Path-Complete Lyapunov Functions for Continuous-Time Switched Systems

Matteo Della Rossa, Mirko Pasquini, David Angeli⋆

LAAS – CNRS, Toulouse, France
Department of Electrical and Electronic Engineering, Imperial College, London

⋆(Eh ouais, le communitarisme au Imperial College c'est bien pire que au 93.)

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Overview

1. Preliminaries
2. Fixed-Time Stability Conditions
3. Dwell-Time Stability Conditions
4. Linear Sub-Dynamics and Example
5. Conclusions

Switching Systems

Consider $\mathcal{I} = \{1, \ldots, K\}$, and a family $\mathcal{F} = \{f_1, \ldots, f_K\} \subset \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ s.t.

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Given a $\sigma \in \mathcal{S}$ we finally have the (time-dependent) switched system

$$\dot{x}(t) = f_{\sigma(t)}(x(t)). \tag{Sw.Sys}$$

Basically, switching systems are a subclass of non-autonomous differential equations, piecewise constants w.r.t the time variable.

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Fixed-Time Switching Signals:

$$\mathcal{S}_{\text{fix}}(\tau) := \left\{ \sigma \in \mathcal{S} \mid \frac{t_i^\sigma - t_{i-1}^\sigma}{\tau} \in \mathbb{N}, \;\; \forall t_i^\sigma > 0 \right\}.$$

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Of course, $\forall \, \tau > 0$,

$$\mathcal{S}_{\text{fix}}(\tau) \subset \mathcal{S}_{\text{dw}}(\tau)$$

Stability Concepts

Stability w.r.t. a set of signals

Consider a set of switching signals $\widehat{\mathcal{S}} \subset \mathcal{S}$. The switched system (Sw.Sys) is said to be *uniformly globally asymptotically stable on* $\widehat{\mathcal{S}}$ (GAS), if there exists an $\beta \in \mathcal{KL}$ such that

$$|x(t, x_0, \sigma)| \leq \beta(|x_0|, t),$$

for all $\sigma \in \widehat{\mathcal{S}}$, for all $x_0 \in \mathbb{R}^n$ and for all $t \geq 0$.

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Notation: $x(t, x_0, \sigma)$ denotes the solution of (Sw.Sys) starting at $x_0 \in \mathbb{R}^n$, with respect to the signal $\sigma \in \mathcal{S}$, evaluated at some time $t \in \mathbb{R}$.

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Given a $\tau > 0$, we want to study stability with respect to $\mathcal{S}_{\text{dw}}(\tau)$ and $\mathcal{S}_{\text{fix}}(\tau)$ using a multiple Lyapunov construction based on graphs.

Path-Complete Graphs

Given a discrete alphabet $\mathcal{I} \subset \mathbb{N}$, a *direct and labeled graph* $\mathcal{G} = (S, E)$ is defined by a finite set S (the set of nodes) and $E \subset S \times S \times \mathcal{I}$ (the set of edges).

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Path-Completeness

A graph $\mathcal{G} = (S, E)$ is *path-complete* for $\mathcal{I}$ if, for any $K \geq 1$ and any "word" $j_1 \, \ldots \, j_K$, with $j_k \in \mathcal{I}$, there exists a *path* $\{(s_k, s_{k+1}, j_k)\}_{1 \leq k \leq K}$ such that $(s_k, s_{k+1}, j_k) \in E$, for each $1 \leq k \leq K$.

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Intuitively, a graph is path complete, if given any sequence in $\mathcal{I}^{\mathbb{N}}$, we can "reconstructing" it by "walking" thorough the (labeled) edges...let's see some pictures, it will be nicer...

Example of Path-Complete Graph

Alphabet $\mathcal{I} = \{1, 2\}$, $S = \{a, b\}$, $E = \{(a, a, 1), (a, b, 1), (b, b, 2), (b, a, 2)\}$,

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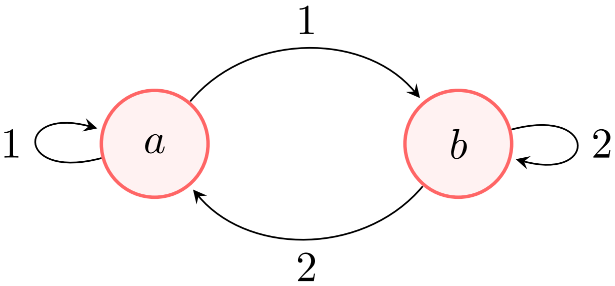
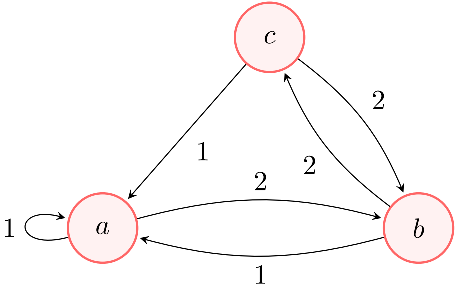


It is path complete! (You can trust me or you can try the infinite (but countable) sequence of 1 and 2.)

Example of Path-Complete Graph 2

Alphabet $\mathcal{I} = \{1, 2\}$, $S = \{a, b, c\}$,
$E = \{(a, a, 1), (a, b, 2), (b, b, 2), (b, a, 1), (b, c, 2), (c, b, 2), (c, a, 1)\}$,



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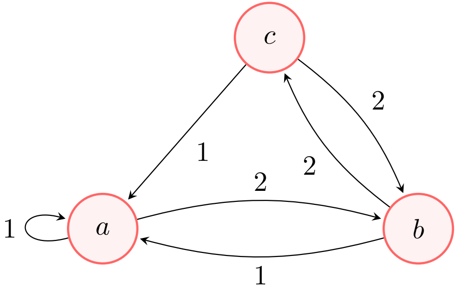

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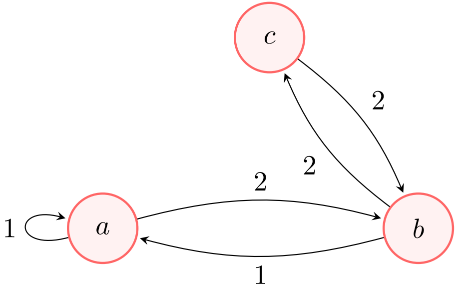

It is NOT path complete! Any word of the form (1,2,2,1,...) can not be reconstructed.

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No, I'm joking! But let's stop playing with graphs. :)

Fixed-Time Policy vs Discrete Time Switched System

Let us recall

$$\mathcal{S}_{\text{fix}}(\tau) := \left\{ \sigma \in \mathcal{S} \mid \frac{t_i^\sigma - t_{i-1}^\sigma}{\tau} \in \mathbb{N}, \ \ \forall t_i^\sigma > 0 \right\},$$

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Thus, stability of switched systems under fixed time is equivalent to stability of the discrete-time switched system

$$x^+ \in \mathsf{co}\left\{\phi_j(\tau, x) \mid j \in \mathcal{I}\right\},$$

where $\phi_j : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ denotes the flow map of the subsystem $\dot{x} = f_j(x)$.

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We can have the following adaptation of results concerning path-complete graph and discrete-time switched system.

Encoding inequalities in labeled and directed graphs

Candidate vector-valued Lyapunov function

Given a finite set S, a *candidate vector-valued Lyapunov function* is a map $V:\mathbb{R}^n \to \mathbb{R}^{|S|}$, such that, $\forall \ell \in S$, $V_\ell \in \mathcal{C}^1(\mathbb{R}^n,\mathbb{R})$ and $\exists\, \underline{\alpha}_\ell, \overline{\alpha}_\ell \in \mathcal{K}_\infty$ such that

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$$\underline{\alpha}_\ell(|x|)\le V_\ell(x)\le\overline{\alpha}_\ell(|x|),\ \ \forall\, x\in\mathbb{R}^n.$$

Given $\tau>0$ and $\mathcal{F}=\{f_j\}_{j\in\mathcal{I}}$, a candidate vector-valued Lyapunov function $V:\mathbb{R}^n\to\mathbb{R}^{|S|}$ and a $\rho\in\mathcal{PD}$; given $a,b\in S$ and $j\in\mathcal{I}$, we define a set of labeled edges "E" between nodes in S according to the rule

$$(a,b,j)_\tau\in E,\quad \textit{means}\quad V_b(\phi_j(\tau,x))-V_a(x)\le-\rho(|x|),\quad \forall x\in\mathbb{R}^n.$$

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Intuitevely: Fixing any $c > 0$ and considering the corresponding sublevel set of V_a defined by $L_a(c) := \{x \in \mathbb{R}^n \mid V_a(x) \leq c\}$, solutions of $\dot{x} = f_j(x)$ reach the sublevel set $L_b(c)$ of V_b by in time $\tau > 0$ (with a margin given by ρ).

Path-Complete Lyapunov Functions (Fixed Time)

Fixed-Time Lyapunov Direct Result

Consider a $\tau > 0$, $\mathcal{F} = \{f_j\}_{j \in \mathcal{I}} \subset \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ a function $\rho \in \mathcal{PD}$, a finite set S, and $V : \mathbb{R}^n \to \mathbb{R}^{|S|}$ a candidate vector-valued Lyapunov function. If the associated graph $\mathcal{G} = (S, E)$ is path-complete for $\mathcal{I}$ then switched system (Sw.Sys) is globally asymptotically stable on $\mathcal{S}_{\mathit{fix}}(\tau)$.

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Sketch of the proof: For any $\sigma \in \mathcal{S}_{\mathsf{fix}}(\tau)$, we "recursively" construct a continuous function $U : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ decreasing along solutions, "gluing" the node functions V_ℓ on intervals of lenght τ, following the "word" accociated to σ.

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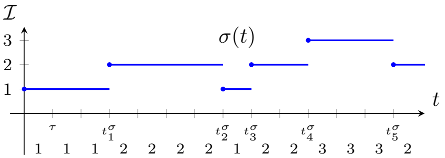

An example of switching signal $\sigma : \mathbb{R}_+ \to \mathcal{I} := \{1, 2, 3\}$, $\sigma \in \mathcal{S}_{\mathsf{fix}}(\tau)$ and the associated word, that is, the sequence $(1, 1, 1, 2, 2, 2, 2, 1, 2, 2, 3, 3, 3, 2, \dots) \in \mathcal{I}^{\mathbb{N}}$.

Relaxed Conditions: "Splitting Edges"

The strength of Lyapunov direct results lies in the fact that (asymptotic) stability is ensured *without* computing the solutions. On the other hand inequality encoded in a generic arch (a, b, j), depends on the solutions of $\dot{x} = f_j(x)$ at time τ.

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Consider $f_j \in \mathcal{F}$, $\tau > 0$ and $K \in \mathbb{N}$. Suppose there exist $V_0, \dots, V_K \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ positive definite ($V_0 \equiv V_a$, $V_K \equiv V_b$) and $\widetilde{\rho} \in \mathcal{PD}$ such that

$$\begin{cases} \nabla V_k(x) \cdot f_j(x) \quad + \frac{K(V_k(x) - V_{k-1}(x))}{\tau} \leq -\widetilde{\rho}(|x|), \;\; \forall\, x \in \mathbb{R}^n, \\ \nabla V_{k-1}(x) \cdot f_j(x) + \frac{K(V_k(x) - V_{k-1}(x))}{\tau} \leq -\widetilde{\rho}(|x|), \;\; \forall\, x \in \mathbb{R}^n. \end{cases}$$

This implies that there exists a $\rho \in \mathcal{PD}$ such that

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Roughly speaking, increasing $K \in \mathbb{N}$, i.e. the number of "auxiliary" functions between $V_a \equiv V_0$ and $V_b \equiv V_K$, we decrease the conservatism in proving $(a, b, j) \in E$

Sketch of the Proof

W.l.o.g. case $K = 1$ (no auxiliary functions between V_a and V_b). Define $W : [0,\tau] \times \mathbb{R}^n \to \mathbb{R}$ by

$$W(t,x) := \frac{\tau - t}{\tau} V_a(x) + \frac{t}{\tau} V_b(x) \qquad \forall t \in [0,\tau].$$

Computing the derivative of W along the solution $x(t) := \phi_j(t,x)$ we have

$$\begin{aligned}
\dot{W}(t,x(t)) &= \langle \frac{\partial W}{\partial x}(t,x(t)), f_j(x(t))\rangle + \frac{\partial W}{\partial t}(t,x(t)) \\
&= \frac{\tau - t}{\tau}\langle \nabla V_a(x(t)), f_j(x(t))\rangle + \frac{t}{\tau}\langle \nabla V_b(x(t)), f_j(x(t))\rangle + \frac{V_b(x(t)) - V_a(x(t))}{\tau} \\
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&= \frac{\tau - t}{\tau} \left(\langle \nabla V_a(x(t)), f_j(x(t)) \rangle + \frac{V_b(x(t)) - V_a(x(t))}{\tau} \right) \\
&\quad + \frac{t}{\tau} \left(\langle \nabla V_b(x(t)), f_j(x(t)) \rangle + \frac{V_b(x(t)) - V_a(x(t))}{\tau} \right) \leq -\widetilde{\rho}(|x(t)|).
\end{aligned}$$

Thus $W(\tau, x(\tau)) - W(0, x(0)) = V_b(x(\tau)) - V_a(x) \leq -\rho(|x|)$.

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Intuitively: Fixing any $c > 0$ and considering the corresponding sublevel set of V_a defined by $L_a(c) := \{x \in \mathbb{R}^n \mid V_a(x) \leq c\}$, solutions $\phi_j(\cdot, x)$ starting in $L_a(c)$ not only reach the sublevel set $L_b(c)$ in time τ, but also remain inside it for at least an interval of length τ.

Stability Result for Dwell-Time

Corollary

Consider a finite set S, and $V : \mathbb{R}^n \to \mathbb{R}^{|S|}$ a candidate vector-valued function. Consider a $\tau > 0$. Suppose the associated graph $\mathcal{G} = (S, E^{\mathsf{dw}})$ is path-complete for $\mathcal{I}$. Then system (Sw.Sys) is GAS on $\mathcal{S}_{dw}(\tau)$.

The proof follows similar ideas, just splitting each interval $[t_i^\sigma, t_{i+1}^\sigma]$, in $\underline{n}(i) - 1$ sub-intervals of length τ, and the last one of length in $[\tau, 2\tau)$. (Defining $\underline{n}(i) := \lfloor \frac{t_i^\sigma - t_{i-1}^\sigma}{\tau} \rfloor$) which is ≥ 1 by definition of $\mathcal{S}_{dw}(\tau)$.

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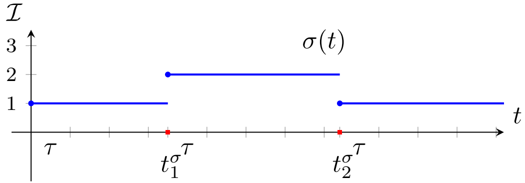

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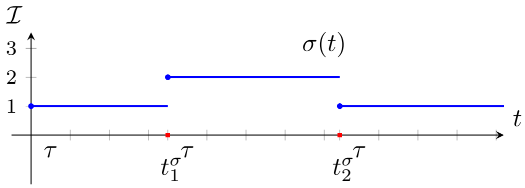

Then the construction of the decreasing $W : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ is the same.

Splitting Edges, Dwell-Time Case

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Splitting Edges

Consider $f_j \in \mathcal{F}$, $\tau > 0$ and $K \in \mathbb{N}$. Suppose there exist $V_0, \dots, V_K \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ ($V_0 \equiv V_a$, $V_K \equiv V_b$) and $\widetilde{\rho} \in \mathcal{PD}$ such that

$$\begin{cases} \nabla V_k(x) \cdot f_j(x) \quad + \frac{K(V_k(x)-V_{k-1}(x))}{s} \leq -\widetilde{\rho}(|x|), \ \ \forall\, x \in \mathbb{R}^n, \\ \nabla V_{k-1}(x) \cdot f_j(x) + \frac{K(V_k(x)-V_{k-1}(x))}{s} \leq -\widetilde{\rho}(|x|), \ \ \forall\, x \in \mathbb{R}^n, \end{cases}$$

for both $s=\tau$ and $s=2\tau$, and for all $k \in \{1, \dots, K\}$. This implies that $(a,b,j)^{\mathsf{dw}}_{\tau} \in E^{\mathsf{dw}}$.

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for both $s=\tau$ *and* $s=2\tau$, and for all $k \in \{1,\dots,K\}$. This implies that $(a,b,j)^{\mathsf{dw}}_\tau \in E^{\mathsf{dw}}$.

Basically, for each edge, we require to verify $4K$ inequalities involving gradients of some auxiliary functions $(V_0,\dots V_K \in \mathcal{C}^1(\mathbb{R}^n,\mathbb{R}))$. Main drawback: the number of inequalities increases rapidly as we increase the number of nodes and K, as required to reduce conservativeness.

Linear Switched Systems

Consider $\mathcal{A} = \{A_1, \dots, A_K\} \subset \mathbb{R}^{n \times n}$, we define the *linear* switched system, as

$$\dot{x}(t) = A_{\sigma(t)} x(t), \qquad \text{(Sw.Lin)}$$

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Given a set S, we consider functions $V : \mathbb{R}^n \to \mathbb{R}^{|S|}$ component-wise *quadratic*, that is

$$V_\ell(x) = x^\top P_\ell x, \quad \forall x \in \mathbb{R}^n,$$

where $P_\ell \in \mathbb{R}^{n\times n}$ are positive definite, for any $\ell \in S$.

Edges and LMIs (I underline LMIs, not BMIs :))

In this framework, once a $\tau > 0$ is fixed, the conditions encoded in edges are LMIs:

- The self loop $(a, a, j) \in E \Rightarrow P_a A_j + A_j^\top P_a < 0$ (already seen somewhere ?)
- Edge $(a, b, j) \in E \;\; \Rightarrow \;\; e^{A_j^\top \tau} P_b e^{A_j \tau} - P_a < 0$, and once we split "$K$-times": existence of $P_0, \dots, P_K > 0$, with $P_0 = P_a$, $P_K = P_b$ such that

$$\begin{cases} P_k A_j + A_j^\top P_k - \frac{K}{\tau}(P_k - P_{k-1}) < 0, \\ P_{k-1} A_j + A_j^\top P_{k-1} - \frac{K}{\tau}(P_k - P_{k-1}) < 0. \end{cases}$$

  for all $k \in \{1, \dots, K\}$.
- Dwell Time Edge $(a, b, j)^{\text{dw}} \in E^{\text{dw}} \Rightarrow$ once we split "K-times": existence of $P_0, \dots, P_K >$, with $P_0 = P_a$, $P_K = P_b$ such that

$$\begin{cases} P_k A_j + A_j^\top P_k & -\frac{K}{s}(P_k - P_{k-1}) < 0, \\ P_{k-1} A_j + A_j^\top P_{k-1} & -\frac{K}{s}(P_k - P_{k-1}) < 0. \end{cases}$$

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Example

Consider $\mathcal{A} = \{A_1,\ A_2\} \subset \mathbb{R}^{2\times 2}$, with:

$$A_1 = \begin{bmatrix} -18 & 17 \\ -9 & 8 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 13 & -79 \\ 4 & -20 \end{bmatrix}.$$

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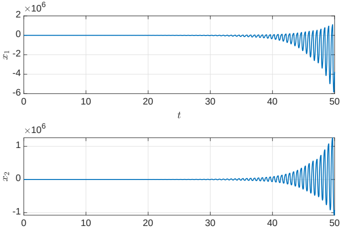
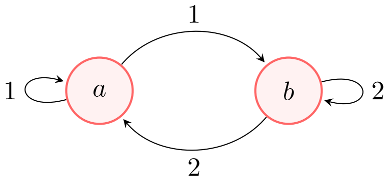
Using the switching signal $\sigma = \{1, 2, 1, 2, \dots\}$, with a fixed switching time $\tau = 0.3125$, the system diverges, implying that $\tau_{\text{dw}} > 0.3125$.

But A_1, A_2 are Hurwitz, we can estimate the minimal dwell time (that we now know bigger than 0.3125).



Extimation of τ_{dw}

Consider the (quite simple) path complete graph



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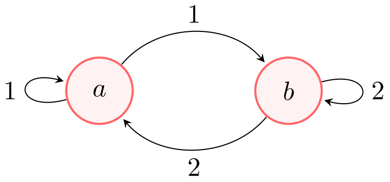

- Splitting 4 times the LMIs are feasible up to $\tau = 0.8$,
- Splitting 50 times the LMIs are feasible up to $\tau = 0.35$,
- Splitting 90 times the LMIs are feasible up to $\tau = 0.345$,
- So we know that $0.3125 < \tau_{\text{dw}} \leq 0.345$.

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Thank you !! Questions ??

Go Aneel, finally you can destroy me.

And, since a certain moment I've to shoot a movie about this for the CDC, feedbacks on the slides are well accepted (of course I will remove all the bêtises.)