Locally Lipschitz Lyapunov Functions for Switching Differential Inclusions

Matteo Della Rossa, Aneel Tanwani and Luca Zaccarian

LAAS – CNRS, Toulouse, France

Séminaires des doctorants MAC, Troisième Épisode (28 Mars 2019)
Un grand merci au comité d'organisation = Matteo Tacchi

Introduction

Main Target

Given a state-dependent switching system, we want to provide sufficient conditions for the asymptotic stability, via locally Lipschitz (and in particular max-min) Lyapunov functions.

Introduction

Main Target

Given a state-dependent switching system, we want to provide sufficient conditions for the asymptotic stability, via locally Lipschitz (and in particular max-min) Lyapunov functions.

- Reference Paper: *Max-Min Lyapunov Functions for Switching Differential Inclusion*. Matteo Della Rossa, Aneel Tanwani, Luca Zaccarian, 57th IEEE-Conference on Decision and Control (CDC 2018), Dec 2018, Miami, United States.
- Extended version: *Max-Min Lyapunov Functions for Switched Systems and the Related Differential Inclusions*. Submitted for publication.
- If you don't want to read, for the next 18 months I'm in the bureau E47.

Systems Class

Let us consider $\mathcal{F} = \{f_1, \dots, f_K\} \subset \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ and a switching signal $\sigma : \mathbb{R}^n \to \{1, \dots, K\}$.

State Dependent Switching System

$$\dot{x} = f_{\sigma(x)}(x)$$

Systems Class

Let us consider
$\mathcal{F} = \{f_1, \ldots, f_K\} \subset \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ and a switching signal $\sigma : \mathbb{R}^n \to \{1, \ldots, K\}$.

State Dependent Switching System

$$\dot{x} = f_{\sigma(x)}(x)$$

Remarks:

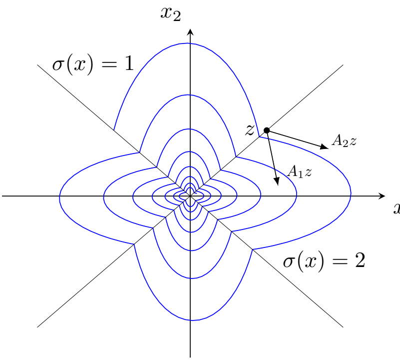
- Discontinuous differential equations, *but*
- $\mathcal{C}^1$ in the regions where the signal σ is constant.

Systems Class

Let us consider $\mathcal{F} = \{f_1, \ldots, f_K\} \subset \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ and a switching signal $\sigma : \mathbb{R}^n \to \{1, \ldots, K\}$.

State Dependent Switching System

$$\dot{x} = f_{\sigma(x)}(x)$$



Remarks:

- Discontinuous differential equations, *but*
- $\mathcal{C}^1$ in the regions where the signal σ is constant.

"Flower" example: Linear switching system, but non convex trajectories/reachable sets. Existence and uniqueness of solution. Existence/Uniqueness in the general case ?

Existence and Uniqueness

We *do not* have existence/uniqueness of Caratheodory solutions in general.

Existence and Uniqueness

We *do not* have existence/uniqueness of Caratheodory solutions in general.

Non existence

$$\dot{x}(t) = \begin{cases} 1, & \text{if } x \leq 0, \\ -1, & \text{if } x > 0. \end{cases}$$

No solution starting at $x_0 = 0$.

Existence and Uniqueness

We *do not* have existence/uniqueness of Caratheodory solutions in general.

Non existence

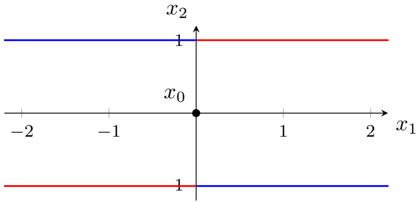
$$\dot{x}(t) = \begin{cases} 1, & \text{if } x \leq 0, \\ -1, & \text{if } x > 0. \end{cases}$$

No solution starting at $x_0 = 0$.

Non uniqueness

$$\dot{x}(t) = \begin{cases} -1, & \text{if } x \leq 0, \\ 1, & \text{if } x > 0. \end{cases}$$

Two solutions starting at $x_0 = 0$.



Filippov Regularization

We need a proper regularization of the system, in order to ensure existence.

Filippov Regularization

We need a proper regularization of the system, in order to ensure existence.

Filippov Regularization of Switching Systems

$$\dot{x} \in F^{\mathrm{sw}}(x) := \overline{\mathrm{co}}\{f_i(x)\,|\,i \in J(x)\}$$

where $J(x) := \{j\,|\,\forall\,\varepsilon > 0, \exists\,\xi \in \mathbb{B}_\varepsilon(x)$ s.t. $\sigma(\xi) = j\}$.

Filippov Regularization

We need a proper regularization of the system, in order to ensure existence.

Filippov Regularization of Switching Systems

$$\dot{x} \in F^{\text{sw}}(x) := \overline{\text{co}}\{f_i(x) \,|\, i \in J(x)\}$$

where $J(x) := \{j \,|\, \forall\, \varepsilon > 0, \exists\, \xi \in \mathbb{B}_\varepsilon(x) \text{ s.t. } \sigma(\xi) = j\}$.

Hypothesis on the switching signal:
$\sigma(x) = j$ on a set D_j, $j = 1, \dots, K$, such that

$$D_j := \{x \in \mathbb{R}^n \,|\, S_j(x) > 0; S_j : \mathbb{R}^n \to \mathbb{R} \text{ is analytic}\} \wedge \text{connected},$$

$$\bigcup_j \overline{D_j} = \mathbb{R}^n, \quad \text{and} \quad D_i \cap D_j = \emptyset,\ i \neq j.$$

Filippov Regularization

We need a proper regularization of the system, in order to ensure existence.

Filippov Regularization of Switching Systems

$$\dot{x} \in F^{\mathrm{sw}}(x) := \overline{\mathrm{co}}\{f_i(x) \,|\, i \in J(x)\}$$

where $J(x) := \{j \,|\, \forall\, \varepsilon > 0, \exists\, \xi \in \mathbb{B}_\varepsilon(x) \text{ s.t. } \sigma(\xi) = j\}$.

Hypothesis on the switching signal:
$\sigma(x) = j$ on a set D_j, $j = 1, \ldots, K$, such that

$$D_j := \{x \in \mathbb{R}^n \,|\, S_j(x) > 0; S_j : \mathbb{R}^n \to \mathbb{R} \text{ is analytic}\} \wedge \text{connected},$$

$$\bigcup_j \overline{D_j} = \mathbb{R}^n, \quad \text{and} \quad D_i \cap D_j = \emptyset,\ i \neq j.$$

The mapping $F^{\mathrm{sw}} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, the Filippov/Krasovskii regularization, is upper semicontinuous, with non empty, compact and convex values.

$\Rightarrow$ Existence of complete solutions **but**
no uniqueness nor continuity w.r.t initial conditions.

Why non-smooth Lyapunov Functions?

We want to study the stabilty of switching system using non-smooth Lyapunov functions.

Why non-smooth Lyapunov Functions?

We want to study the stabilty of switching system using non-smooth Lyapunov functions.
Under our conditions the existence of a smooth Lyapunov functions is **necessary and sufficient** for asymptotic stability of a compact set.

Why non-smooth Lyapunov Functions?

We want to study the stabilty of switching system using non-smooth Lyapunov functions.
Under our conditions the existence of a smooth Lyapunov functions is **necessary and sufficient** for asymptotic stability of a compact set.
On the other hand, in many situations:

- A nonsmooth function V may be easier to describe and construct;

Why non-smooth Lyapunov Functions?

We want to study the stabilty of switching system using non-smooth Lyapunov functions.

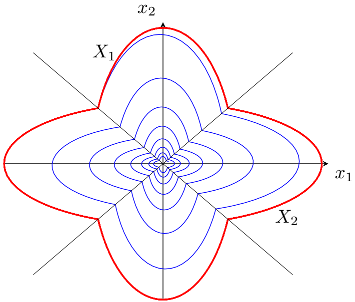
Under our conditions the existence of a smooth Lyapunov functions is **necessary and sufficient** for asymptotic stability of a compact set.

On the other hand, in many situations:

- A nonsmooth function V may be easier to describe and construct;
- The piecewise structure "fits well" with our problem;

Example/Spoiler: It will be easy to "imagine" (construct) a non-smooth Lyapunov function for the flower system. (In red the level set).

A smooth one exists but it is not so easy to construct.



Notions of Derivative

Problem: Non smooth function ⇒ The gradient is not defined!

Notions of Derivative

Problem: Non smooth function $\Rightarrow$ The gradient is not defined!
Let us consider a locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}$,

Notions of Derivative

Problem: Non smooth function $\Rightarrow$ The gradient is not defined!
Let us consider a locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}$,

- V is almost everywhere differentiable (Rademacher Theorem);

Notions of Derivative

Problem: Non smooth function $\Rightarrow$ The gradient is not defined!
Let us consider a locally Lipschitz function $V:\mathbb{R}^n \to \mathbb{R}$,

- V is almost everywhere differentiable (Rademacher Theorem);
- **Clarke's gradient:**

$$\partial V(x) = \overline{\text{co}}\left\{\lim_{k\to\infty} \nabla V(x_k)\,|\,x_k \to x,\, x_k \notin \mathcal{N}\cup\mathcal{S}\right\}$$

- if V is differentiable at x, $\partial V(x) = \{\nabla V(x)\}$,

Notions of Derivative

Problem: Non smooth function $\Rightarrow$ The gradient is not defined!
Let us consider a locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}$,

- V is almost everywhere differentiable (Rademacher Theorem);
- **Clarke's gradient:**

$$\partial V(x) = \overline{\text{co}} \left\{ \lim_{k\to\infty} \nabla V(x_k) \,|\, x_k \to x,\, x_k \notin \mathcal{N} \cup \mathcal{S} \right\}$$

- if V is differentiable at x, $\partial V(x) = \{\nabla V(x)\}$,
- **Directional Derivative:** Given $x \in \mathbb{R}^n$ and a direction $w \in \mathbb{R}^n$,

$$V'(x,w) \in \{\langle p, w\rangle \mid p \in \partial V(x)\}$$

Derivative with respect to a differential inclusion

Let us consider a differential inclusion $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$,
Clarke's derivative: $\dot{V}_F(x) := \{\langle p, f\rangle \mid p \in \partial V(x),\ f \in F(x)\,\}$.
Lie's derivative: $\dot{\overline{V}}_F(x) := \{a \in \mathbb{R} \,|\, \exists f \in F(x) : a = \langle p, f\rangle, \forall p \in \partial V(x)\}$.

They are compact intervals.

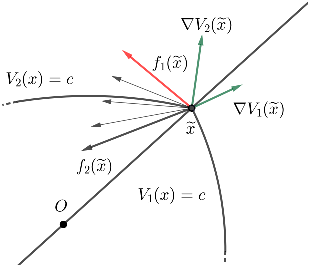
In general: $\dot{\overline{V}}_F(x) \subset \dot{V}_F(x)$

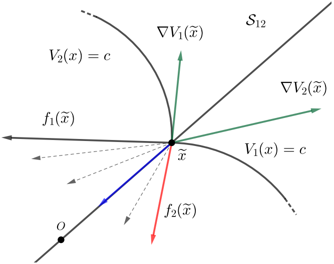
Derivative with respect to a differential inclusion

Let us consider a differential inclusion $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$,
Clarke's derivative: $\dot{V}_F(x) := \{\langle p, f\rangle \mid p \in \partial V(x),\ f \in F(x)\}$.
Lie's derivative: $\dot{\overline{V}}_F(x) := \{a \in \mathbb{R} \,|\, \exists f \in F(x) : a = \langle p, f\rangle, \forall p \in \partial V(x)\}$.

They are compact intervals.

In general: $\dot{\overline{V}}_F(x) \subset \dot{V}_F(x)$





Stability Conditions

Theorem (Lie derivative based condition)

Given a locally Lipschitz and regular function $V : \mathbb{R}^n \to \mathbb{R}$. If $\exists$ a class $\mathcal{K}$ function γ, and a scalar $\delta > 0$, such that, for every x with $|x| < \delta$,

$$\max \overline{\dot{V}}_F(x) \leq -\gamma(|x|),$$

then the origin of $\dot{x} \in F(x)$ is (locally) asymp. stable.

Proof Sketch:

- Main Step: It holds for almost every $t \geq 0$ that

$$\frac{d}{dt} V(\varphi(t)) \in \overline{\dot{V}}_F(\varphi(t))$$

- By assumption, $\overline{\dot{V}}_F(\varphi(t)) \leq -\gamma(|\varphi(t)|)$, and one can derive local and global versions.

Function Class

[Angeli-Philippe-Athanasopoulos-Jungers '17]

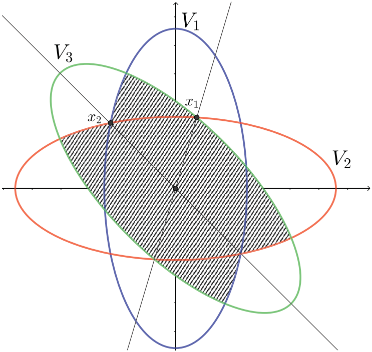
Max-Min Function

Given $V_i \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$, $i = 1, \dots, K$, let

$$V_{\mathbf{Mm}}(x) := \max_{j \in \{1,\dots,J\}} \left\{ \min_{k \in S_j} \{V_k(x)\} \right\},$$

where $S_j \subset \{1, \dots, K\}$.

- (Possibly) non-convex level sets
- Locally Lipschitz, and hence differentiable <u>almost</u> everywhere



"Active Index Set" for V is $\alpha_V : \mathbb{R}^n \rightrightarrows \{1, \cdots, K\}$,

$$\alpha_V(x) := \left\{ \begin{array}{c} \ell \mid \forall \text{ neighborhood } \mathcal{U} \text{ of } x, \exists \text{ an open } \mathcal{V} \subset \mathcal{U} \\ \text{s.t. } V(z) = V_\ell(z), \forall z \in \mathcal{V} \end{array} \right\}$$

Max-Min and Level Sets

Given $V : \mathbb{R}^n \to \mathbb{R}$ and $c \in \mathbb{R}$ we define the level set

$$\mathcal{E}_V(c) := \{x \in \mathbb{R}^n : V(x) \leq c\}.$$

Max Function:

$V_{\mathtt{M}}(x) := \max\{V_1(x), \dots V_K(x)\} \Rightarrow \mathcal{E}_{V_{\mathtt{M}}}(c) = \mathcal{E}_{V_1}(c) \cap \dots \cap \mathcal{E}_{V_K}(c).$

Min Function:

$V_{\mathtt{m}}(x) := \min\{V_1(x), \dots V_K(x)\} \Rightarrow \mathcal{E}_{V_{\mathtt{m}}}(c) = \mathcal{E}_{V_1}(c) \cup \dots \cup \mathcal{E}_{V_K}(c).$

Max-Min and Level Sets

Given $V:\mathbb{R}^n\to\mathbb{R}$ and $c\in\mathbb{R}$ we define the level set

$$\mathcal{E}_V(c):=\{x\in\mathbb{R}^n : V(x)\le c\}.$$

Max Function:
$V_{\mathtt{M}}(x):=\max\{V_1(x),\dots V_K(x)\}\Rightarrow \mathcal{E}_{V_{\mathtt{M}}}(c)=\mathcal{E}_{V_1}(c)\cap\dots\cap\mathcal{E}_{V_K}(c).$
Min Function:
$V_{\mathtt{m}}(x):=\min\{V_1(x),\dots V_K(x)\}\Rightarrow \mathcal{E}_{V_{\mathtt{m}}}(c)=\mathcal{E}_{V_1}(c)\cup\dots\cup\mathcal{E}_{V_K}(c).$
Max-Min Function:

$$V_{\mathbf{Mm}}(x):=\max_{j\in\{1,\dots,J\}}\left\{\min_{k\in S_j}\{V_k(x)\}\right\}\Rightarrow\ \mathcal{E}_{V_{\mathbf{Mm}}}(c)=\bigcap_{j=1}^{J}\Big(\bigcup_{k\in S_j}\mathcal{E}_{V_k}(c)\Big).$$

Stability conditions for Max-Min Functions

Given $V \in \mathbf{Mm}(V_1, \dots, V_K)$, the following equality holds

$$\partial V(x) = \overline{\text{co}}\{\nabla V_\ell(x) \,|\, \ell \in \alpha_V(x)\}. \tag{1}$$

In particular, given $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, we have

$$\dot{\overline{V}}_F(x) = \{a \in \mathbb{R} \,|\, \exists f \in F(x) : a = \langle \nabla V_\ell(x), f\rangle, \forall \ell \in \alpha_V(x)\}. \tag{2}$$

Corollary (Lie conditions for Max-Min Functions)

Given K positive-definite functions $V_1, \dots, V_K \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$, let $V \in \mathbf{Mm}(V_1, \dots, V_K)$. If $\exists$ a class $\mathcal{K}$ function γ, and a scalar $\delta > 0$, such that, for every x with $|x| < \delta$,

$$\max \dot{\overline{V}}_F(x) \leq -\gamma(|x|),$$

then the origin of $\dot{x} \in F(x)$ is (locally) asymp. stable.

Geometric Interpretation of $\dot{\overline{V}}_F(x)$

Consider $V \in \mathbf{Mm}(V_1, \dots, V_K)$ and an $x \in \mathbb{R}^n$ such that $\alpha_V(x) = \{\ell_1, \dots, \ell_p\}$. We have $x \in \mathcal{S} := \{y \in \mathbb{R}^n \mid V_{\ell_1}(y) = \cdots = V_{\ell_p}(y)\}$, a discontinuity surface of the function V.

Geometric Interpretation of $\dot{\overline{V}}_F(x)$

Consider $V \in \mathbf{Mm}(V_1, \dots, V_K)$ and an $x \in \mathbb{R}^n$ such that $\alpha_V(x) = \{\ell_1, \dots, \ell_p\}$. We have $x \in \mathcal{S} := \{y \in \mathbb{R}^n \,|\, V_{\ell_1}(y) = \cdots = V_{\ell_p}(y)\}$, a discontinuity surface of the function V.

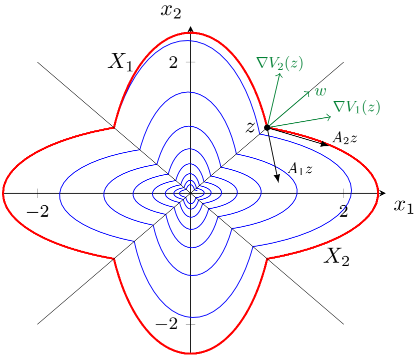
Geometry of $\dot{\overline{V}}_F(x)$

If x is a regular point of the discontinuity surface $\mathcal{S}$, then

$$\dot{\overline{V}}_F(x) = \{\langle \nabla V_\ell(x), f \rangle \,|\, \forall \ell \in \alpha_V(x),\ \forall f \in T_{\mathcal{S}}(x) \cap F(x)\}.$$

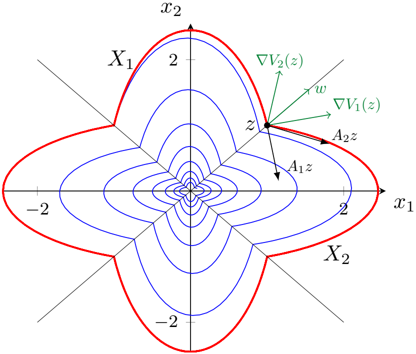
If the discontinuity surfaces of V and f_σ coincide, $f \in T_{\mathcal{S}}(x)$ represent a sliding direction.

Revisiting the Flower Example



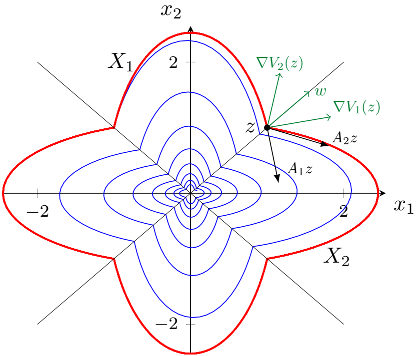
- System class: $\dot{x} = A_{\sigma(x)}(x)$, with $\sigma : \mathbb{R}^2 \to \{1, 2\}$

Revisiting the Flower Example



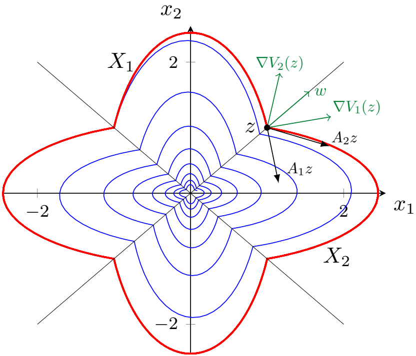
- System class: $\dot{x} = A_{\sigma(x)}(x)$, with $\sigma : \mathbb{R}^2 \to \{1, 2\}$
- $\sigma(x) = i$ on X_i, a cone.

Revisiting the Flower Example

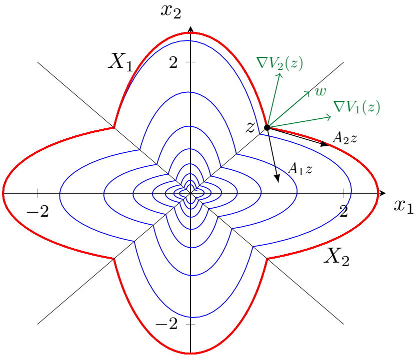


- System class: $\dot{x} = A_{\sigma(x)}(x)$, with $\sigma : \mathbb{R}^2 \to \{1, 2\}$
- $\sigma(x) = i$ on X_i, a cone.
- Lyapunov function: $x \mapsto V(x) := \min\{x^\top P_1 x, x^\top P_2 x\}$,

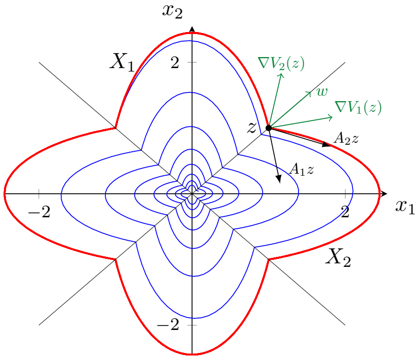
Proving Stability



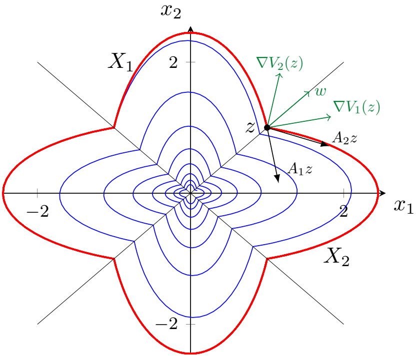
Proving Stability



Proving Stability

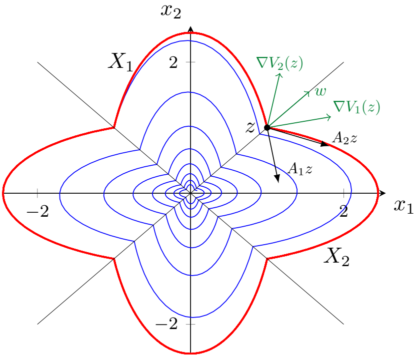


Proving Stability



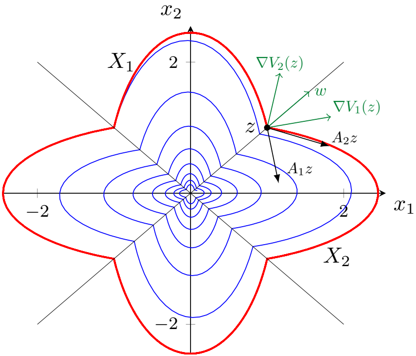
- $P_i A_i + A_i^\top P_i < 0$, for $i \in \{1, 2\}$ that implies $\langle \nabla V(x), A_i x \rangle < 0$ for all $x \in \text{int}(X_i)$,

Proving Stability



- $P_i A_i + A_i^\top P_i < 0$, for $i \in \{1,2\}$ that implies $\langle \nabla V(x), A_i x \rangle < 0$ for all $x \in \text{int}(X_i)$,
- In the discontinuity point we have $\overline{\dot{V}}_F(x) = \emptyset$, nothing has to be checked.

Proving Stability



- $P_i A_i + A_i^\top P_i < 0$, for $i \in \{1, 2\}$ that implies $\langle \nabla V(x), A_i x \rangle < 0$ for all $x \in \text{int}(X_i)$,
- In the discontinuity point we have $\dot{\overline{V}}_F(x) = \emptyset$, nothing has to be checked.
- The Clarke's conditions are not satisfied.

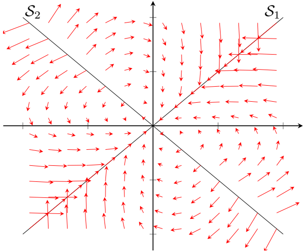
A Nonlinear Example, Sliding Modes

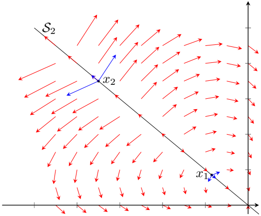
$$\dot{x} = \begin{cases} f_1(x) := A_1 x + b\widetilde{g}(x) & \text{if } x^\top Q x < 0, \\ f_2(x) := A_2 x + b\widetilde{g}(x) & \text{if } x^\top Q x > 0, \end{cases}$$

Stability analysis:

Step 1: For some P_1, P_2, $P_2 - P_1 = 4Q$,

$$V(x) = \min\{x^\top P_1 x, x^\top P_2 x\}.$$





A Nonlinear Example, Sliding Modes

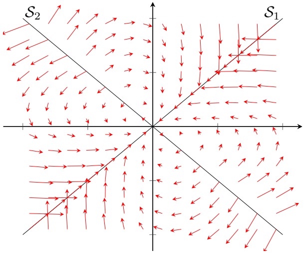
$$\dot{x} = \begin{cases} f_1(x) := A_1 x + b\widetilde{g}(x) & \text{if } x^\top Q x < 0, \\ f_2(x) := A_2 x + b\widetilde{g}(x) & \text{if } x^\top Q x > 0, \end{cases}$$

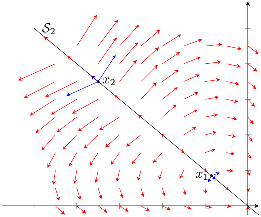
Stability analysis:

Step 1: For some P_1, P_2, $P_2 - P_1 = 4Q$,

$$V(x) = \min\{x^\top P_1 x, x^\top P_2 x\}.$$

Step 2: Stability condition in the interior of the domain





A Nonlinear Example, Sliding Modes

$$\dot{x} = \begin{cases} f_1(x) := A_1 x + b\widetilde{g}(x) & \text{if } x^\top Q x < 0, \\ f_2(x) := A_2 x + b\widetilde{g}(x) & \text{if } x^\top Q x > 0, \end{cases}$$

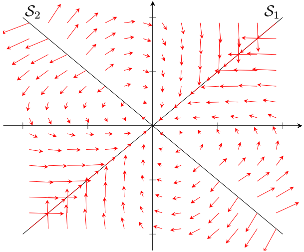
Stability analysis:

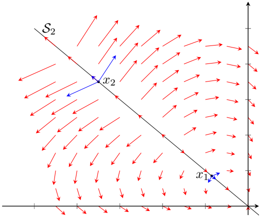
Step 1: For some P_1, P_2, $P_2 - P_1 = 4Q$,

$$V(x) = \min\{x^\top P_1 x, x^\top P_2 x\}.$$

Step 2: Stability condition in the interior of the domain

Step 3: Analyze the surface $\mathcal{S}_1$ by finding the right convex combination





A Nonlinear Example, Sliding Modes

$$\dot{x} = \begin{cases} f_1(x) := A_1 x + b\widetilde{g}(x) & \text{if } x^\top Q x < 0, \\ f_2(x) := A_2 x + b\widetilde{g}(x) & \text{if } x^\top Q x > 0, \end{cases}$$

Stability analysis:

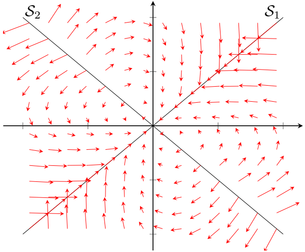
Step 1: For some P_1, P_2, $P_2 - P_1 = 4Q$,

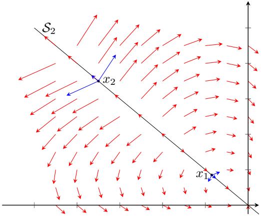
$$V(x) = \min\{x^\top P_1 x, x^\top P_2 x\}.$$

Step 2: Stability condition in the interior of the domain

Step 3: Analyze the surface $\mathcal{S}_1$ by finding the right convex combination

Step 4: Analyze the surface $\mathcal{S}_2$ and analyze dominant terms near origin





Linear Case with Arbitrary Switching

System dynamics (LDI): $\dot{x} \in \overline{\text{co}}\{A_i x \mid i \in \{1, \dots, M\}\}$
Base functions: K quadratic forms $P_1, \dots, P_K$
Ordering: For $\rho = (j_1, \dots, j_K) \in \mathbb{S}_K$ (symmetric group of order K)

$$C_\rho := \left\{x \in \mathbb{R}^n \mid x^\top P_{j_1} x < \cdots < x^\top P_{j_K} x\right\},$$

Linear Case with Arbitrary Switching

System dynamics (LDI): $\dot{x} \in \overline{\mathrm{co}}\{A_i x \mid i \in \{1, \dots, M\}\}$
Base functions: K quadratic forms $P_1, \dots, P_K$
Ordering: For $\rho = (j_1, \dots, j_K) \in \mathbb{S}_K$ (symmetric group of order K)

$$C_\rho := \left\{x \in \mathbb{R}^n \mid x^\top P_{j_1} x < \cdots < x^\top P_{j_K} x\right\},$$

Corollary (Sufficient Condition using Clarke's gradient)

Let $V \in \mathbf{Mm}\{V_1, \dots, V_K\}$. *If, for each* $i \in \{1, \dots, M\}$, *and for each* $\rho = (j_1, \dots, j_K) \in \mathbb{S}_K$, *there exist* $\tau_{j_1}, \dots, \tau_{j_{K-1}} \geq 0$ *such that*

$$A_i^\top P_\ell + P_\ell A_i + \sum_{k=1}^{K-1} \tau_{j_k}(P_{j_{k+1}} - P_{j_k}) < 0, \quad \ell = \alpha_V(C_\rho),$$

then V *is a radially unbounded Lyapunov function and (LDI) is GAS.*

Linear Case with Arbitrary Switching

System dynamics (LDI): $\dot{x} \in \overline{\text{co}}\{A_i x \mid i \in \{1, \ldots, M\}\}$
Base functions: K quadratic forms $P_1, \ldots, P_K$
Ordering: For $\rho = (j_1, \ldots, j_K) \in \mathbb{S}_K$ (symmetric group of order K)

$$C_\rho := \left\{x \in \mathbb{R}^n \mid x^\top P_{j_1} x < \cdots < x^\top P_{j_K} x\right\},$$

Corollary (Sufficient Condition using Clarke's gradient)

Let $V \in \mathbf{Mm}\{V_1, \ldots, V_K\}$. If, for each $i \in \{1, \ldots, M\}$, and for each $\rho = (j_1, \ldots, j_K) \in \mathbb{S}_K$, there exist $\tau_{j_1}, \ldots, \tau_{j_{K-1}} \geq 0$ such that

$$A_i^\top P_\ell + P_\ell A_i + \sum_{k=1}^{K-1} \tau_{j_k}(P_{j_{k+1}} - P_{j_k}) < 0, \quad \ell = \alpha_V(C_\rho),$$

then V is a radially unbounded Lyapunov function and (LDI) is GAS.

- Since $|\mathbb{S}_K| = K!$ finding a max-min V requires solving $M \cdot K!$ inequalities, which involve $M(K-1)K!$ non-negative scalars and K symmetric positive-definite matrices.

A Case-Study with Three Quadratics

Base functions: $\{x^\top P_1 x, x^\top P_2 x, x^\top P_3 x\}$

- Common Lyapunov function: $V = \max\{\min\{P_i\}\}$;
- Min of 3 quadratics: $V = \max\{\min\{P_1, P_2, P_3\}\}$;
- Max of 3 quadratics: $V = \max\{\min\{P_1\}, \min\{P_2\}, \min\{P_3\}\}$;
- Quasi-max functions: $V = \max\{\min\{P_1\}, \min\{P_2, P_3\}\}$;
- Quasi-min functions: $V = \max\{\min\{P_1, P_3\}, \min\{P_2, P_3\}\}$;
- Mid-of-quadratics: $V = \max\{\min\{P_1, P_2\}, \min\{P_2, P_3\}\min\{P_1, P_3\}\}$.

A Case-Study with Three Quadratics

Base functions: $\{x^\top P_1 x, x^\top P_2 x, x^\top P_3 x\}$

BMIs for Quasi-Max Function

$$V = \max\{\min\{P_1\}, \min\{P_2, P_3\}\}$$

For each $i \in \{1, \dots, M\}$, find the scalars $\tau_{12}, \tau_{13}, \tau_{21}, \tau_{23}, \tau_{31}, \tau_{32} \geq 0$, and $\tilde{\tau}_{12}, \tilde{\tau}_{13}, \tilde{\tau}_{21}, \tilde{\tau}_{23}, \tilde{\tau}_{31}, \tilde{\tau}_{32} \geq 0$ such that

- $A_i^\top P_2 + P_2 A_i + \tau_{21}(P_2 - P_1) + \tau_{32}(P_3 - P_2) < 0$, over C_{123}
- $A_i^\top P_3 + P_3 A_i + \tau_{31}(P_3 - P_1) + \tau_{23}(P_2 - P_3) < 0$, over C_{132}
- $A_i^\top P_1 + P_1 A_i + \tilde{\tau}_{31}(P_1 - P_3) + \tilde{\tau}_{21}(P_2 - P_1) < 0$, over C_{312}
- $A_i^\top P_1 + P_1 A_i + \tau_{12}(P_1 - P_2) + \tilde{\tau}_{31}(P_3 - P_1) < 0$, over C_{213}
- $A_i^\top P_1 + P_1 A_i + \tilde{\tau}_{32}(P_3 - P_2) + \tau_{13}(P_1 - P_3) < 0$, over C_{231}
- $A_i^\top P_1 + P_1 A_i + \tilde{\tau}_{23}(P_2 - P_3) + \tilde{\tau}_{12}(P_1 - P_2) < 0$, over C_{321}.

A Case-Study with Three Quadratics

Base functions: $\{x^\top P_1 x, x^\top P_2 x, x^\top P_3 x\}$

BMIs for Quasi-Max Function

$$V = \max\{\min\{P_1\}, \min\{P_2, P_3\}\}$$

For each $i \in \{1, \ldots, M\}$, find the scalars $\tau_{12}, \tau_{13}, \tau_{21}, \tau_{23}, \tau_{31}, \tau_{32} \geq 0$, and $\tilde{\tau}_{12}, \tilde{\tau}_{13}, \tilde{\tau}_{21}, \tilde{\tau}_{23}, \tilde{\tau}_{31}, \tilde{\tau}_{32} \geq 0$ such that

- $A_i^\top P_2 + P_2 A_i + \tau_{21}(P_2 - P_1) + \tau_{32}(P_3 - P_2) < 0$, over C_{123}
- $A_i^\top P_3 + P_3 A_i + \tau_{31}(P_3 - P_1) + \tau_{23}(P_2 - P_3) < 0$, over C_{132}
- $A_i^\top P_1 + P_1 A_i + \tilde{\tau}_{31}(P_1 - P_3) + \tilde{\tau}_{21}(P_2 - P_1) < 0$, over C_{312}
- $A_i^\top P_1 + P_1 A_i + \tau_{12}(P_1 - P_2) + \tilde{\tau}_{31}(P_3 - P_1) < 0$, over C_{213}
- $A_i^\top P_1 + P_1 A_i + \tilde{\tau}_{32}(P_3 - P_2) + \tau_{13}(P_1 - P_3) < 0$, over C_{231}
- $A_i^\top P_1 + P_1 A_i + \tilde{\tau}_{23}(P_2 - P_3) + \tilde{\tau}_{12}(P_1 - P_2) < 0$, over C_{321}

- $\exists\, \tilde{\lambda} \geq 0$ s.t. $A_i^\top P_1 + P_1 A_i + \tilde{\lambda}(P_1 - P_2) < 0$, over $(C_{213} \cup C_{231} \cup C_{321})$

A Case-Study with Three Quadratics

Base functions: $\{x^\top P_1 x, x^\top P_2 x, x^\top P_3 x\}$

BMIs for Mid of 3 Quadratics

$$V = \max\{\min\{P_1, P_2\}, \min\{P_2, P_3\}, \min\{P_3, P_1\}\}$$

For each $i \in \{1, \ldots, M\}$, find the scalars $\tau_{12}, \tau_{13}, \tau_{21}, \tau_{23}, \tau_{31}, \tau_{32} \geq 0$, and $\tilde{\tau}_{12}, \tilde{\tau}_{13}, \tilde{\tau}_{21}, \tilde{\tau}_{23}, \tilde{\tau}_{31}, \tilde{\tau}_{32} \geq 0$ such that

- $A_i^\top P_2 + P_2 A_i + \tau_{21}(P_2 - P_1) + \tau_{32}(P_3 - P_2) < 0$, over C_{123}
- $A_i^\top P_3 + P_3 A_i + \tau_{31}(P_3 - P_1) + \tau_{23}(P_2 - P_3) < 0$, over C_{132}
- $A_i^\top P_1 + P_1 A_i + \tilde{\tau}_{31}(P_1 - P_3) + \tilde{\tau}_{21}(P_2 - P_1) < 0$, over C_{312}
- $A_i^\top P_1 + P_1 A_i + \tau_{12}(P_1 - P_2) + \tilde{\tau}_{31}(P_3 - P_1) < 0$, over C_{213}
- $A_i^\top P_3 + P_3 A_i + \tilde{\tau}_{32}(P_3 - P_2) + \tau_{13}(P_1 - P_3) < 0$, over C_{231}
- $A_i^\top P_2 + P_2 A_i + \tilde{\tau}_{23}(P_2 - P_3) + \tilde{\tau}_{12}(P_1 - P_2) < 0$, over C_{321}

A Case-Study with Three Quadratics

Base functions: $\{x^\top P_1 x, x^\top P_2 x, x^\top P_3 x\}$

BMIs for Max of 3 Quadratics

$$V = \max\{\min\{P_1\}, \min\{P_2\}, \min\{P_3\}\}.$$

For each $i \in \{1, \dots, M\}$, find the scalars $\tau_{12}, \tau_{13}, \tau_{21}, \tau_{23}, \tau_{31}, \tau_{32} \geq 0$, and $\tilde{\tau}_{12}, \tilde{\tau}_{13}, \tilde{\tau}_{21}, \tilde{\tau}_{23}, \tilde{\tau}_{31}, \tilde{\tau}_{32} \geq 0$ such that

- $A_i^\top P_3 + P_3 A_i + \tau_{21}(P_2 - P_1) + \tau_{32}(P_3 - P_2) < 0$, over C_{123}
- $A_i^\top P_3 + P_3 A_i + \tau_{12}(P_1 - P_2) + \tilde{\tau}_{31}(P_3 - P_1) < 0$, over C_{213}
- $A_i^\top P_2 + P_2 A_i + \tau_{31}(P_3 - P_1) + \tau_{23}(P_2 - P_3) < 0$, over C_{132}
- $A_i^\top P_2 + P_2 A_i + \tilde{\tau}_{31}(P_1 - P_3) + \tilde{\tau}_{21}(P_2 - P_1) < 0$, over C_{312}
- $A_i^\top P_1 + P_1 A_i + \tilde{\tau}_{32}(P_3 - P_2) + \tau_{13}(P_1 - P_3) < 0$, over C_{231}
- $A_i^\top P_1 + P_1 A_i + \tilde{\tau}_{23}(P_2 - P_3) + \tilde{\tau}_{12}(P_1 - P_2) < 0$, over C_{321}

A Case-Study with Three Quadratics

Base functions: $\{x^\top P_1 x, x^\top P_2 x, x^\top P_3 x\}$

BMIs for Max of 3 Quadratics

$$V = \max\{\min\{P_1\}, \min\{P_2\}, \min\{P_3\}\}.$$

For each $i \in \{1, \ldots, M\}$, find $\lambda_{12}, \lambda_{13}, \lambda_{21}, \lambda_{23}, \lambda_{31}, \lambda_{32} \geq 0$ such that

- $A_i^\top P_3 + P_3 A_i + \lambda_{31}(P_3 - P_1) + \lambda_{32}(P_3 - P_2) < 0$, over $C_{123} \cup C_{213}$
- $A_i^\top P_2 + P_2 A_i + \lambda_{21}(P_2 - P_1) + \tau_{23}(P_2 - P_3) < 0$, over $C_{132} \cup C_{312}$
- $A_i^\top P_1 + P_1 A_i + \lambda_{12}(P_1 - P_2) + \lambda_{13}(P_1 - P_3) < 0$, over $C_{231} \cup C_{321}$

Example with max of quadratics

Arbitrary switching system with two modes:

$$\dot{x}(t) \in \overline{\text{co}}\{A_1 x(t), A_2(a)x(t)\},$$

where $a > 0$, and $A_1 = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$, $\quad A_2(a) = \begin{bmatrix} -1 & -a \\ 1/a & -1 \end{bmatrix}$

- In [Dayawansa-Martin '99]: $\exists$ a common quadratic Lyapunov function for $1 < a < 3 + \sqrt{8}$, but the system is GUES for $3 + \sqrt{8} < a \lessapprox 10$
- In [Goebl et. al '06]: Max of 7 quadratics gives a up to 10.1081
- Our approach: consistent but more classes of Lyapunov functions

	CLF	Max of 2	Min of 2
$a_{\max}$	$3+\sqrt{8}$	8.10	6.78

	Quasi-max	Quasi-min	Max of 3
$a_{\max}$	8.32	8.02	8.89

Conclusion

Summary:

- A class of locally Lipschitz functions for switched systems

Conclusion

Summary:

- A class of locally Lipschitz functions for switched systems

- Different notions of derivatives
- Lie derivative is less conservative but more demanding computationally
- Clarke gradient based conditions are conservative but relatively less demanding

Conclusion

Summary:

- A class of locally Lipschitz functions for switched systems

- Different notions of derivatives
- Lie derivative is less conservative but more demanding computationally
- Clarke gradient based conditions are conservative but relatively less demanding

- Max-Min functions as "intuitive" subclass of locally Lipschitz functions.

Conclusion

Summary:

- A class of locally Lipschitz functions for switched systems

- Different notions of derivatives
- Lie derivative is less conservative but more demanding computationally
- Clarke gradient based conditions are conservative but relatively less demanding

- Max-Min functions as "intuitive" subclass of locally Lipschitz functions.
- BMIs are not easy to solve

Conclusion

Summary:

- A class of locally Lipschitz functions for switched systems

- Different notions of derivatives
- Lie derivative is less conservative but more demanding computationally
- Clarke gradient based conditions are conservative but relatively less demanding

- Max-Min functions as "intuitive" subclass of locally Lipschitz functions.
- BMIs are not easy to solve

Thank you !!

Questions ??