Co-design of dynamic allocation functions and anti-windup

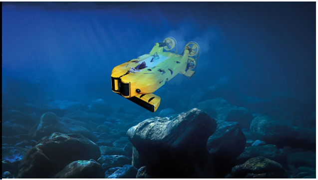
Thiago Lima, Sophie Tarbouriech

Seminaire des Doctorants

November 13, 2020

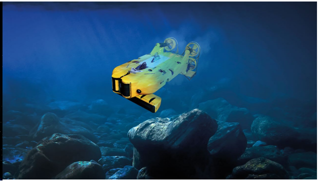
Overview

- Many applications as Aircrafts, Satellites and Underwater Vehicles.



- Many applications as Aircrafts, Satellites and Underwater Vehicles.

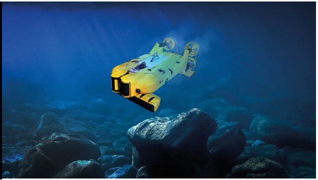




- Regulated plant generally contains torques/forces as inputs, also called virtual inputs.

- Many applications as Aircrafts, Satellites and Underwater Vehicles.





- Regulated plant generally contains torques/forces as inputs, also called virtual inputs.
- The virtual input is generated by a set of multiple actuators, for example microthrusters in space applications.

- Many applications as Aircrafts, Satellites and Underwater Vehicles.

- Regulated plant generally contains torques/forces as inputs, also called virtual inputs.
- The virtual input is generated by a set of multiple actuators, for example microthrusters in space applications.
- Each actuator might present constraints.

- Needs to search for an adequate algorithm that:

- Needs to search for an adequate algorithm that:
  - Is capable of generating (as much as possible) the computed control effort in view of the constraints.

- Needs to search for an adequate algorithm that:
  - Is capable of generating (as much as possible) the computed control effort in view of the constraints.
  - Takes advantage of redundancy to include additional optimization criteria.

- Needs to search for an adequate algorithm that:
  - Is capable of generating (as much as possible) the computed control effort in view of the constraints.
  - Takes advantage of redundancy to include additional optimization criteria.
  - Of course, guarantees stability and some level of performance.

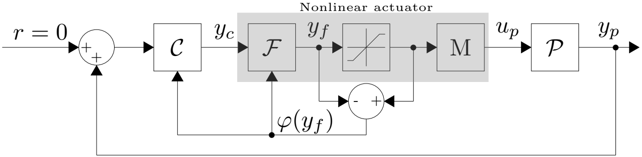


Figure: General view of control allocation problem.

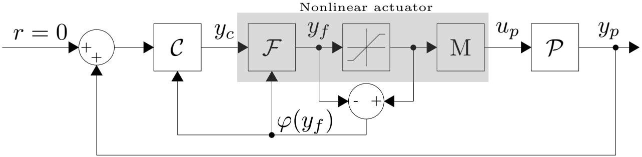


Figure: General view of control allocation problem.

- Subsystems $\mathcal{C}$, $\mathcal{F}$, and $\mathcal{P}$ are the controller, the control allocator, and the plant, respectively.

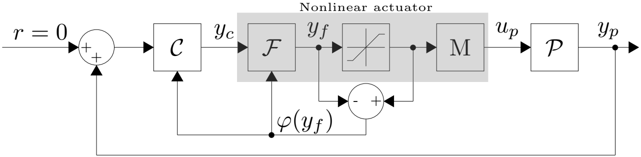


Figure: General view of control allocation problem.

- Subsystems $\mathcal{C}$, $\mathcal{F}$, and $\mathcal{P}$ are the controller, the control allocator, and the plant, respectively.
- The plant is driven by u_p in $\mathbb{R}^{m_c}$ inputs. The controller computes a set of desired y_c in $\mathbb{R}^{m_c}$ efforts to be injected in ideal conditions.

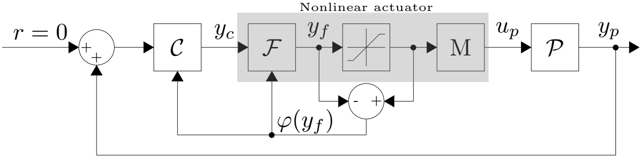


Figure: General view of control allocation problem.

- Subsystems $\mathcal{C}$, $\mathcal{F}$, and $\mathcal{P}$ are the controller, the control allocator, and the plant, respectively.
- The plant is driven by u_p in $\mathbb{R}^{m_c}$ inputs. The controller computes a set of desired y_c in $\mathbb{R}^{m_c}$ efforts to be injected in ideal conditions.
- $m_a \geq m_c$ actuators, represented by the signal y_f in $\mathbb{R}^{m_a}$.

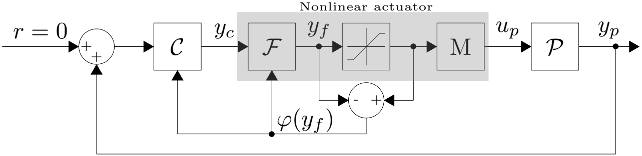


Figure: General view of control allocation problem.

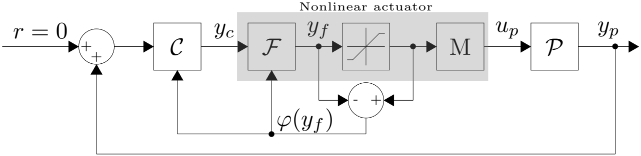


Figure: General view of control allocation problem.

- The plant input is given by $u_p = \mathrm{M}sat(y_f)$ with the decentralized saturation function being defined as

$$sat(y_{f(i)}) = sign(y_{f(i)}) \min\{|y_{f(i)}|, \bar{u}_{(i)}\}, \bar{u}_{(i)} > 0, \quad (1)$$

for $i = 1, \ldots, m_a$, where $\bar{u}_{(i)}$ denotes the amplitude bound in each actuator.

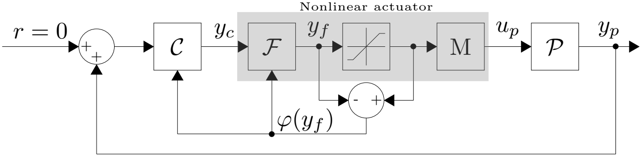


Figure: General view of control allocation problem.

- The plant input is given by $u_p = \mathrm{M}sat(y_f)$ with the decentralized saturation function being defined as

$$sat(y_{f(i)}) = sign(y_{f(i)}) \min\{|y_{f(i)}|, \bar{u}_{(i)}\}, \bar{u}_{(i)} > 0, \quad (1)$$

for $i = 1, \ldots, m_a$, where $\bar{u}_{(i)}$ denotes the amplitude bound in each actuator.

- The influence matrix M in $\mathbb{R}^{m_c \times m_a}$ maps how each individual effort of the m_a actuators combines to generate the inputs acting on the plant.

- The simplest allocation function often considered is given by:

$$\mathcal{F} = \mathrm{M}^{\dagger}, \text{ with } \mathrm{M}\mathrm{M}^{\dagger} = \mathrm{I} \tag{2}$$

- The simplest allocation function often considered is given by:

$$\mathcal{F} = \mathrm{M}^{\dagger}, \text{ with } \mathrm{M}\mathrm{M}^{\dagger} = \mathrm{I} \tag{2}$$

- No allocation error is produced when the system is not saturated due to $u_p = \mathrm{M}\mathrm{M}^{\dagger} y_c = y_c \implies e = u_p - y_c = 0$. However:

- The simplest allocation function often considered is given by:

$$\mathcal{F} = \mathrm{M}^{\dagger}, \text{ with } \mathrm{M}\mathrm{M}^{\dagger} = \mathrm{I} \tag{2}$$

- No allocation error is produced when the system is not saturated due to $u_p = \mathrm{M}\mathrm{M}^{\dagger} y_c = y_c \implies e = u_p - y_c = 0$. However:
  - Does not take advantage of the multi-actuated nature of the system to redistribute control effort among desired actuators.

- The simplest allocation function often considered is given by:

$$\mathcal{F} = \mathrm{M}^{\dagger}, \text{ with } \mathrm{M}\mathrm{M}^{\dagger} = \mathrm{I} \tag{2}$$

- No allocation error is produced when the system is not saturated due to $u_p = \mathrm{M}\mathrm{M}^{\dagger} y_c = y_c \implies e = u_p - y_c = 0$. However:
  - Does not take advantage of the multi-actuated nature of the system to redistribute control effort among desired actuators.
  - In the presence of nonlinearities as saturation, the produced error is no longer null and guarantees of stability of the closed loop, as well as estimation of regions of safe operation, need to be assured.

- The simplest allocation function often considered is given by:

$$\mathcal{F} = \mathrm{M}^{\dagger}, \text{ with } \mathrm{M}\mathrm{M}^{\dagger} = \mathrm{I} \tag{2}$$

- No allocation error is produced when the system is not saturated due to $u_p = \mathrm{M}\mathrm{M}^{\dagger} y_c = y_c \implies e = u_p - y_c = 0$. However:
  - Does not take advantage of the multi-actuated nature of the system to redistribute control effort among desired actuators.
  - In the presence of nonlinearities as saturation, the produced error is no longer null and guarantees of stability of the closed loop, as well as estimation of regions of safe operation, need to be assured.
- Therefore, more complex allocation functions with the ability to handle redundancy and constraints should be applied.

- The plant is described by

$$\mathcal{P} \sim \begin{cases} \dot{x}_p = \mathrm{A}_p x_p + \mathrm{B}_p u_p, \\ y_p = \mathrm{C}_p x_p, \end{cases} \tag{3}$$

where x_p in $\mathbb{R}^{n_p}$ is the plant state vector, u_p in $\mathbb{R}^{m_c}$ is the plant input, y_p in $\mathbb{R}^q$ is the measured output.

- The plant is described by

$$\mathcal{P} \sim \begin{cases} \dot{x}_p = \mathrm{A}_p x_p + \mathrm{B}_p u_p, \\ y_p = \mathrm{C}_p x_p, \end{cases} \tag{3}$$

where x_p in $\mathbb{R}^{n_p}$ is the plant state vector, u_p in $\mathbb{R}^{m_c}$ is the plant input, y_p in $\mathbb{R}^{q}$ is the measured output.

- The controller is described by

$$\mathcal{C} \sim \begin{cases} \dot{x}_c = \mathrm{A}_c x_c + \mathrm{B}_c y_p + v_{aw}, \\ y_c = \mathrm{C}_c x_c + \mathrm{D}_c y_p, \end{cases} \tag{4}$$

where x_c in $\mathbb{R}^{n_c}$ is the controller state vector and y_c in $\mathbb{R}^{m_c}$ is the controller output.

- The controller is linearly designed via the connection $u_p = y_c$, that is without taking into account the saturation and with $\mathcal{F} = \mathrm{M}^{\dagger}$.

- The controller is linearly designed via the connection $u_p = y_c$, that is without taking into account the saturation and with $\mathcal{F} = \mathrm{M}^{\dagger}$.

- The anti-windup compensation signal $v_{aw} = \mathrm{E}_c\varphi(y_f)$, E_c in $\mathbb{R}^{n_c \times m_a}$, is added in order to mitigate the undesired effects of saturation, with the deadzone $\varphi(y_f)$ defined as

$$\varphi(y_f) = sat(y_f) - y_f, \tag{5}$$

- The controller is linearly designed via the connection $u_p = y_c$, that is without taking into account the saturation and with $\mathcal{F} = \mathrm{M}^{\dagger}$.
- The anti-windup compensation signal $v_{aw} = \mathrm{E}_c \varphi(y_f)$, E_c in $\mathbb{R}^{n_c \times m_a}$, is added in order to mitigate the undesired effects of saturation, with the deadzone $\varphi(y_f)$ defined as

$$\varphi(y_f) = sat(y_f) - y_f, \tag{5}$$

Remark 1

By construction, the linear connection plant-controller is supposed to be stable. In other words, the controller (4) (with $v_{aw} = 0$) stabilizes the plant (3) through the linear interconnection $u_p = y_c$ and therefore the matrix

$$\mathrm{A}_0 = \begin{bmatrix} \mathrm{A}_p + \mathrm{B}_p \mathrm{D}_c \mathrm{C}_p & \mathrm{B}_p \mathrm{C}_c \\ \mathrm{B}_c \mathrm{C}_p & \mathrm{A}_c \end{bmatrix} \text{ in } \mathbb{R}^{(n_p+n_c)\times(n_p+n_c)} \tag{6}$$

is Hurwitz.

- Let N in $\mathbb{R}^{m_a \times n_f}$, $n_f = m_a - m_c$, be a basis for the Kernel of M, i.e. $\mathrm{MN} = 0$. We consider the following dynamic allocation function

$$\mathcal{F} \sim \begin{cases} \dot{x}_f = \mathrm{K}_f \mathrm{N}^\top \mathrm{W} \mathrm{N} x_f + \mathrm{K}_f \mathrm{N}^\top \mathrm{W} \mathrm{M}^\dagger y_c + \mathrm{E}_f \varphi(y_f), \\ y_f = \mathrm{N} x_f + \mathrm{M}^\dagger y_c, \end{cases} \tag{7}$$

where x_f in $\mathbb{R}^{n_f}$ is the allocator state vector, and y_f in $\mathbb{R}^{m_a}$ is the allocator output.

- Let N in $\mathbb{R}^{m_a \times n_f}$, $n_f = m_a - m_c$, be a basis for the Kernel of M, i.e. $\mathrm{MN} = 0$. We consider the following dynamic allocation function

$$\mathcal{F} \sim \begin{cases} \dot{x}_f = \mathrm{K}_f \mathrm{N}^\top \mathrm{W} \mathrm{N} x_f + \mathrm{K}_f \mathrm{N}^\top \mathrm{W} \mathrm{M}^\dagger y_c + \mathrm{E}_f \varphi(y_f), \\ y_f = \mathrm{N} x_f + \mathrm{M}^\dagger y_c, \end{cases} \tag{7}$$

where x_f in $\mathbb{R}^{n_f}$ is the allocator state vector, and y_f in $\mathbb{R}^{m_a}$ is the allocator output.

- W=diag$(w_1, w_2, \ldots, w_{m_a})$ in $\mathbb{S}^+_{m_a}$ is a matrix which receives the weightings that penalizes the use of each actuator.

- Let N in $\mathbb{R}^{m_a \times n_f}$, $n_f = m_a - m_c$, be a basis for the Kernel of M, i.e. $\mathrm{MN} = 0$. We consider the following dynamic allocation function

$$\mathcal{F} \sim \begin{cases} \dot{x}_f = \mathrm{K}_f \mathrm{N}^\top \mathrm{W} \mathrm{N} x_f + \mathrm{K}_f \mathrm{N}^\top \mathrm{W} \mathrm{M}^\dagger y_c + \mathrm{E}_f \varphi(y_f), \\ y_f = \mathrm{N} x_f + \mathrm{M}^\dagger y_c, \end{cases} \tag{7}$$

  where x_f in $\mathbb{R}^{n_f}$ is the allocator state vector, and y_f in $\mathbb{R}^{m_a}$ is the allocator output.

- W=diag$(w_1, w_2, \ldots, w_{m_a})$ in $\mathbb{S}^+_{m_a}$ is a matrix which receives the weightings that penalizes the use of each actuator.

- Matrices K_f in $\mathbb{R}^{n_f \times n_f}$ and E_f in $\mathbb{R}^{n_f \times m_a}$ must be designed to achieve desired behavior.

- This allocator is in some sense optimal in terms of both the allocation error and actuators usage, as explained in the next two remarks.

- This allocator is in some sense optimal in terms of both the allocation error and actuators usage, as explained in the next two remarks.

Remark 2

Consider the general expression $y_f = \mathrm{C}_f x_f + \mathrm{D}_f y_c$*, and let us define the allocator error as* $e = u_p - y_c$*. Then using the definition of* $\varphi(y_f)$ *in* (5)*, the expression* $e = (\mathrm{MD}_f - \mathrm{I})\, y_c + \mathrm{MC}_f x_f + \mathrm{M}\varphi(y_f)$ *is easily obtained. It is straightforward to see that the choice* $\mathrm{D}_f = \mathrm{M}^{\dagger}$, $\mathrm{C}_f = \mathrm{N}$ *leads to* $e = \mathrm{M}\varphi(y_f)$*, therefore the error is null in absence of saturation. Furthermore, by guaranteeing convergence of the extended vector* $x = \begin{bmatrix} x_p^\top & x_c^\top & x_f^\top \end{bmatrix}^\top$ *to the origin, we always obtain* $e^* = 0$*, where* e^* *is the steady-state value of* e*.*

- This allocator is in some sense optimal in terms of both the allocation error and actuators usage, as explained in the next two remarks.

Remark 3

Consider the cost function

$$\min_{x_f} \mathrm{T}(y_f) = y_f^\top \mathrm{W} y_f \text{ subject to } y_f = \mathrm{N}x_f + \mathrm{M}^\dagger y_c^*, \quad (8)$$

where y_c^* is any controller output. The optimal solution to (8) is given by $x_f = -(\mathrm{N}^\top \mathrm{W}^\top \mathrm{N})^{-1}\mathrm{N}^\top \mathrm{W}\mathrm{M}^\dagger y_c^*$, which corresponds to the steady-state value of x_f in (7).

- The allocator in this work generalizes the one from [1]:

[1] L. Zaccarian, "Dynamic allocation for input redundant control systems," *Automatica*, vol. 45, no. 6, pp. 1431 –1438, 2009, ISSN: 0005-1098.

- The allocator in this work generalizes the one from [1]:
  - Considers the case $m_a \geq m_c$ and influence matrix M.

[1] L. Zaccarian, "Dynamic allocation for input redundant control systems," *Automatica*, vol. 45, no. 6, pp. 1431 –1438, 2009, ISSN: 0005-1098.

- The allocator in this work generalizes the one from [1]:
  - Considers the case $m_a \geq m_c$ and influence matrix $\mathbb{M}$.
  - Adds anti-windup gains directly to the formulation.

[1] L. Zaccarian, "Dynamic allocation for input redundant control systems," *Automatica*, vol. 45, no. 6, pp. 1431 –1438, 2009, ISSN: 0005-1098.

- The allocator in this work generalizes the one from [1]:
  - Considers the case $m_a \geq m_c$ and influence matrix M.
  - Adds anti-windup gains directly to the formulation.
  - Methodology to do co-design via linear matrix inequalities.

[1] L. Zaccarian, "Dynamic allocation for input redundant control systems," *Automatica*, vol. 45, no. 6, pp. 1431 –1438, 2009, ISSN: 0005-1098.

- The allocator in this work generalizes the one from [1]:
  - Considers the case $m_a \geq m_c$ and influence matrix M.
  - Adds anti-windup gains directly to the formulation.
  - Methodology to do co-design via linear matrix inequalities.

Remark 4 (Case when $m_a = m_c$ and $\mathrm{M} = \mathrm{I}$)

In some papers the influence matrix M enters the plant model. In this case, $m_a = m_c$, the system has more inputs than states ($m_c > n_p$) and the input-redundancy nature of the plant is explicit. All the results in this paper can straightforwardly be applied in this case by making $\mathrm{M} = \mathrm{I}$ and choosing N as a base for the null space of B_p, that is, $\mathrm{B}_p\mathrm{N} = 0$.

[1] L. Zaccarian, "Dynamic allocation for input redundant control systems," *Automatica*, vol. 45, no. 6, pp. 1431 –1438, 2009, ISSN: 0005-1098.

- The complete closed-loop system with $x = \begin{bmatrix} x_p^\top & x_c^\top & x_f^\top \end{bmatrix}^\top$ in $\mathbb{R}^n$, $n = n_p + n_c + n_f$, can be written as

$$\begin{cases} \dot{x} = (\mathrm{A} + \mathrm{L}_f \mathrm{K}_f \bar{\mathrm{C}})x + (\mathrm{B} + \mathrm{LE})\varphi(y_f) \\ y_f = \mathrm{C}x \end{cases} \tag{9}$$

where

$$\mathrm{A} = \begin{bmatrix} \mathrm{A}_0 & 0 \\ 0 & 0 \end{bmatrix}, \mathrm{B} = \begin{bmatrix} \mathrm{B}_p\mathrm{M} \\ 0 \end{bmatrix}, \mathrm{E} = \begin{bmatrix} \mathrm{E}_c \\ \mathrm{E}_f \end{bmatrix}, \bar{\mathrm{C}} = \mathrm{N}^\top \mathrm{WC}$$

$$\mathrm{L} = \begin{bmatrix} 0_{n_p \times n_c} & \overbrace{0_{n_p \times n_f}}^{\mathrm{L}_f =} \\ \mathrm{I}_{n_c} & 0_{n_c \times n_f} \\ 0_{n_f \times n_c} & \mathrm{I}_{n_f} \end{bmatrix}, \mathrm{C} = \begin{bmatrix} \mathrm{M}^\dagger \mathrm{D}_c \mathrm{C}_p & \mathrm{M}^\dagger \mathrm{C}_c & \mathrm{N} \end{bmatrix},$$

with A_0 defined in (6).

- The presence of the deadzone in the closed-loop dynamics (9) implies to characterize a suitable region of the state space in which the stability is ensured.

- The presence of the deadzone in the closed-loop dynamics (9) implies to characterize a suitable region of the state space in which the stability is ensured.

- Then the problem we intend to solve can be summarized as follows.

- The presence of the deadzone in the closed-loop dynamics (9) implies to characterize a suitable region of the state space in which the stability is ensured.
- Then the problem we intend to solve can be summarized as follows.

Problem 1

Given the controller matrices A_c, B_c, C_c, D_c, *and the weighting matrix* W, *design matrices* K_f, E_f *and* E_c, *such that*

1. *the regional asymptotic stability of the closed-loop system (9) is ensured and the estimate of the region of attraction is maximized.*
2. *the total energy consumption of the actuators over time is minimized.*

- The following theorem provides a solution to Problem 1.

Theorem 1

Assume the existence of matrices $\bar{\mathrm{P}}$ *in* $\mathbb{S}_n^+$, J_o *in* $\mathbb{R}^{(n_p+n_c)\times(n_p+n_c)}$, J_f *in* $\mathbb{R}^{n_f\times n}$, $\bar{\mathrm{K}}_f$ *in* $\mathbb{R}^{n_f\times n_f}$, K_e *in* $\mathbb{R}^{(n_c+n_f)\times m_a}$, $\bar{\mathrm{G}}$ *in* $\mathbb{R}^{m_a\times n}$, *diagonal matrix* $\mathrm{S}=\mathrm{S}^\top$ *in* $\mathbb{S}_{m_a}^+$ *and positive scalar* γ *such that*

$$\Psi=\begin{bmatrix} -\bar{\mathrm{J}}-\bar{\mathrm{J}}^\top & \bar{\mathrm{P}}+\mathrm{A}\bar{\mathrm{J}}^\top+\mathrm{Z}-\bar{\mathrm{J}} & \Psi_{13} & 0 \\ \star & \mathrm{A}\bar{\mathrm{J}}^\top+\mathrm{Z}+\bar{\mathrm{J}}\mathrm{A}^\top+\mathrm{Z}^\top & \Psi_{23} & \bar{\mathrm{J}}\mathrm{C}^\top\mathrm{W}^{\frac{1}{2}} \\ \star & \star & -2\mathrm{S} & \mathrm{S}\mathrm{W}^{\frac{1}{2}} \\ \star & \star & \star & -\gamma\mathrm{I} \end{bmatrix}\prec 0,\ \begin{bmatrix} \bar{\mathrm{P}} & \bar{\mathrm{G}}_{(i)}^\top \\ \star & \bar{u}_{(i)}^2 \end{bmatrix}\succeq 0, \tag{10}$$

hold with $\Psi_{13}=\mathrm{BS}+\mathrm{LK}_e$, $\Psi_{23}=\Psi_{13}+\bar{\mathrm{G}}^\top-\bar{\mathrm{J}}\mathrm{C}^\top$ *and where* $\bar{\mathrm{J}}=\begin{bmatrix}\bar{\mathrm{C}}^\perp\mathrm{J}_o^\top & \mathrm{J}_f^\top\end{bmatrix}^\top$ *in* $\mathbb{R}^{n\times n}$, $\bar{\mathrm{C}}^\perp$ *in* $\mathbb{R}^{n\times(n_p+n_c)}$ *is a matrix such that* $\bar{\mathrm{C}}\bar{\mathrm{C}}^\perp=0$, *and* $\mathrm{Z}=$*diag*$(0_{n_p+n_c},\bar{\mathrm{K}}_f)$.

Theorem 1 - Continuation

Then, matrices $\mathrm{E} = \begin{bmatrix} \mathrm{E}_c^\top & \mathrm{E}_f^\top \end{bmatrix}^\top = \mathrm{K}_e \mathrm{S}^{-1}$, $\mathrm{K}_f = \bar{\mathrm{K}}_f \left(\bar{\mathrm{C}} \mathrm{J}_f^\top \right)^{-1}$ are solution to Problem 1. In other words:

1. the closed-loop system (9) is asymptotically stable in the ellipsoid $\varepsilon(\mathrm{P}, 1) = \{x \text{ in } \mathbb{R}^n; x^\top \mathrm{P} x \leq 1\}$, with $\mathrm{P} = \mathrm{J} \bar{\mathrm{P}} \mathrm{J}^\top$ and $\mathrm{J} = \bar{\mathrm{J}}^{-1}$;
2. the energy of the actuators usage signal is limited and given by $\int_0^\infty sat(y_f(\tau))^\top \mathrm{W} sat(y_f(\tau)) d\tau \leq \gamma$.

- The proof is based in the application of the following inequality:

$$\overbrace{\dot{\mathrm{V}}(x)}^{1} - \overbrace{2\varphi^{\top}(y_f)\mathrm{S}^{-1}[\varphi(y_f)+\theta]}^{2} + \overbrace{\gamma^{-1} sat(y_f)^{\top}\mathrm{W} sat(y_f)}^{3} < 0, \quad (11)$$

- The proof is based in the application of the following inequality:

$$\overbrace{\dot{\mathrm{V}}(x)}^{1} - \overbrace{2\varphi^\top(y_f)\mathrm{S}^{-1}[\varphi(y_f) + \theta]}^{2} + \overbrace{\gamma^{-1} sat(y_f)^\top \mathrm{W} sat(y_f)}^{3} < 0, \quad (11)$$

  - 1 comes from a quadratic Lyapunov function $\mathrm{V}(x) = x^\top \mathrm{P} x$, with $\mathrm{P} \succ 0$.

- The proof is based in the application of the following inequality:

$$\overbrace{\dot{\mathrm{V}}(x)}^{1} - \overbrace{2\varphi^\top(y_f)\mathrm{S}^{-1}[\varphi(y_f)+\theta]}^{2} + \overbrace{\gamma^{-1}sat(y_f)^\top \mathrm{W} sat(y_f)}^{3} < 0, \quad (11)$$

  - 1 comes from a quadratic Lyapunov function $\mathrm{V}(x) = x^\top \mathrm{P} x$, with $\mathrm{P} \succ 0$.

  - 2 comes from the application of the generalized sector condition, with $\theta = \mathrm{C}x - \mathrm{G}x$ and S a diagonal matrix in $\mathbb{S}_n^+$. The first item in Theorem 1 is guaranteed by (11) and a inclusion of the level set $\varepsilon(\mathrm{P},1) = \{x$ in $\mathbb{R}^n; x^\top \mathrm{P} x \leq 1\}$ in a set obtained from the application of the generalized sector condition.

- The proof is based in the application of the following inequality:

$$\overbrace{\dot{\mathrm{V}}(x)}^{1} - \overbrace{2\varphi^\top(y_f)\mathrm{S}^{-1}[\varphi(y_f) + \theta]}^{2} + \overbrace{\gamma^{-1} sat(y_f)^\top \mathrm{W} sat(y_f)}^{3} < 0, \quad (11)$$

  - 1 comes from a quadratic Lyapunov function $\mathrm{V}(x) = x^\top \mathrm{P} x$, with $\mathrm{P} \succ 0$.

  - 2 comes from the application of the generalized sector condition, with $\theta = \mathrm{C}x - \mathrm{G}x$ and S a diagonal matrix in $\mathbb{S}_n^+$. The first item in Theorem 1 is guaranteed by (11) and a inclusion of the level set $\varepsilon(\mathrm{P}, 1) = \{x$ in $\mathbb{R}^n; x^\top \mathrm{P} x \leq 1\}$ in a set obtained from the application of the generalized sector condition.

  - 3 is used to ensure some bound in the energy of signal $sat(y_f)$ and leads to the second item in Theorem 1.

- Similarly to the problem of SOF (static output feedback) design, the presence of the term $\bar{\mathrm{C}}$ in $\mathbb{R}^{n_f \times n}$ multiplying K_f in the closed loop (9) could complicate the gathering of convex conditions for the computation of the unknown variables.

[2] T. A. Lima, S. Tarbouriech, F. G. Nogueira, *et al.*, "Co-design of dynamic allocation functions and anti-windup," working paper or preprint, 2020.

- Similarly to the problem of SOF (static output feedback) design, the presence of the term $\bar{\mathrm{C}}$ in $\mathbb{R}^{n_f \times n}$ multiplying K_f in the closed loop (9) could complicate the gathering of convex conditions for the computation of the unknown variables.
- This problem was overcame by the application of Finsler's Lemma with a special format given to the Lagrange multipliers.

[2] T. A. Lima, S. Tarbouriech, F. G. Nogueira, *et al.*, "Co-design of dynamic allocation functions and anti-windup," working paper or preprint, 2020.

- Similarly to the problem of SOF (static output feedback) design, the presence of the term $\bar{\mathrm{C}}$ in $\mathbb{R}^{n_f \times n}$ multiplying K_f in the closed loop (9) could complicate the gathering of convex conditions for the computation of the unknown variables.

- This problem was overcame by the application of Finsler's Lemma with a special format given to the Lagrange multipliers.

  - Details on the proof can be found in [2].

[2] T. A. Lima, S. Tarbouriech, F. G. Nogueira, *et al.*, "Co-design of dynamic allocation functions and anti-windup," working paper or preprint, 2020.

Remark 5 (On the choice of matrix W)

It can be noted from Remark 3 and item ii) of Theorem 1 that the entries of the matrix W *are inversely proportional to the level of usage of the actuator. Although the user can specify any desired value* $w_i > 0$*, one promising choice in the case the level of saturation of the different actuators is different is to make* $w_i = \bar{u}_{(i)}^{-1}$.

Remark 5 (On the choice of matrix W)

It can be noted from Remark 3 and item ii) of Theorem 1 that the entries of the matrix W are inversely proportional to the level of usage of the actuator. Although the user can specify any desired value $w_i > 0$, one promising choice in the case the level of saturation of the different actuators is different is to make $w_i = \bar{u}_{(i)}^{-1}$.

Remark 6 (Global stability case)

In case the plant state matrix A_p is Hurwitz stable, global stability of the closed loop can be achieved and the design of K_f, E_f, E_c can also be realized by solving LMI (10) with $\overline{\mathrm{G}} = 0$.

- Minimization of γ leads to minimization of the energy of $sat(y_f(t))$.

- Minimization of γ leads to minimization of the energy of $sat(y_f(t))$.
- Minimization of the trace of P leads to maximization of the ellipsoid $\varepsilon(\mathrm{P}, 1)$.

- Minimization of γ leads to minimization of the energy of $sat(y_f(t))$.
- Minimization of the trace of P leads to maximization of the ellipsoid $\varepsilon(\mathrm{P}, 1)$.
- By considering an auxiliary matrix $\mathrm{P}_0 \succ 0$, and inequality

$$\begin{bmatrix} \mathrm{P}_0 & \mathrm{I} \\ \star & \bar{\mathrm{J}} + \bar{\mathrm{J}}^\top - \bar{\mathrm{P}} \end{bmatrix} \succeq 0, \qquad (12)$$

the following optimization problem can be formulated:

- Minimization of γ leads to minimization of the energy of $sat(y_f(t))$.
- Minimization of the trace of P leads to maximization of the ellipsoid $\varepsilon(\mathrm{P}, 1)$.
- By considering an auxiliary matrix $\mathrm{P}_0 \succ 0$, and inequality

$$\begin{bmatrix} \mathrm{P}_0 & \mathrm{I} \\ \star & \bar{\mathrm{J}} + \bar{\mathrm{J}}^\top - \bar{\mathrm{P}} \end{bmatrix} \succeq 0, \tag{12}$$

the following optimization problem can be formulated:

Optimization problem

Consider weighting parameters ρ_1, ρ_2. Then the following optimization procedure takes place in case of Theorem 1

$$\begin{aligned} &\min\ \rho_1\lambda + \rho_2\gamma \\ &\text{subject to (10), (12)}, \mathrm{P}_0 \preceq \lambda \mathrm{I} \end{aligned} \tag{13}$$

- Plant from [1], with saturation limits given by $\bar{u} = \begin{bmatrix} 1 & 0.01 & 0.02 \end{bmatrix}^\top$.

$$\left[\begin{array}{c|c} \mathrm{A}_p & \mathrm{B}_p \\ \hline \mathrm{C}_p & \mathrm{D}_p \end{array}\right] = \left[\begin{array}{cc|ccc} -0.157 & -0.094 & 0.87 & 0.253 & 0.743 \\ -0.416 & -0.45 & 0.39 & 0.354 & 0.65 \\ \hline 0 & 1 & 0 & 0 & 0 \end{array}\right].$$

[1] L. Zaccarian, "Dynamic allocation for input redundant control systems," *Automatica*, vol. 45, no. 6, pp. 1431 –1438, 2009, ISSN: 0005-1098.

- Plant from [1], with saturation limits given by $\bar{u} = \begin{bmatrix} 1 & 0.01 & 0.02 \end{bmatrix}^\top$.

$$\left[\begin{array}{c|c} \mathrm{A}_p & \mathrm{B}_p \\ \hline \mathrm{C}_p & \mathrm{D}_p \end{array}\right] = \left[\begin{array}{cc|ccc} -0.157 & -0.094 & 0.87 & 0.253 & 0.743 \\ -0.416 & -0.45 & 0.39 & 0.354 & 0.65 \\ \hline 0 & 1 & 0 & 0 & 0 \end{array}\right].$$

- [1] inserts an integrator and designs a stabilizing LQG controller which purposefully only uses the first two input channels. The resulting controller is given by

$$\left[\begin{array}{c|c} \mathrm{A}_c & \mathrm{B}_c \\ \hline \mathrm{C}_c & \mathrm{D}_c \end{array}\right] = \left[\begin{array}{cccc|c} -1.57 & 0.5767 & 0.822 & -0.65 & 0 \\ -0.9 & -0.501 & -0.94 & 0.802 & 0 \\ 0 & 1 & -1.61 & 1.614 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ \hline 1.81 & -1.2 & -0.46 & 0 & 0 \\ -0.62 & 1.47 & 0.89 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

[1] L. Zaccarian, "Dynamic allocation for input redundant control systems," *Automatica*, vol. 45, no. 6, pp. 1431 –1438, 2009, ISSN: 0005-1098.

- For this example, $m_a = m_c$ and $\mathrm{M} = \mathrm{I}$. We select then N as the Kernel of B_p, resulting in $\mathrm{N} = \begin{bmatrix} -0.4726 & -1.3143 & 1 \end{bmatrix}^\top$.

- For this example, $m_a = m_c$ and $\mathrm{M} = \mathrm{I}$. We select then N as the Kernel of B_p, resulting in $\mathrm{N} = \left[\begin{array}{ccc} -0.4726 & -1.3143 & 1 \end{array} \right]^\top$.

- The entries of matrix W are chosen as $w_i = \bar{u}_{(i)}^{-1}$.

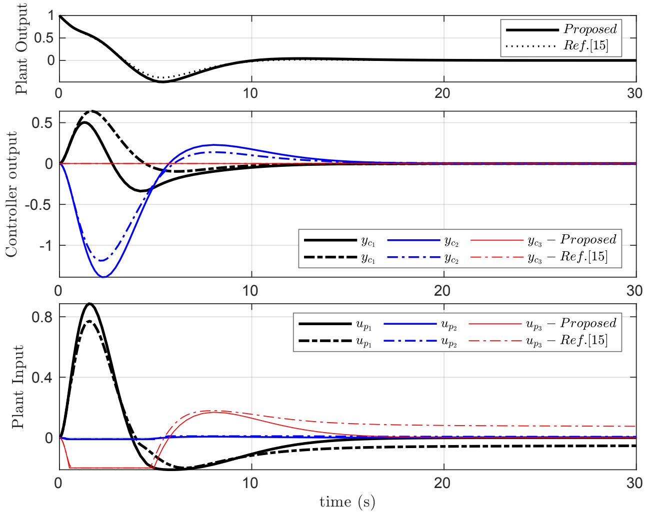
- For this example, $m_a = m_c$ and $\mathrm{M} = \mathrm{I}$. We select then N as the Kernel of B_p, resulting in $\mathrm{N} = \begin{bmatrix} -0.4726 & -1.3143 & 1 \end{bmatrix}^\top$.

- The entries of matrix W are chosen as $w_i = \bar{u}_{(i)}^{-1}$.

- By running the developed methods we obtain $\mathrm{K}_f = -2.4992$ and

$$\left[\begin{array}{c} \mathrm{E}_c \\ \hline \mathrm{E}_f \end{array}\right] = \left[\begin{array}{ccc} -0.8972 & -0.1642 & -0.7012 \\ -0.3176 & -0.3523 & -0.6356 \\ -0.5494 & 0.0361 & -0.0159 \\ -0.5607 & 0.2415 & 0.1140 \\ \hline -0.5958 & -0.0456 & -0.6322 \end{array}\right].$$

- We simulate the system response for an initial condition $x_p(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^\top$, with $x_c(0) = 0$ and $x_f(0) = 0$



- Satellite formation flying control problem from [3].

[3] J. Boada, C. Prieur, S. Tarbouriech, *et al.*, "Formation flying control for satellites: Anti-windup based approach," in *Modeling and Optimization in Space Engineering*, G. Fasano and J. D. Pintér, Eds., New York, NY: Springer New York, 2013, pp. 61–83, ISBN: 978-1-4614-4469-5.

- Satellite formation flying control problem from [3].
- y_p represents the relative position between two satellites in $z - axis$.

[3] J. Boada, C. Prieur, S. Tarbouriech, *et al.*, "Formation flying control for satellites: Anti-windup based approach," in *Modeling and Optimization in Space Engineering*, G. Fasano and J. D. Pintér, Eds., New York, NY: Springer New York, 2013, pp. 61–83, ISBN: 978-1-4614-4469-5.

- Satellite formation flying control problem from [3].
- y_p represents the relative position between two satellites in $z - axis$.
- The process can be represented by the following model

$$\left[\begin{array}{c|c} \mathrm{A}_p & \mathrm{B}_p \\ \hline \mathrm{C}_p & \mathrm{D}_p \end{array}\right] = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & m_1^{-1} & -m_2^{-1} \\ \hline 1 & 0 & 0 & 0 \end{array}\right],$$

[3] J. Boada, C. Prieur, S. Tarbouriech, *et al.*, "Formation flying control for satellites: Anti-windup based approach," in *Modeling and Optimization in Space Engineering*, G. Fasano and J. D. Pintér, Eds., New York, NY: Springer New York, 2013, pp. 61–83, ISBN: 978-1-4614-4469-5.

- Satellite formation flying control problem from [3].
- y_p represents the relative position between two satellites in $z - axis$.
- The process can be represented by the following model

$$\left[\begin{array}{c|c} \mathrm{A}_p & \mathrm{B}_p \\ \hline \mathrm{C}_p & \mathrm{D}_p \end{array}\right] = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & m_1^{-1} & -m_2^{-1} \\ \hline 1 & 0 & 0 & 0 \end{array}\right],$$

- Two forces act individually in each satellite, and are generate by a set of 8 thrusters.

[3] J. Boada, C. Prieur, S. Tarbouriech, *et al.*, "Formation flying control for satellites: Anti-windup based approach," in *Modeling and Optimization in Space Engineering*, G. Fasano and J. D. Pintér, Eds., New York, NY: Springer New York, 2013, pp. 61–83, ISBN: 978-1-4614-4469-5.

- The influence matrix is given by $\mathrm{M} = \begin{bmatrix} \mathrm{M}_1 & 0 \\ 0 & \mathrm{M}_2 \end{bmatrix}$, with $\mathrm{M}_1 = \mathrm{M}_2 = \begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix}$

- The influence matrix is given by $\mathrm{M} = \begin{bmatrix} \mathrm{M}_1 & 0 \\ 0 & \mathrm{M}_2 \end{bmatrix}$, with $\mathrm{M}_1 = \mathrm{M}_2 = \begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix}$
- $\bar{u}_i = 50\ mN, i = 1, \ldots, 8.$

- The influence matrix is given by $\mathrm{M} = \begin{bmatrix} \mathrm{M}_1 & 0 \\ 0 & \mathrm{M}_2 \end{bmatrix}$, with $\mathrm{M}_1 = \mathrm{M}_2 = \begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix}$
- $\bar{u}_i = 50\ mN, i = 1, \ldots, 8.$
- After choosing $m_1 = m_2 = 1000\ kg$, a stabilizing LQG controller is designed.

$$\left[\begin{array}{c|c} \mathrm{A}_c & \mathrm{B}_c \\ \hline \mathrm{C}_c & \mathrm{D}_c \end{array}\right] = \left[\begin{array}{cc|c} -1.7321 & 1 & 1.7321 \\ -1.0014 & -0.0532 & 1 \\ \hline -0.7071 & -26.6009 & 0 \\ 0.7071 & 26.6009 & 0 \end{array}\right].$$

- We then compute $\mathrm{M}^{\dagger}=0.25\text{diag}(\mathrm{M}_1^{\top}, \mathrm{M}_2^{\top})$, $\mathrm{N} = \text{diag}(\mathrm{N}_1, \mathrm{N}_2)$, with $\mathrm{N}_1=\mathrm{N}_2=\begin{bmatrix} 1 & 1 & -1 \\ & \mathrm{I}_3 & \end{bmatrix}$

- We then compute $\mathrm{M}^{\dagger}=0.25\text{diag}(\mathrm{M}_1^\top, \mathrm{M}_2^\top)$, $\mathrm{N} = \text{diag}(\mathrm{N}_1, \mathrm{N}_2)$, with $\mathrm{N}_1=\mathrm{N}_2=\begin{bmatrix} 1 & 1 & -1 \\ & \mathrm{I}_3 & \end{bmatrix}$
- We choose $\mathrm{W}=\text{diag}(100, 1, \ldots, 1)$, that is we want to penalize the use of the first actuator.

- We then compute $\mathrm{M}^{\dagger}=0.25\mathrm{diag}(\mathrm{M}_1^{\top},\mathrm{M}_2^{\top})$, $\mathrm{N} = \mathrm{diag}(\mathrm{N}_1,\mathrm{N}_2)$, with $\mathrm{N}_1=\mathrm{N}_2=\begin{bmatrix} 1 & 1 & -1 \\ & \mathrm{I}_3 & \end{bmatrix}$

- We choose $\mathrm{W}=\mathrm{diag}(100, 1, \ldots, 1)$, that is we want to penalize the use of the first actuator.

- Using optimization procedure (13) with weights $\rho_1 = 1, \rho_2 = 0.15$, we obtain

$$\mathrm{K}_f = \begin{bmatrix} -1.1684 & 0.6813 & -0.4766 & 0.0034 & 0.0034 & -0.0034 \\ 0.7282 & -1.0438 & -0.3054 & 0.0249 & 0.0249 & -0.0249 \\ -0.4528 & -0.3418 & -0.8017 & 0.0284 & 0.0284 & -0.0284 \\ -0.0200 & 0.0792 & 0.0584 & -0.8628 & 0.1381 & -0.1381 \\ -0.0200 & 0.0792 & 0.0584 & 0.1381 & -0.8628 & -0.1381 \\ 0.0200 & -0.0792 & -0.0584 & -0.1381 & -0.1381 & -0.8628 \end{bmatrix}, \tag{14}$$

$$
\left[\begin{array}{c} \mathrm{E}_c \\ \hline \mathrm{E}_f \end{array}\right] =
$$

$$
\left[\begin{array}{cccccccc}
0.0019 & -0.0000 & 0.0394 & -0.0193 & 0.0325 & -0.0411 & -0.0411 & 0.0411 \\
-0.0002 & -0.0047 & 0.0142 & -0.0043 & 0.0118 & -0.0160 & -0.0160 & 0.0160 \\
\hline
1.2781 & 0.0144 & 0.1663 & -0.0741 & 0.1006 & 0.1297 & 0.1297 & -0.1297 \\
-0.6243 & -0.0044 & -0.0738 & 0.3141 & 0.2881 & -0.0918 & -0.0918 & 0.0918 \\
0.7725 & 0.0088 & 0.0736 & 0.2114 & 0.3749 & 0.0720 & 0.0720 & -0.0720 \\
-0.9763 & -0.0119 & 0.0949 & -0.0674 & 0.0721 & 0.9519 & -0.3357 & 0.3357 \\
-0.9763 & -0.0119 & 0.0949 & -0.0674 & 0.0721 & -0.3357 & 0.9519 & 0.3357 \\
0.9763 & 0.0119 & -0.0949 & 0.0674 & -0.0721 & 0.3357 & 0.3357 & 0.9519
\end{array}\right] \tag{15}
$$

- Simulation for $x_p(0) = \begin{bmatrix} -0.25 & 0 \end{bmatrix}^\top$, with $x_c(0) = 0$ and $x_f(0) = 0$.

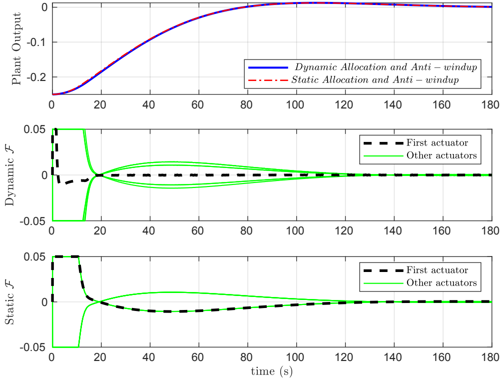


Figure: Example 2: Output and actuators .

- Simulation for $x_p(0) = \begin{bmatrix} -0.25 & 0 \end{bmatrix}^\top$, with $x_c(0) = 0$ and $x_f(0) = 0$.

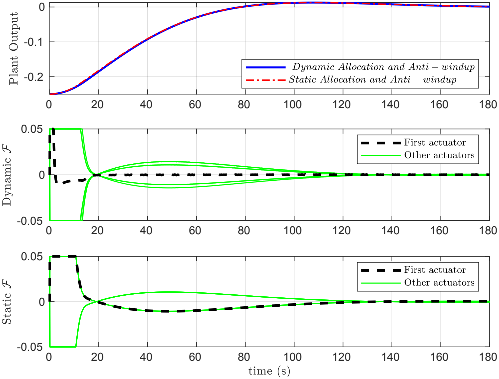


Figure: Example 2: Output and actuators .

- Both strategies stabilize the system, however the dynamic allocation successfully reduces the usage of the penalized actuator.

Contributions

- Co-design of dynamic allocation functions along with anti-windup.
- The Allocation+AW problem is solved simultaneously, unlike previous formulations.
- Introduction of influence matrix M to the dynamic allocator formulation from [1], allowing to deal with broader range of cases.
- Guaranteed convergence of the allocator error to zero, avoiding waste of energy.

[1] L. Zaccarian, "Dynamic allocation for input redundant control systems," *Automatica*, vol. 45, no. 6, pp. 1431 –1438, 2009, ISSN: 0005-1098.

Future Research

- Consideration of other nonlinearities.
- The case of event-triggered control.

Summary of LAAS séjour

Allocation

- "Energy based design of dynamic allocation in the presence of saturating actuators," *Accepted for Proceedings of the 24th International Symposium on MTNS, August 2021*.
- "Co-design of dynamic allocation functions and anti-windup," *preprint submitted to IEEE CSS Letters*.

Time delays

- "Analysis and experimental application of a dead-time compensator for input saturated processes with output time-varying delays," *Accepted for publication at IET Control Theory and Applications*.
- "New predictor-based stabilization for systems with time-varying delays,", preprint under work.

Merci beaucoup !