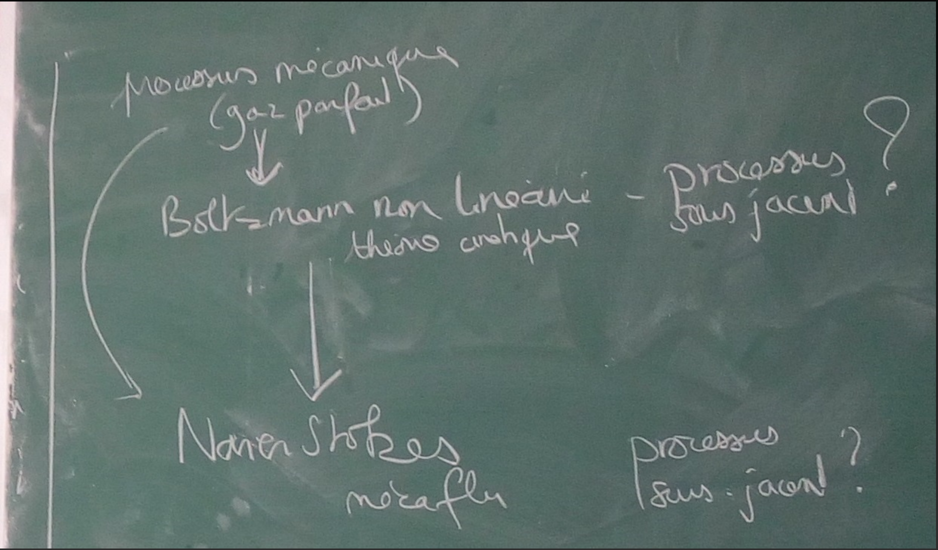
Boltzmann linear equation: $(\partial_t + v.\nabla_x)f = L(f)$
$t \mapsto (x(t), v(t))$ some Markov stochastic process.

$$\downarrow \quad m^4 \log(|\log r|) \gg 1 \text{ when } Nr^2 \sim 1 \text{ as } N \to +\infty, m \to 0$$

Fokker-Planck equation: $(\partial_t + v.\nabla_x)F = \frac{\sigma^2}{2}\Delta_v F + \nabla_v.(\omega v f)$
Ornstein-Uhlenbeck stochastic process $t \mapsto (x(t), v(t))$.

$$\downarrow \quad \frac{\text{collision term}}{\text{transprt term}} \to 0 \text{ as } \sigma, \omega \to 0$$

Diffusion equation: $(\partial_t - D\Delta_x)\rho = 0$
$\rho(t,x) = \mathbb{E}(\rho_0(x + W_{2Dt}))$ where $t \mapsto W_t$ is the standard brownian movement in $\mathbb{R}^3$. $t \mapsto x(t)$ stochastic process.


Processus mécanique
(gaz parfait)
Boltzmann non linéaire
théorie cinétique
processus sous-jacent ?
Navier Stokes
mécaflu
processus sous-jacent ?

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