Internship defense at LAAS
Title: Extracting Information from Moments: Nonnegativity Certification, Polynomial Systems Resolution and Generalized Christoffel-Darboux Kernels

LAAS CNRS, MAC team, advisors: Jean-Bernard Lasserre, Victor Magron

MAI Ngoc Hoang Anh

LAAS

July 1, 2019

Outline

Nonnegativity certificates for polynomials on non-compact sets

Systems of polynomial equations: A new algorithm converting stationary points which are not global minima to singularities

Polynomial systems: A new algorithm adding sphere inequalities constraints

Christoffel functions: Extracting information from moments combined with Newton's method

Nonnegativity certificates for polynomials on non-compact sets

Sum of squares (SOS) and Semidefinite programming (SDP)

$\mathbb{R}[x]$: the set of all polynomials $f = v_d^T c$ where

- $v_d := (x^\alpha)_{\alpha \in \mathbb{N}_d^n}$ (vector of monomials),
- $c \in \mathbb{R}^{s(d)}$ (vector of coefficients) with
$$s(d) := \begin{pmatrix} n+d \\ d \end{pmatrix}.$$

Sum of squares (SOS) and Semidefinite programming (SDP)

$\mathbb{R}[x]$: the set of all polynomials $f = v_d^T c$ where

- $v_d := (x^\alpha)_{\alpha \in \mathbb{N}_d^n}$ (vector of monomials),
- $c \in \mathbb{R}^{s(d)}$ (vector of coefficients) with
  $$s(d) := \binom{n+d}{d}.$$

Let $g_i, h_j \in \mathbb{R}[x]$.

Basis semialgebraic set

$S(g,h) := \{x \in \mathbb{R}^n : \ g_i(x) \geq 0, \ h_j(x) = 0\}.$

$\mathbb{R}[x]$: the set of all polynomials $f = v_d^T c$ where

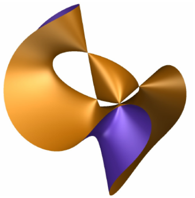
- $v_d := (x^\alpha)_{\alpha \in \mathbb{N}_d^n}$ (vector of monomials),
- $c \in \mathbb{R}^{s(d)}$ (vector of coefficients) with

$$s(d) := \begin{pmatrix} n+d \\ d \end{pmatrix}.$$

Let $g_i, h_j \in \mathbb{R}[x]$.

Basis semialgebraic set

$$S(g,h) := \{x \in \mathbb{R}^n : \ g_i(x) \geq 0, \ h_j(x) = 0\}.$$



Sum of squares (SOS) and Semidefinite programming (SDP)

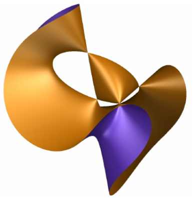
$\mathbb{R}[x]$: the set of all polynomials $f = v_d^T c$ where

- $v_d := (x^\alpha)_{\alpha \in \mathbb{N}_d^n}$ (vector of monomials),
- $c \in \mathbb{R}^{s(d)}$ (vector of coefficients) with
$$s(d) := \begin{pmatrix} n+d \\ d \end{pmatrix}.$$

Let $g_i, h_j \in \mathbb{R}[x]$.

Basis semialgebraic set

$$S(g,h) := \{x \in \mathbb{R}^n : \; g_i(x) \geq 0, \; h_j(x) = 0\}.$$



SOS

$f = f_1^2 + \ldots + f_m^2$, where $f_1, \ldots, f_m \in \mathbb{R}[x]$.

Sum of squares (SOS) and Semidefinite programming (SDP)

$\mathbb{R}[x]$: the set of all polynomials $f = v_d^T c$ where

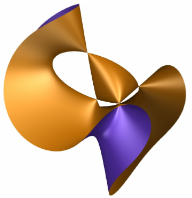
- $v_d := (x^\alpha)_{\alpha \in \mathbb{N}_d^n}$ (vector of monomials),
- $c \in \mathbb{R}^{s(d)}$ (vector of coefficients) with

$$s(d) := \begin{pmatrix} n+d \\ d \end{pmatrix}.$$

Let $g_i, h_j \in \mathbb{R}[x]$.

Basis semialgebraic set

$S(g,h) := \{x \in \mathbb{R}^n : \ g_i(x) \geq 0, \ h_j(x) = 0\}.$



SOS

$f = f_1^2 + ... + f_m^2$, where $f_1, ..., f_m \in \mathbb{R}[x]$.

$\Sigma[x]$: the the set of all SOSs.

SDP

$$\begin{aligned} \min_{X \in \mathbb{S}^N} \quad & \langle C, X \rangle_{\mathbb{S}^N} \\ \text{s.t.} \quad & \langle A_k, X \rangle_{\mathbb{S}^N} = b_k, \quad k = 1, \ldots, l \\ & X \succeq 0 \end{aligned}$$

$\mathbb{R}[x]$: the set of all polynomials $f = v_d^T c$ where

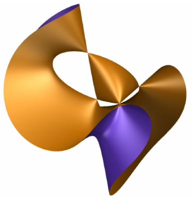
- $v_d := (x^\alpha)_{\alpha \in \mathbb{N}_d^n}$ (vector of monomials),
- $c \in \mathbb{R}^{s(d)}$ (vector of coefficients) with

$$s(d) := \begin{pmatrix} n+d \\ d \end{pmatrix}.$$

Let $g_i, h_j \in \mathbb{R}[x]$.

Basis semialgebraic set

$S(g,h) := \{x \in \mathbb{R}^n : \ g_i(x) \geq 0, \ h_j(x) = 0\}.$



SOS

$f = f_1^2 + ... + f_m^2$, where $f_1, ..., f_m \in \mathbb{R}[x]$.

$\Sigma[x]$: the the set of all SOSs.

SDP

$$\begin{aligned} \min_{X \in \mathbb{S}^N} \quad & \langle C, X \rangle_{\mathbb{S}^N} \\ \text{s.t.} \quad & \langle A_k, X \rangle_{\mathbb{S}^N} = b_k, \quad k = 1, \ldots, l \\ & X \succeq 0 \end{aligned}$$

Relation between SOS and SDP

f is SOS $\Leftrightarrow \exists G \succeq 0 : f = v_d^T G v_d$.

Sum of squares (SOS) and Semidefinite programming (SDP)

$\mathbb{R}[x]$: the set of all polynomials $f = v_d^T c$ where

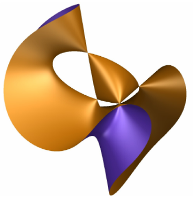
- $v_d := (x^\alpha)_{\alpha \in \mathbb{N}_d^n}$ (vector of monomials),
- $c \in \mathbb{R}^{s(d)}$ (vector of coefficients) with

$$s(d) := \begin{pmatrix} n+d \\ d \end{pmatrix}.$$

Let $g_i, h_j \in \mathbb{R}[x]$.

Basis semialgebraic set

$S(g,h) := \{x \in \mathbb{R}^n : \ g_i(x) \geq 0, \ h_j(x) = 0\}$.



SOS

$f = f_1^2 + \ldots + f_m^2$, where $f_1, \ldots, f_m \in \mathbb{R}[x]$.

$\Sigma[x]$: the the set of all SOSs.

SDP

$$\begin{aligned} \min_{X \in \mathbb{S}^N} \quad & \langle C, X \rangle_{\mathbb{S}^N} \\ \text{s.t.} \quad & \langle A_k, X \rangle_{\mathbb{S}^N} = b_k, \quad k = 1, \ldots, l \\ & X \succeq 0 \end{aligned}$$

Relation between SOS and SDP

f is SOS $\Leftrightarrow \exists G \succeq 0 : f = v_d^T G v_d$.

$\mathbb{R}[x]_d$: the set of all polynomial of degree at most d
$\Sigma[x]_d$: the set of all SOS of degree at most $2d$.

Representation theorems

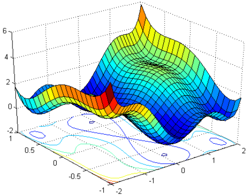
Representation theorems

Issue

How to certify that a polynomial p is nonnegative on $S(g, h)$ with $g_0 = 1$?

Issue

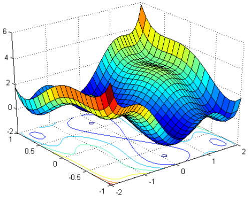
How to certify that a polynomial p is nonnegative on $S(g, h)$ with $g_0 = 1$?



Representation theorems

Issue

How to certify that a polynomial p is nonnegative on $S(g, h)$ with $g_0 = 1$?



$Q(g, h)$: (quadratic module) the set of all

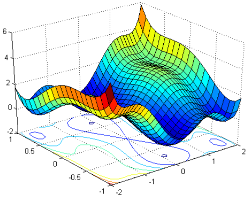
$$\sum_{i=0}^{m} \sigma_i g_i + \sum_{j=1}^{l} \phi_j h_j$$

where $\sigma_i \in \Sigma[x]$ and $\phi_j \in \mathbb{R}[x]$.

Representation theorems

Issue

How to certify that a polynomial p is nonnegative on $S(g, h)$ with $g_0 = 1$?



$Q(g, h)$: (quadratic module) the set of all

$$\sum_{i=0}^{m} \sigma_i g_i + \sum_{j=1}^{l} \phi_j h_j$$

where $\sigma_i \in \Sigma[x]$ and $\phi_j \in \mathbb{R}[x]$.

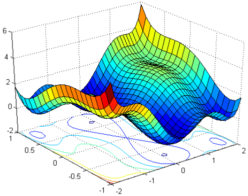
Putinar's Positivstellensatz for compact $S(g, h)$ (with Archimedian condition: $L - |x|^2 \in Q(g, h)$)

$p > 0$ on $S(g, h) \Rightarrow p \in Q(g, h)$.

Representation theorems

Issue

How to certify that a polynomial p is nonnegative on $S(g,h)$ with $g_0 = 1$?



$Q(g,h)$: (quadratic module) the set of all

$$\sum_{i=0}^{m} \sigma_i g_i + \sum_{j=1}^{l} \phi_j h_j$$

where $\sigma_i \in \Sigma[x]$ and $\phi_j \in \mathbb{R}[x]$.

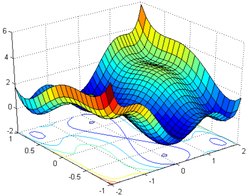
Putinar's Positivstellensatz for compact $S(g,h)$ (with Archimedian condition: $L - |x|^2 \in Q(g,h)$)

$p > 0$ on $S(g,h) \Rightarrow p \in Q(g,h)$.

$\theta := 1 + |x|^2$.

Issue

How to certify that a polynomial p is nonnegative on $S(g,h)$ with $g_0 = 1$?



$Q(g,h)$: (quadratic module) the set of all

$$\sum_{i=0}^{m} \sigma_i g_i + \sum_{j=1}^{l} \phi_j h_j$$

where $\sigma_i \in \Sigma[x]$ and $\phi_j \in \mathbb{R}[x]$.

Putinar's Positivstellensatz for compact $S(g,h)$ (with Archimedian condition: $L - |x|^2 \in Q(g,h)$)

$p > 0$ on $S(g,h) \Rightarrow p \in Q(g,h)$.

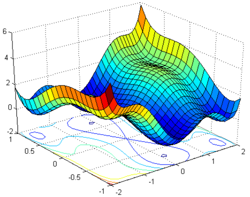
$\theta := 1 + |x|^2$.

Result for non-compact $S(g,h)$ (without Archimedian condition)

$p \geq 0$ on $S(g,h) \Rightarrow \forall \varepsilon > 0,\ \exists K_\varepsilon \in \mathbb{N}$:
$\left(p + \varepsilon \theta^{1+\lceil \deg(p)/2 \rceil}\right) \theta^{K_\varepsilon} \in Q(g,h)$.

Issue

How to certify that a polynomial p is nonnegative on $S(g,h)$ with $g_0 = 1$?



$Q(g,h)$: (quadratic module) the set of all

$$\sum_{i=0}^{m} \sigma_i g_i + \sum_{j=1}^{l} \phi_j h_j$$

where $\sigma_i \in \Sigma[x]$ and $\phi_j \in \mathbb{R}[x]$.

Putinar's Positivstellensatz for compact $S(g,h)$ (with Archimedian condition: $L - |x|^2 \in Q(g,h)$)

$p > 0$ on $S(g,h) \Rightarrow p \in Q(g,h)$.

$\theta := 1 + |x|^2$.

Result for non-compact $S(g,h)$ (without Archimedian condition)

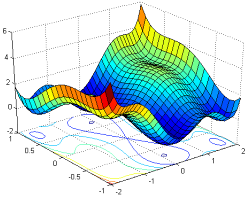
$p \geq 0$ on $S(g,h) \Rightarrow \forall \varepsilon > 0,\ \exists K_\varepsilon \in \mathbb{N}$:
$\left(p + \varepsilon \theta^{1+\lceil \deg(p)/2 \rceil}\right) \theta^{K_\varepsilon} \in Q(g,h)$.

When $S(g,h) = \mathbb{R}^n$:

Representation theorems

Issue

How to certify that a polynomial p is nonnegative on $S(g,h)$ with $g_0 = 1$?



$Q(g,h)$: (quadratic module) the set of all

$$\sum_{i=0}^{m} \sigma_i g_i + \sum_{j=1}^{l} \phi_j h_j$$

where $\sigma_i \in \Sigma[x]$ and $\phi_j \in \mathbb{R}[x]$.

Putinar's Positivstellensatz for compact $S(g,h)$ (with Archimedian condition: $L - |x|^2 \in Q(g,h)$)

$p > 0$ on $S(g,h) \Rightarrow p \in Q(g,h)$.

$\theta := 1 + |x|^2$.

Result for non-compact $S(g,h)$ (without Archimedian condition)

$p \geq 0$ on $S(g,h) \Rightarrow \forall \varepsilon > 0,\ \exists K_\varepsilon \in \mathbb{N}$: $\left(p + \varepsilon \theta^{1+\lceil \deg(p)/2 \rceil}\right) \theta^{K_\varepsilon} \in Q(g,h)$.

When $S(g,h) = \mathbb{R}^n$:

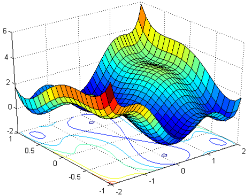
Reznick's representation (with assumption p is homogeneous)

$p > 0$ on $\mathbb{R}^n \setminus \{0\} \Rightarrow \exists K \in \mathbb{N}$: $|x|^{2K} p \in \Sigma[x]$.

Representation theorems

Issue

How to certify that a polynomial p is nonnegative on $S(g,h)$ with $g_0 = 1$?



$Q(g,h)$: (quadratic module) the set of all

$$\sum_{i=0}^{m} \sigma_i g_i + \sum_{j=1}^{l} \phi_j h_j$$

where $\sigma_i \in \Sigma[x]$ and $\phi_j \in \mathbb{R}[x]$.

Putinar's Positivstellensatz for compact $S(g,h)$ (with Archimedian condition: $L - |x|^2 \in Q(g,h)$)

$p > 0$ on $S(g,h) \Rightarrow p \in Q(g,h)$.

$\theta := 1 + |x|^2$.

Result for non-compact $S(g,h)$ (without Archimedian condition)

$p \geq 0$ on $S(g,h) \Rightarrow \forall \varepsilon > 0,\ \exists K_\varepsilon \in \mathbb{N}$: $\left(p + \varepsilon \theta^{1+\lceil \deg(p)/2 \rceil}\right) \theta^{K_\varepsilon} \in Q(g,h)$.

When $S(g,h) = \mathbb{R}^n$:

Reznick's representation (with assumption p is homogeneous)

$p > 0$ on $\mathbb{R}^n \backslash \{0\} \Rightarrow \exists K \in \mathbb{N}: \ |x|^{2K} p \in \Sigma[x]$.

Result for global nonnegativity (without homogeneous assumption)

$p \geq 0$ on $\mathbb{R}^n \Rightarrow \forall \varepsilon > 0,\ \exists K_\varepsilon \in \mathbb{N}$: $\left(p + \varepsilon \theta^{\deg(p)/2}\right) \theta^{K_\varepsilon} \in \Sigma[x]$.

Lasserre's SOS approach for polynomial optimization problems (POPs)

J.-B Lasserre.
Global optimization with polynomials and the problem of moments.
In *SIAM Journal on optimization*, 2001.

Lasserre's SOS approach for polynomial optimization problems (POPs)

J.-B Lasserre.
Global optimization with polynomials and the problem of moments.
In *SIAM Journal on optimization*, 2001.

- Consider POP

$$f^* = \inf_{x \in S(g,h)} f(x).$$

with $S(g, h)$ satisfying Archimedian condition.

Lasserre's SOS approach for polynomial optimization problems (POPs)

J.-B Lasserre.
Global optimization with polynomials and the problem of moments.
In *SIAM Journal on optimization*, 2001.

- Consider POP

$$f^* = \inf_{x \in S(g,h)} f(x).$$

with $S(g,h)$ satisfying Archimedian condition.

- It is equivalent to the problem $f^* = \sup\{\lambda \in \mathbb{R} : f - \lambda > 0 \text{ on } S(g,h)\}$.

J.-B Lasserre.
Global optimization with polynomials and the problem of moments.
In *SIAM Journal on optimization*, 2001.

- Consider POP

$$f^* = \inf_{x \in S(g,h)} f(x).$$

with $S(g,h)$ satisfying Archimedian condition.

- It is equivalent to the problem $f^* = \sup \{\lambda \in \mathbb{R} : f - \lambda > 0 \text{ on } S(g,h)\}$.
- By using Putinar's representation, $f^* = \sup \{\lambda \in \mathbb{R} : f - \lambda \in Q(g,h)\}$.

J.-B Lasserre.
Global optimization with polynomials and the problem of moments.
In *SIAM Journal on optimization*, 2001.

- Consider POP

$$f^* = \inf_{x \in S(g,h)} f(x).$$

with $S(g,h)$ satisfying Archimedian condition.

- It is equivalent to the problem $f^* = \sup \{\lambda \in \mathbb{R} : f - \lambda > 0 \text{ on } S(g,h)\}$.
- By using Putinar's representation, $f^* = \sup \{\lambda \in \mathbb{R} : f - \lambda \in Q(g,h)\}$.
- Set $v_i := \lceil \deg(g_i)/2 \rceil$ and $u_j := \lceil \deg(h_j)/2 \rceil$.

Lasserre's SOS approach for polynomial optimization problems (POPs)

J.-B Lasserre.
Global optimization with polynomials and the problem of moments.
In *SIAM Journal on optimization*, 2001.

- Consider POP

$$f^* = \inf_{x \in S(g,h)} f(x).$$

with $S(g,h)$ satisfying Archimedian condition.

- It is equivalent to the problem $f^* = \sup \{\lambda \in \mathbb{R} : f - \lambda > 0 \text{ on } S(g,h)\}$.
- By using Putinar's representation, $f^* = \sup \{\lambda \in \mathbb{R} : f - \lambda \in Q(g,h)\}$.
- Set $v_i := \lceil \deg(g_i)/2 \rceil$ and $u_j := \lceil \deg(h_j)/2 \rceil$.
- $Q(g,h)_d$: (truncated quadratic module) the set of all

$$\sum_{i=0}^{m} \sigma_i g_i + \sum_{j=1}^{l} \phi_j h_j$$

where $\sigma_i \in \Sigma[x]_{d-v_i}$ and $\phi_j \in \mathbb{R}[x]_{d-u_j}$.

Lasserre's SOS approach for polynomial optimization problems (POPs)

J.-B Lasserre.
Global optimization with polynomials and the problem of moments.
In *SIAM Journal on optimization*, 2001.

- Consider POP

$$f^* = \inf_{x \in S(g,h)} f(x).$$

with $S(g,h)$ satisfying Archimedian condition.

- It is equivalent to the problem $f^* = \sup\{\lambda \in \mathbb{R} : f - \lambda > 0 \text{ on } S(g,h)\}$.
- By using Putinar's representation, $f^* = \sup\{\lambda \in \mathbb{R} : f - \lambda \in Q(g,h)\}$.

- Set $v_i := \lceil \deg(g_i)/2 \rceil$ and $u_j := \lceil \deg(h_j)/2 \rceil$.
- $Q(g,h)_d$: (truncated quadratic module) the set of all

$$\sum_{i=0}^{m} \sigma_i g_i + \sum_{j=1}^{l} \phi_j h_j$$

where $\sigma_i \in \Sigma[x]_{d-v_i}$ and $\phi_j \in \mathbb{R}[x]_{d-u_j}$.

Lasserre's hierarchy

Consider the hierarchy of SDPs for every $k \in \mathbb{N}$

$$\rho_k = \sup\left\{\lambda \in \mathbb{R} : f - \lambda \in Q(g,h)_k\right\}.$$

Then $\rho_k \uparrow f^*$ as $k \to \infty$.

Lasserre's moment approach for POPs

Let μ be finite Borel measure.

Lasserre's moment approach for POPs

Let μ be finite Borel measure.

Moments

$y_\alpha = \int x^\alpha d\mu, \ \alpha \in \mathbb{N}^n$.

Lasserre's moment approach for POPs

Let μ be finite Borel measure.

Moments

$y_\alpha = \int x^\alpha d\mu, \ \alpha \in \mathbb{N}^n.$

Moment matrix

$M_d(y) = \left(y_{\alpha+\beta}\right)_{\alpha+\beta \in \mathbb{N}_d^n}.$

Lasserre's moment approach for POPs

Let μ be finite Borel measure.

Moments

$y_\alpha = \int x^\alpha d\mu, \ \alpha \in \mathbb{N}^n.$

Moment matrix

$M_d(y) = \left(y_{\alpha+\beta}\right)_{\alpha+\beta \in \mathbb{N}^n_d}.$

Localizing matrix

$$M_d(gy) = \left(\sum_\gamma g_\gamma y_{\gamma+\alpha+\beta}\right)_{\alpha,\beta \in \mathbb{N}^n_d}.$$

Lasserre's moment approach for POPs

Let μ be finite Borel measure.

Moments

$y_\alpha = \int x^\alpha d\mu, \ \alpha \in \mathbb{N}^n.$

Moment matrix

$M_d(y) = \left(y_{\alpha+\beta}\right)_{\alpha+\beta \in \mathbb{N}^n_d}.$

Localizing matrix

$$M_d(gy) = \left(\sum_\gamma g_\gamma y_{\gamma+\alpha+\beta}\right)_{\alpha,\beta \in \mathbb{N}^n_d}.$$

Riesz linear functional
$L_y : \mathbb{R}[x] \to \mathbb{R}$

$L_y(f) := \sum_\alpha f_\alpha y_\alpha.$

Lasserre's moment approach for POPs

Let μ be finite Borel measure.

Moments

$y_\alpha = \int x^\alpha d\mu, \ \alpha \in \mathbb{N}^n.$

Moment matrix

$M_d(y) = \left(y_{\alpha+\beta}\right)_{\alpha+\beta\in\mathbb{N}^n_d}.$

Localizing matrix

$$M_d(gy) = \left(\sum_\gamma g_\gamma y_{\gamma+\alpha+\beta}\right)_{\alpha,\beta\in\mathbb{N}^n_d}.$$

Riesz linear functional
$L_y : \mathbb{R}[x] \to \mathbb{R}$

$L_y(f) := \sum_\alpha f_\alpha y_\alpha.$

$M(S(g,h))$: the set of all finite Borel measures supported on $S(g,h)$.

Lasserre's moment approach for POPs

Let μ be finite Borel measure.

Moments

$y_\alpha = \int x^\alpha d\mu, \ \alpha \in \mathbb{N}^n.$

Moment matrix

$M_d(y) = \left(y_{\alpha+\beta}\right)_{\alpha+\beta \in \mathbb{N}^n_d}.$

Localizing matrix

$M_d(gy) = \left(\sum_\gamma g_\gamma y_{\gamma+\alpha+\beta}\right)_{\alpha,\beta \in \mathbb{N}^n_d}.$

Riesz linear functional $L_y : \mathbb{R}[x] \to \mathbb{R}$

$L_y(f) := \sum_\alpha f_\alpha y_\alpha.$

$M(S(g,h))$: the set of all finite Borel measures supported on $S(g,h)$.

- Equivalent problem over measure $f^* = \inf_{\mu \in M(S(g,h))} \int f d\mu.$

Lasserre's moment approach for POPs

Let μ be finite Borel measure.

Moments

$y_\alpha = \int x^\alpha d\mu, \ \alpha \in \mathbb{N}^n.$

Moment matrix

$M_d(y) = (y_{\alpha+\beta})_{\alpha+\beta \in \mathbb{N}^n_d}.$

Localizing matrix

$$M_d(gy) = \left(\sum_\gamma g_\gamma y_{\gamma+\alpha+\beta}\right)_{\alpha,\beta\in\mathbb{N}^n_d}.$$

Riesz linear functional $L_y : \mathbb{R}[x] \to \mathbb{R}$

$L_y(f) := \sum_\alpha f_\alpha y_\alpha.$

$M(S(g,h))$: the set of all finite Borel measures supported on $S(g,h)$.

- Equivalent problem over measure $f^* = \inf_{\mu \in M(S(g,h))} \int f d\mu.$
- One has

$$f^* = \inf L_y(f)$$
$$\text{s.t. } y \in \mathbb{R}^\infty \text{ has representing measure in } M(S(g,h))$$

Lasserre's moment approach for POPs

Let μ be finite Borel measure.

Moments

$y_\alpha = \int x^\alpha d\mu, \ \alpha \in \mathbb{N}^n.$

Moment matrix

$M_d(y) = \left(y_{\alpha+\beta}\right)_{\alpha+\beta\in\mathbb{N}_d^n}.$

Localizing matrix

$M_d(gy) = \left(\sum_\gamma g_\gamma y_{\gamma+\alpha+\beta}\right)_{\alpha,\beta\in\mathbb{N}_d^n}.$

Riesz linear functional $L_y : \mathbb{R}[x] \to \mathbb{R}$

$L_y(f) := \sum_\alpha f_\alpha y_\alpha.$

$M(S(g,h))$: the set of all finite Borel measures supported on $S(g,h)$.

- Equivalent problem over measure $f^* = \inf_{\mu\in M(S(g,h))} \int f d\mu.$
- One has

$$f^* = \inf L_y(f)$$
$$\text{s.t. } y \in \mathbb{R}^\infty \text{ has representing measure in } M(S(g,h))$$

- We obtain

$$\begin{aligned} f^* = \inf_{y\in\mathbb{R}^\infty} \quad & L_y(f) \\ \text{s.t.} \quad & M_{k-v_i}(g_i y) \succeq 0, \ \forall k \in \mathbb{N}, \\ & M_{k-u_j}(h_j y) = 0, \ \forall k \in \mathbb{N}. \end{aligned}$$

Lasserre's moment approach for POPs

Let μ be finite Borel measure.

Moments

$y_\alpha = \int x^\alpha d\mu,\ \alpha \in \mathbb{N}^n.$

Moment matrix

$M_d(y) = \left(y_{\alpha+\beta}\right)_{\alpha+\beta\in\mathbb{N}_d^n}.$

Localizing matrix

$M_d(gy) = \left(\sum_\gamma g_\gamma y_{\gamma+\alpha+\beta}\right)_{\alpha,\beta\in\mathbb{N}_d^n}.$

Riesz linear functional $L_y : \mathbb{R}[x] \to \mathbb{R}$

$L_y(f) := \sum_\alpha f_\alpha y_\alpha.$

$M(S(g,h))$: the set of all finite Borel measures supported on $S(g,h)$.

- Equivalent problem over measure $f^* = \inf_{\mu\in M(S(g,h))} \int f d\mu.$
- One has

$$f^* = \inf L_y(f)$$
$$\text{s.t. } y \in \mathbb{R}^\infty \text{ has representing measure in } M(S(g,h))$$

- We obtain

$$\begin{aligned} f^* = \inf_{y\in\mathbb{R}^\infty} \quad & L_y(f) \\ \text{s.t.} \quad & M_{k-v_i}(g_i y) \succeq 0,\ \forall k\in\mathbb{N}, \\ & M_{k-u_j}(h_j y) = 0,\ \forall k\in\mathbb{N}. \end{aligned}$$

Lasserre's hierarchy

For every $k \in \mathbb{N}$

$$\begin{aligned} \tau_k = \inf_{y\in\mathbb{R}^{s(2k)}} \quad & L_y(f) \\ \text{s.t.} \quad & M_{k-v_i}(g_i y) \succeq 0, \\ & M_{k-u_j}(h_j y) = 0. \end{aligned}$$

is the dual of the previous SOS problem with value ρ_k.

Application for polynomial optimization over non-compact semialgebraic sets

Application for polynomial optimization over non-compact semialgebraic sets

- Consider POP

$$f^* = \inf_{x \in S(g,h)} f(x).$$

attained at x^*.

Application for polynomial optimization over non-compact semialgebraic sets

- Consider POP

$$f^* = \inf_{x \in S(g,h)} f(x).$$

  attained at x^*.
- Set $d := 1 + \lceil \deg(f)/2 \rceil$.

Application for polynomial optimization over non-compact semialgebraic sets

- Consider POP

$$f^* = \inf_{x \in S(g,h)} f(x) .$$

attained at x^*.

- Set $d := 1 + \lceil \deg(f)/2 \rceil$.

Application for polynomial optimization over non-compact semialgebraic sets

- Consider POP

$$f^* = \inf_{x \in S(g,h)} f(x) .$$

attained at x^*.
- Set $d := 1 + \lceil \deg(f)/2 \rceil$.

A new Lasserre's hierarchy

Let $\bar{\varepsilon} > 0$ be fixed. Consider the hierarchy of semidefinite programs for every $k \in \mathbb{N}$

$$\rho_k = \sup \left\{ \lambda \in \mathbb{R} : \ \theta^k \left(f + \bar{\varepsilon}\theta^d - \lambda \right) \in Q(g,h)_{k+d} \right\} .$$

Then $\exists K \in N : \forall k \geq K, \ \rho_k \in \left[f^*, f^* + \varepsilon\theta(x^*)^d \right]$.

Application for polynomial optimization over non-compact semialgebraic sets

- Consider POP

$$f^* = \inf_{x \in S(g,h)} f(x).$$

attained at x^*.

- Set $d := 1 + \lceil \deg(f)/2 \rceil$.

A new Lasserre's hierarchy

Let $\bar{\varepsilon} > 0$ be fixed. Consider the hierarchy of semidefinite programs for every $k \in \mathbb{N}$

$$\rho_k = \sup \left\{ \lambda \in \mathbb{R} : \ \theta^k \left(f + \bar{\varepsilon}\theta^d - \lambda \right) \in Q(g,h)_{k+d} \right\}.$$

Then $\exists K \in N : \forall k \geq K, \ \rho_k \in \left[f^*, f^* + \varepsilon\theta(x^*)^d \right].$

The dual of the problem of value ρ_k

$$\begin{aligned} \tau_k = \inf \quad & L_y\left(\theta^k\left(f + \bar{\varepsilon}\theta^d\right)\right) \\ \text{s.t.} \quad & M_{k+d-v_i}(g_i y) \succeq 0, \\ & M_{k+d-u_j}(h_j y) = 0, \\ & L_y\left(\theta^k\right) = 1. \end{aligned}$$

Application for polynomial optimization over non-compact semialgebraic sets

- Consider POP

$$f^* = \inf_{x \in S(g,h)} f(x).$$

attained at x^*.

- Set $d := 1 + \lceil \deg(f)/2 \rceil$.

A new Lasserre's hierarchy

Let $\bar{\varepsilon} > 0$ be fixed. Consider the hierarchy of semidefinite programs for every $k \in \mathbb{N}$

$$\rho_k = \sup \left\{ \lambda \in \mathbb{R} : \ \theta^k \left(f + \bar{\varepsilon}\theta^d - \lambda \right) \in Q(g,h)_{k+d} \right\}.$$

Then $\exists K \in N : \forall k \geq K, \ \rho_k \in \left[f^*, f^* + \varepsilon\theta(x^*)^d \right]$.

The dual of the problem of value ρ_k

$$\begin{aligned} \tau_k = \inf \quad & L_y \left(\theta^k \left(f + \bar{\varepsilon}\theta^d \right) \right) \\ \text{s.t.} \quad & M_{k+d-v_i} \left(g_i y \right) \succeq 0, \\ & M_{k+d-u_j} \left(h_j y \right) = 0, \\ & L_y \left(\theta^k \right) = 1. \end{aligned}$$

For unconstrained case: $S(g,h) := \mathbb{R}^n$, $d := \deg(f)/2$ and $Q(g,h)_{k+d} := \Sigma[x]_{k+d}$.

Application for polynomial optimization over non-compact semialgebraic sets

- Consider POP

$$f^* = \inf_{x \in S(g,h)} f(x).$$

attained at x^*.

- Set $d := 1 + \lceil \deg(f)/2 \rceil$.

A new Lasserre's hierarchy

Let $\bar{\varepsilon} > 0$ be fixed. Consider the hierarchy of semidefinite programs for every $k \in \mathbb{N}$

$$\rho_k = \sup \left\{ \lambda \in \mathbb{R} : \ \theta^k \left(f + \bar{\varepsilon}\theta^d - \lambda \right) \in Q(g,h)_{k+d} \right\}.$$

Then $\exists K \in N : \forall k \geq K, \ \rho_k \in \left[f^*, f^* + \varepsilon\theta(x^*)^d \right].$

The dual of the problem of value ρ_k

$$\begin{aligned} \tau_k = \inf \quad & L_y\left(\theta^k\left(f + \bar{\varepsilon}\theta^d\right)\right) \\ \text{s.t.} \quad & M_{k+d-v_i}\left(g_i y\right) \succeq 0, \\ & M_{k+d-u_j}\left(h_j y\right) = 0, \\ & L_y\left(\theta^k\right) = 1. \end{aligned}$$

For unconstrained case: $S(g,h) := \mathbb{R}^n$, $d := \deg(f)/2$ and $Q(g,h)_{k+d} := \Sigma[x]_{k+d}$.

Algorithm

Begin with $k = 0$ and do:

1. Solve SDP to get ρ_k.
2. If $S(g \cup \{\rho_k - f\}, h) = \emptyset$, set $k := k+1$ and do again step 1. If $S(g \cup \{\rho_k - f\}, h) \neq \emptyset$, take a point $\bar{x} \in S(g \cup \{\rho_k - f\}, h)$ and stop.

Systems of polynomial equations: A new algorithm converting stationary points which are not global minima to singularities

Convergence theorem

Convergence theorem

Problem

Find a real solution of system of polynomial equations

$$f_i(x) = 0, \; i = 1, ..., m.$$

Convergence theorem

Problem

Find a real solution of system of polynomial equations

$$f_i(x) = 0, \; i = 1, ..., m.$$

P Aubry, F Rouillier, M Safey El Din.
Real solving for positive dimensional systems.
Journal of Symbolic Computation, 2002.

AJ Sommese, J Verschelde, CW Wampler.
Introduction to numerical algebraic geometry.
Solving polynomial equations, 2005.

B Mourrain, P Trebuchet.
Generalized normal forms and polynomial system solving.
Proceedings of the 2005 international symposium on Symbolic and algebraic computation, 2005.

Convergence theorem

Problem

Find a real solution of system of polynomial equations

$$f_i(x) = 0, \; i = 1, ..., m.$$

P Aubry, F Rouillier, M Safey El Din.
Real solving for positive dimensional systems.
Journal of Symbolic Computation, 2002.

AJ Sommese, J Verschelde, CW Wampler.
Introduction to numerical algebraic geometry.
Solving polynomial equations, 2005.

B Mourrain, P Trebuchet.
Generalized normal forms and polynomial system solving.
Proceedings of the 2005 international symposium on Symbolic and algebraic computation, 2005.

Remark

Previous methods depend on the degrees of f_i.

Convergence theorem

Problem

Find a real solution of system of polynomial equations

$$f_i(x) = 0,\ i = 1, \ldots, m.$$

P Aubry, F Rouillier, M Safey El Din.
Real solving for positive dimensional systems.
Journal of Symbolic Computation, 2002.

AJ Sommese, J Verschelde, CW Wampler.
Introduction to numerical algebraic geometry.
Solving polynomial equations, 2005.

B Mourrain, P Trebuchet.
Generalized normal forms and polynomial system solving.
Proceedings of the 2005 international symposium on Symbolic and algebraic computation, 2005.

Remark

Previous methods depend on the degrees of f_i.

- Set $\varphi := f_1^2 + \ldots + f_m^2$. Assume that $n \geq 2$ and the set of stationary points of φ, $\{x \in \mathbb{R}^n : \nabla\varphi(x) = 0\}$, is a zero dimensional.

Convergence theorem

Problem

Find a real solution of system of polynomial equations

$$f_i(x) = 0, \ i = 1, \dots, m.$$

P Aubry, F Rouillier, M Safey El Din.
Real solving for positive dimensional systems.
Journal of Symbolic Computation, 2002.

AJ Sommese, J Verschelde, CW Wampler.
Introduction to numerical algebraic geometry.
Solving polynomial equations, 2005.

B Mourrain, P Trebuchet.
Generalized normal forms and polynomial system solving.
Proceedings of the 2005 international symposium on Symbolic and algebraic computation, 2005.

Remark

Previous methods depend on the degrees of f_i.

- Set $\varphi := f_1^2 + \dots + f_m^2$. Assume that $n \geq 2$ and the set of stationary points of φ, $\{x \in \mathbb{R}^n : \ \nabla\varphi(x) = 0\}$, is a zero dimensional.
- For every $\varepsilon > 0$, let $I_\varepsilon : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$I_\varepsilon(x) = \begin{cases} \varphi(x)\log\left(\delta_\varepsilon\psi(x) + 1\right)\exp\left(\frac{\varepsilon}{2}|x|^2\right) & \text{if } \nabla\varphi(x) \neq 0; \\ 0 & \text{if } \nabla\varphi(x) = 0 \text{ and } \varphi(x) = 0; \\ \infty & \text{if } \nabla\varphi(x) = 0 \text{ and } \varphi(x) > 0 \end{cases}$$

where
$\psi(x) = 2|\nabla\varphi(x)|^{-2}, \ \forall x \in \mathbb{R}^n : \ \nabla\varphi(x) \neq 0.$

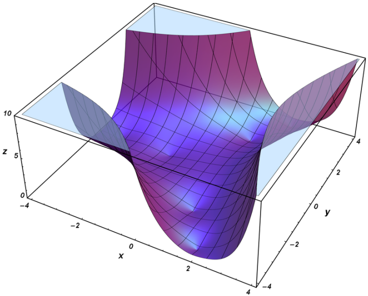
Convergence theorem

Lemma

1. $l_\varepsilon \geq 0$ *on* $\mathbb{R}^n$ *and* $l_\varepsilon(x) = 0 \Leftrightarrow \varphi(x) = 0$ *.*
2. $W := \{x \in {}^n : \nabla\varphi(x) = 0,\ \varphi(x) > 0\}$ *is the set of all stationary points which are not global minima of* $\min\{\varphi(x) : x \in \mathbb{R}^n\}$*.*
3. $l_\varepsilon \in C^1(\mathbb{R}^n \setminus W)$*.*
4. $\lim\limits_{|x|\to\infty} l_\varepsilon(x) = \infty$ *and* $\lim\limits_{x\to a\in W} l_\varepsilon(x) = \infty$*.*

Lemma

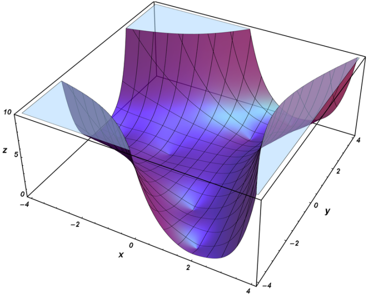
1. $l_\varepsilon \geq 0$ *on* $\mathbb{R}^n$ *and* $l_\varepsilon(x) = 0 \Leftrightarrow \varphi(x) = 0$ *.*
2. $W := \{x \in {}^n : \nabla\varphi(x) = 0,\ \varphi(x) > 0\}$ *is the set of all stationary points which are not global minima of* $\min\{\varphi(x) : x \in \mathbb{R}^n\}$*.*
3. $l_\varepsilon \in C^1(\mathbb{R}^n \setminus W)$*.*
4. $\lim_{|x|\to\infty} l_\varepsilon(x) = \infty$ *and* $\lim_{x\to a\in W} l_\varepsilon(x) = \infty$*.*



Convergence theorem

Lemma

1. $l_\varepsilon \geq 0$ *on* $\mathbb{R}^n$ *and* $l_\varepsilon(x) = 0 \Leftrightarrow \varphi(x) = 0$ *.*
2. $W := \{x \in {}^n : \nabla\varphi(x) = 0, \ \varphi(x) > 0\}$ *is the set of all stationary points which are not global minima of* $\min\{\varphi(x) : x \in \mathbb{R}^n\}$*.*
3. $l_\varepsilon \in C^1(\mathbb{R}^n \setminus W)$*.*
4. $\lim\limits_{|x|\to\infty} l_\varepsilon(x) = \infty$ *and* $\lim\limits_{x\to a\in W} l_\varepsilon(x) = \infty$*.*



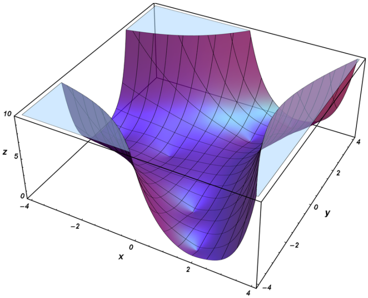
Remark

The properties of l_ε rely on the different rates between polynomials, logarithm function and exponential function.

Convergence theorem

Lemma

1. $l_\varepsilon \geq 0$ *on* $\mathbb{R}^n$ *and* $l_\varepsilon(x) = 0 \Leftrightarrow \varphi(x) = 0$ *.*
2. $W := \{x \in {}^n : \ \nabla\varphi(x) = 0, \ \varphi(x) > 0\}$ *is the set of all stationary points which are not global minima of* $\min\{\varphi(x) : x \in \mathbb{R}^n\}$*.*
3. $l_\varepsilon \in C^1(\mathbb{R}^n \backslash W)$*.*
4. $\lim\limits_{|x|\to\infty} l_\varepsilon(x) = \infty$ *and* $\lim\limits_{x\to a\in W} l_\varepsilon(x) = \infty$*.*



Remark

The properties of l_ε rely on the different rates between polynomials, logarithm function and exponential function.

Theorem

Let $(\varepsilon_k)_{k\in\mathbb{N}} \subset \mathbb{R}_+^*$ *such that* $\varepsilon_k \downarrow 0$ *as* $k \to \infty$*. Let* $M > 0$*. For every* $k \in \mathbb{N}$*, denote* $l_k := l_{\varepsilon_k}$ *and denote*

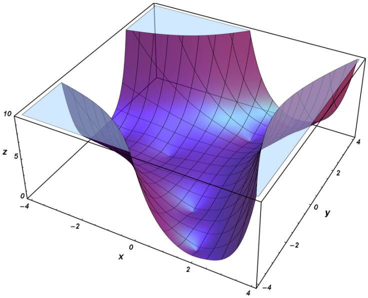
$$J_k(x) := l_k(x) - \log(M - l_0(x)), \ x \in \mathbb{R}^n.$$

Let $(x_k)_{k\in\mathbb{N}} \subset \mathbb{R}^n \backslash W$ *such that* $l_0(x_k) \leq M$ *and* $\nabla J_k(x_k) = 0$*.*

Convergence theorem

Lemma

1. $l_\varepsilon \geq 0$ *on* $\mathbb{R}^n$ *and* $l_\varepsilon(x) = 0 \Leftrightarrow \varphi(x) = 0$ *.*
2. $W := \{x \in {}^n : \nabla\varphi(x) = 0, \ \varphi(x) > 0\}$ *is the set of all stationary points which are not global minima of* $\min\{\varphi(x) : x \in \mathbb{R}^n\}$*.*
3. $l_\varepsilon \in C^1(\mathbb{R}^n \backslash W)$*.*
4. $\lim_{|x| \to \infty} l_\varepsilon(x) = \infty$ *and* $\lim_{x \to a \in W} l_\varepsilon(x) = \infty$*.*



Remark

The properties of l_ε rely on the different rates between polynomials, logarithm function and exponential function.

Theorem

Let $(\varepsilon_k)_{k \in \mathbb{N}} \subset \mathbb{R}_+^*$ *such that* $\varepsilon_k \downarrow 0$ *as* $k \to \infty$*. Let* $M > 0$*. For every* $k \in \mathbb{N}$*, denote* $l_k := l_{\varepsilon_k}$ *and denote*

$$J_k(x) := l_k(x) - \log(M - l_0(x)), \ x \in \mathbb{R}^n.$$

Let $(x_k)_{k \in \mathbb{N}} \subset \mathbb{R}^n \backslash W$ *such that* $l_0(x_k) \leq M$ *and* $\nabla J_k(x_k) = 0$*.*
Then $(x_k)_{k \in}$ *is a bounded sequence and every convergent subsequence of* $(x_k)_{k \in \mathbb{N}}$ *converges to a global minimum of* $\min\{\varphi(x) : \ x \in \mathbb{R}^n\}$*.*

Building algorithm

Building algorithm

Algorithm

Let $z_0 \in^n$ such that $\nabla\varphi(z_0) \neq 0$. Let $\tau \in (0, 1)$ be small. Set $M = l_0(z_0) + 1$. Set $k = 0$ and do the following steps:

Building algorithm

Algorithm

Let $z_0 \in^n$ such that $\nabla\varphi(z_0) \neq 0$. Let $\tau \in (0,1)$ be small. Set $M = l_0(z_0) + 1$. Set $k = 0$ and do the following steps:

1. Find a local minimum z of unconstrained problem $\min\limits_{x\in\mathbb{R}^n} J(x)$, where

$$J(x) = \begin{cases} J_k(x) & \text{if } l_0(x) \neq M; \\ \infty & \text{if } l_0(x) = M, \end{cases}$$

   by quasi-Newton method with initial point z_0. Go to step 2.

Building algorithm

Algorithm

Let $z_0 \in^n$ such that $\nabla\varphi(z_0) \neq 0$. Let $\tau \in (0, 1)$ be small. Set $M = l_0(z_0) + 1$. Set $k = 0$ and do the following steps:

1. Find a local minimum z of unconstrained problem $\min\limits_{x\in\mathbb{R}^n} J(x)$, where

$$J(x) = \begin{cases} J_k(x) & \text{if } l_0(x) \neq M; \\ \infty & \text{if } l_0(x) = M, \end{cases}$$

   by quasi-Newton method with initial point z_0. Go to step 2.
2. If $\varphi(z) \leq \tau$, stop. Otherwise, set $k := k + 1$ and go to step 1.

Building algorithm

Algorithm

Let $z_0 \in^n$ such that $\nabla\varphi(z_0) \neq 0$. Let $\tau \in (0,1)$ be small. Set $M = l_0(z_0) + 1$. Set $k = 0$ and do the following steps:

1. Find a local minimum z of unconstrained problem $\min\limits_{x \in \mathbb{R}^n} J(x)$, where

$$J(x) = \begin{cases} J_k(x) & \text{if } l_0(x) \neq M; \\ \infty & \text{if } l_0(x) = M, \end{cases}$$

   by quasi-Newton method with initial point z_0. Go to step 2.

2. If $\varphi(z) \leq \tau$, stop. Otherwise, set $k := k + 1$ and go to step 1.

Example

Economics problem

$$\begin{cases} x_k^2 + x_1(x_2 + \ldots + x_6) - 2x_1x_k - 4x_1^2 = 0, \\ \hfill k = 1, \ldots, 6. \end{cases}$$

An approximation of solution:

$$(2.5876, 2.5340, 2.6294, 2.6373, 2.5098, 2.6261).$$

Computing time: 0.603346 seconds.

Building algorithm

Algorithm

Let $z_0 \in^n$ such that $\nabla\varphi(z_0) \neq 0$. Let $\tau \in (0,1)$ be small. Set $M = l_0(z_0) + 1$. Set $k = 0$ and do the following steps:

1. Find a local minimum z of unconstrained problem $\min_{x\in\mathbb{R}^n} J(x)$, where

$$J(x) = \begin{cases} J_k(x) & \text{if } l_0(x) \neq M; \\ \infty & \text{if } l_0(x) = M, \end{cases}$$

   by quasi-Newton method with initial point z_0. Go to step 2.
2. If $\varphi(z) \leq \tau$, stop. Otherwise, set $k := k + 1$ and go to step 1.

Example

Economics problem

$$\begin{cases} x_k^2 + x_1(x_2 + \ldots + x_6) - 2x_1x_k - 4x_1^2 = 0, \\ \qquad k = 1, \ldots, 6. \end{cases}$$

An approximation of solution:

$$(2.5876, 2.5340, 2.6294, 2.6373, 2.5098, 2.6261).$$

Computing time: 0.603346 seconds.

Remark

This algorithm does not depend on the degree of f_i.

Polynomial systems: A new algorithm adding sphere inequalities constraints

Adding sphere inequalities constraints (ASIC)

Adding sphere inequalities constraints (ASIC)

Challenge

Find an approximate real point in $S(g, h)$ if it is nonempty?

Adding sphere inequalities constraints (ASIC)

Challenge

Find an approximate real point in $S(g, h)$ if it is nonempty?

Assume that $S(g, h) \neq \emptyset$.

Challenge

Find an approximate real point in $S(g, h)$ if it is nonempty?

Assume that $S(g, h) \neq \emptyset$.
Let $(a_t)_{t=0,1,\ldots,n} \subset \mathbb{R}^n$ such that $a_t - a_0,\ t = 1, ..., n$ are linear independent.

Adding sphere inequalities constraints (ASIC)

Challenge

Find an approximate real point in $S(g,h)$ if it is nonempty?

Assume that $S(g,h) \neq \emptyset$.
Let $(a_t)_{t=0,1,\ldots,n} \subset \mathbb{R}^n$ such that $a_t - a_0,\ t = 1, ..., n$ are linear independent.

$$r_0 = d(a_0, S(g,h));$$

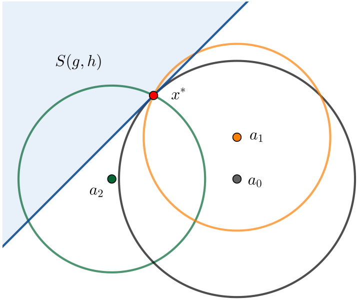
$$r_t = d\left(a_t, S(g,h) \cap \overline{B(a_0,r_0)} \cap ... \cap \overline{B(a_{t-1},r_{t-1})}\right), \quad t = 1, ..., n.$$

Challenge

Find an approximate real point in $S\left(g, h\right)$ if it is nonempty?

Assume that $S\left(g, h\right) \neq \emptyset$.
Let $\left(a_t\right)_{t=0,1,\ldots,n} \subset \mathbb{R}^n$ such that $a_t - a_0,\ t = 1, ..., n$ are linear independent.

$$r_0 = d\left(a_0, S\left(g, h\right)\right);$$
$$r_t = d\left(a_t, S\left(g, h\right) \cap \overline{B\left(a_0, r_0\right)} \cap \ldots \cap \overline{B\left(a_{t-1}, r_{t-1}\right)}\right), \quad t = 1, ..., n.$$



Challenge

Find an approximate real point in $S(g,h)$ if it is nonempty?

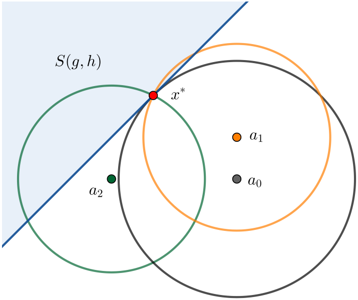
Assume that $S(g,h) \neq \emptyset$.
Let $(a_t)_{t=0,1,\ldots,n} \subset \mathbb{R}^n$ such that $a_t - a_0,\ t = 1,\ldots,n$ are linear independent.

$$r_0 = d\left(a_0, S(g,h)\right);$$
$$r_t = d\left(a_t, S(g,h) \cap \overline{B(a_0,r_0)} \cap \ldots \cap \overline{B(a_{t-1},r_{t-1})}\right), \quad t = 1,\ldots,n.$$

Lemma

$S(g,h) \cap \overline{B(a_0,r_0)} \cap \ldots \cap \overline{B(a_n,r_n)} = \{x^*\}$ and $x^* = A^{-1}b$ with
$A = \begin{pmatrix} a_1 - a_0 & \ldots & a_n - a_0 \end{pmatrix}$ and

$$b = -\frac{1}{2}\begin{pmatrix} r_1^2 - r_0^2 - |a_1|^2 + |a_0|^2 \\ \ldots \\ r_n^2 - r_0^2 - |a_n|^2 + |a_0|^2 \end{pmatrix}.$$



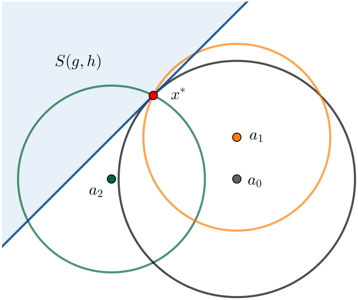
Adding sphere inequalities constraints (ASIC)

Challenge

Find an approximate real point in $S(g,h)$ if it is nonempty?

Assume that $S(g,h) \neq \emptyset$.
Let $(a_t)_{t=0,1,\dots,n} \subset \mathbb{R}^n$ such that $a_t - a_0,\ t = 1, \dots, n$ are linear independent.

$$r_0 = d\left(a_0, S(g,h)\right);$$
$$r_t = d\left(a_t, S(g,h) \cap \overline{B(a_0,r_0)} \cap \dots \cap \overline{B(a_{t-1}, r_{t-1})}\right), \quad t = 1, \dots, n.$$



Lemma

$S(g,h) \cap \overline{B(a_0,r_0)} \cap \dots \cap \overline{B(a_n,r_n)} = \{x^*\}$ *and* $x^* = A^{-1}b$ *with* $A = \begin{pmatrix} a_1 - a_0 & \dots & a_n - a_0 \end{pmatrix}$ *and*

$$b = -\frac{1}{2}\begin{pmatrix} r_1^2 - r_0^2 - |a_1|^2 + |a_0|^2 \\ \dots \\ r_n^2 - r_0^2 - |a_n|^2 + |a_0|^2 \end{pmatrix}.$$

$\varphi_a^\rho := \rho - |x-a|^2$ for $a \in \mathbb{R}^n$ and $\rho \geq 0$.

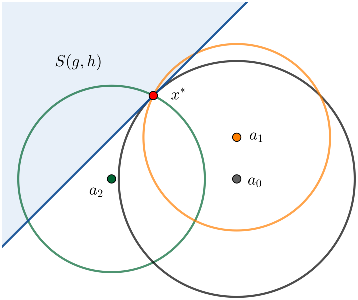
Adding sphere inequalities constraints (ASIC)

Challenge

Find an approximate real point in $S(g,h)$ if it is nonempty?

Assume that $S(g,h) \neq \emptyset$.
Let $(a_t)_{t=0,1,\ldots,n} \subset \mathbb{R}^n$ such that $a_t - a_0,\ t = 1, \ldots, n$ are linear independent.

$$r_0 = d\left(a_0, S(g,h)\right);$$
$$r_t = d\left(a_t, S(g,h) \cap \overline{B(a_0, r_0)} \cap \ldots \cap \overline{B(a_{t-1}, r_{t-1})}\right), \quad t = 1, \ldots, n.$$



Lemma

$S(g,h) \cap \overline{B(a_0, r_0)} \cap \ldots \cap \overline{B(a_n, r_n)} = \{x^*\}$ and $x^* = A^{-1}b$ with $A = \begin{pmatrix} a_1 - a_0 & \ldots & a_n - a_0 \end{pmatrix}$ and

$$b = -\frac{1}{2}\begin{pmatrix} r_1^2 - r_0^2 - |a_1|^2 + |a_0|^2 \\ \ldots \\ r_n^2 - r_0^2 - |a_n|^2 + |a_0|^2 \end{pmatrix}.$$

$\varphi_a^\rho := \rho - |x - a|^2$ for $a \in \mathbb{R}^n$ and $\rho \geq 0$. Then

$$r_0^2 = \min_{x \in S(g,h)} |x - a_0|^2;$$
$$r_t^2 = \min_{x \in S\left(g \cup \left\{\varphi_{a_0}^{r_0^2}, \ldots, \varphi_{a_{t-1}}^{r_{t-1}^2}\right\}, h\right)} |x - a_t|^2, \quad t = 1, \ldots, n.$$

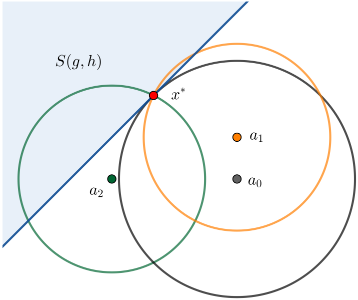
Adding sphere inequalities constraints (ASIC)

Challenge

Find an approximate real point in $S(g,h)$ if it is nonempty?

Assume that $S(g,h) \neq \emptyset$.
Let $(a_t)_{t=0,1,\dots,n} \subset \mathbb{R}^n$ such that $a_t - a_0,\ t = 1,\dots,n$ are linear independent.

$$r_0 = d(a_0, S(g,h));$$
$$r_t = d\left(a_t, S(g,h) \cap \overline{B(a_0,r_0)} \cap \dots \cap \overline{B(a_{t-1},r_{t-1})}\right),\quad t = 1,\dots,n.$$



Lemma

$S(g,h) \cap \overline{B(a_0,r_0)} \cap \dots \cap \overline{B(a_n,r_n)} = \{x^*\}$ and $x^* = A^{-1}b$ with
$A = \begin{pmatrix} a_1 - a_0 & \dots & a_n - a_0 \end{pmatrix}$ *and*

$$b = -\frac{1}{2}\begin{pmatrix} r_1^2 - r_0^2 - |a_1|^2 + |a_0|^2 \\ \dots \\ r_n^2 - r_0^2 - |a_n|^2 + |a_0|^2 \end{pmatrix}.$$

$\varphi_a^\rho := \rho - |x-a|^2$ for $a \in \mathbb{R}^n$ and $\rho \geq 0$.
Then

$$r_0^2 = \min_{x \in S(g,h)} |x - a_0|^2;$$
$$r_t^2 = \min_{x \in S\left(g \cup \left\{\varphi_{a_0}^{r_0^2},\dots,\varphi_{a_{t-1}}^{r_{t-1}^2}\right\},h\right)} |x - a_t|^2,\quad t = 1,\dots,n.$$

Idea: Use the Lasserre's hierarchy to find approximations of r_t.

Numerical scheme of radius

Numerical scheme of radius

Let $\bar{\varepsilon} > 0$ be fixed. For every $\alpha = (\alpha_0, ..., \alpha_n) \in \mathbb{N}^{n+1}$,

Numerical scheme of radius

Let $\bar{\varepsilon} > 0$ be fixed. For every $\alpha = (\alpha_0, ..., \alpha_n) \in \mathbb{N}^{n+1}$,

$$\rho_0(\alpha_0) := \sup\left\{\lambda : \theta^{\alpha_0}\left(|x - a_0|^2 + \bar{\varepsilon}\theta - \lambda\right) \in Q(g,h)_{\alpha_0+1}\right\}$$

$$\rho_t(\alpha_0, .., \alpha_t) := \sup\left\{\lambda : |x - a_t|^2 - \lambda \in Q\left(g \cup \left\{\varphi_{a_0}^{\rho_0(\alpha_0)}, ..., \varphi_{a_{t-1}}^{\rho_{t-1}(\alpha_0,..,\alpha_{t-1})}\right\}, h\right)_{\alpha_t+1}\right\}$$

$$t = 1, ..., n.$$

Numerical scheme of radius

Let $\bar{\varepsilon} > 0$ be fixed. For every $\alpha = (\alpha_0, ..., \alpha_n) \in \mathbb{N}^{n+1}$,

$$\rho_0\left(\alpha_0\right) := \sup\left\{\lambda : \theta^{\alpha_0}\left(|x - a_0|^2 + \bar{\varepsilon}\theta - \lambda\right) \in Q(g,h)_{\alpha_0+1}\right\}$$

$$\rho_t\left(\alpha_0, .., \alpha_t\right) := \sup\left\{\lambda : |x - a_t|^2 - \lambda \in Q\left(g \cup \left\{\varphi_{a_0}^{\rho_0(\alpha_0)}, ..., \varphi_{a_{t-1}}^{\rho_{t-1}(\alpha_0,..,\alpha_{t-1})}\right\}, h\right)_{\alpha_t+1}\right\}$$

$$t = 1, ..., n.$$

Theorem

There exists a function $\delta : \mathbb{R}_+^* \to \mathbb{R}_+$ *such that* $\delta\left(\varepsilon\right) \downarrow 0$ *as* $\varepsilon \downarrow 0$ *which satisfies*

Numerical scheme of radius

Let $\bar{\varepsilon} > 0$ be fixed. For every $\alpha = (\alpha_0, ..., \alpha_n) \in \mathbb{N}^{n+1}$,

$$\rho_0(\alpha_0) := \sup\left\{\lambda : \theta^{\alpha_0}\left(|x - a_0|^2 + \bar{\varepsilon}\theta - \lambda\right) \in Q(g,h)_{\alpha_0+1}\right\}$$

$$\rho_t(\alpha_0, .., \alpha_t) := \sup\left\{\lambda : |x - a_t|^2 - \lambda \in Q\left(g \cup \left\{\varphi_{a_0}^{\rho_0(\alpha_0)}, ..., \varphi_{a_{t-1}}^{\rho_{t-1}(\alpha_0, .., \alpha_{t-1})}\right\}, h\right)_{\alpha_t+1}\right\}$$

$$t = 1, ..., n.$$

Theorem

There exists a function $\delta : \mathbb{R}_+^* \to \mathbb{R}_+$ *such that* $\delta(\varepsilon) \downarrow 0$ *as* $\varepsilon \downarrow 0$ *which satisfies*

$$\exists \Lambda \in \mathbb{N}^{n+1} : \rho_t(\Lambda_0, .., \Lambda_t) \in \left[r_t^2 - \delta(\bar{\varepsilon}), r_t^2 + \delta(\bar{\varepsilon})\right], \quad t = 0, ..., n.$$

Numerical scheme of radius

Let $\bar{\varepsilon} > 0$ be fixed. For every $\alpha = (\alpha_0, ..., \alpha_n) \in \mathbb{N}^{n+1}$,

$$\rho_0(\alpha_0) := \sup\left\{\lambda : \theta^{\alpha_0}\left(|x - a_0|^2 + \bar{\varepsilon}\theta - \lambda\right) \in Q(g, h)_{\alpha_0+1}\right\}$$

$$\rho_t(\alpha_0, .., \alpha_t) := \sup\left\{\lambda : |x - a_t|^2 - \lambda \in Q\left(g \cup \left\{\varphi_{a_0}^{\rho_0(\alpha_0)}, ..., \varphi_{a_{t-1}}^{\rho_{t-1}(\alpha_0, .., \alpha_{t-1})}\right\}, h\right)_{\alpha_t+1}\right\}$$

$$t = 1, ..., n.$$

Theorem

There exists a function $\delta : \mathbb{R}_+^* \to \mathbb{R}_+$ *such that* $\delta(\varepsilon) \downarrow 0$ *as* $\varepsilon \downarrow 0$ *which satisfies*

$$\exists \Lambda \in \mathbb{N}^{n+1} : \rho_t(\Lambda_0, .., \Lambda_t) \in \left[r_t^2 - \delta(\bar{\varepsilon}), r_t^2 + \delta(\bar{\varepsilon})\right], \quad t = 0, ..., n.$$

Lemma

If $\operatorname{rank}(M_d(y)) = 1$ *(with* $y_0 = 1$*), then* y *has representing (1-atomic) measure supported on* $x^* = (y_{e_1}, ..., y_{e_n})$ *where* e_i *is natural basis of* $\mathbb{R}^n$*.*

Numerical scheme of radius

Let $\bar{\varepsilon} > 0$ be fixed. For every $\alpha = (\alpha_0, ..., \alpha_n) \in \mathbb{N}^{n+1}$,

$$\rho_0(\alpha_0) := \sup\left\{\lambda : \theta^{\alpha_0}\left(|x - a_0|^2 + \bar{\varepsilon}\theta - \lambda\right) \in Q(g,h)_{\alpha_0+1}\right\}$$

$$\rho_t(\alpha_0, .., \alpha_t) := \sup\left\{\lambda : |x - a_t|^2 - \lambda \in Q\left(g \cup \left\{\varphi_{a_0}^{\rho_0(\alpha_0)}, ..., \varphi_{a_{t-1}}^{\rho_{t-1}(\alpha_0,..,\alpha_{t-1})}\right\}, h\right)_{\alpha_t+1}\right\}$$

$$t = 1, ..., n.$$

Theorem

There exists a function $\delta : \mathbb{R}_+^* \to \mathbb{R}_+$ such that $\delta(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$ which satisfies
$$\exists \Lambda \in \mathbb{N}^{n+1} : \rho_t(\Lambda_0, .., \Lambda_t) \in \left[r_t^2 - \delta(\bar{\varepsilon}), r_t^2 + \delta(\bar{\varepsilon})\right], \quad t = 0, ..., n.$$

Lemma

If $rank(M_d(y)) = 1$ (with $y_0 = 1$), then y has representing (1-atomic) measure supported on $x^* = (y_{e_1}, ..., y_{e_n})$ where e_i is natural basis of $\mathbb{R}^n$.

ASIC algorithm

For every $\alpha \in \mathbb{N}^{n+1}$, set $t = 0$ and do:

1. Solve the dual of SDP to get $\rho_t(\alpha_0, .., \alpha_t)$ and $M_d(y)$.
   If $t = 1$ or $t \leq n - 1$ and $rank(M_d(y)) > 1$, set $t = t + 1$ and do again step 1.
   If $2 \leq t \leq n - 1$ and $rank(M_d(y)) = 1$, go to step 2.
   If $t = n$, go to step 3.
2. Extract $\bar{x}$ from first moment submatrix of $M_d(y)$ and stop.
3. Solve $\bar{x} = A^{-1}\bar{b}$ with $\bar{b}$ is formed from b by replacing r_t^2 by $\rho_t(\alpha_0, .., \alpha_t)$ and stop.

Recall previous method obtaining optimizer(s) of POP by flat-extension condition

Recall previous method obtaining optimizer(s) of POP by flat-extension condition

M Laurent.
Revisiting two theorems of Curto and Fialkow on moment matrices.
Proceedings of the American Mathematical Society, 2005.

D Henrion, JB Lasserre.
Detecting global optimality and extracting solutions in GloptiPoly.
Positive polynomials in control - Springer, 2005.

Recall previous method obtaining optimizer(s) of POP by flat-extension condition

M Laurent.
Revisiting two theorems of Curto and Fialkow on moment matrices.
Proceedings of the American Mathematical Society, 2005.

D Henrion, JB Lasserre.
Detecting global optimality and extracting solutions in GloptiPoly.
Positive polynomials in control - Springer, 2005.

Lemma

Assume solution y of SPD relaxation of optimal value τ_k satisfies the flat-extension condition

$$\operatorname{rank}\left(M_k\left(y\right)\right)=\operatorname{rank}\left(M_{k-1}\left(y\right)\right).$$

Then $\tau_k = f^*$ there exist $\lambda_1, \ldots, \lambda_r \in$ and $x_1^*, \ldots, x_r^* \in \mathbb{R}^n$ such that y has the representing r-atomic measure

$$\mu=\sum_{i=1}^{r}\lambda_i\delta_{x_i^*}.$$

Moreover, $\operatorname{supp}\left(\mu\right)=\left\{x_i^*:\ i=1,...,r\right\}$, which belongs to the set of all minimizers of $f^*=\min\left\{f\left(x\right):\ x\in S\left(g,h\right)\right\}$.

Recall previous method obtaining optimizer(s) of POP by flat-extension condition

M Laurent.
Revisiting two theorems of Curto and Fialkow on moment matrices.
Proceedings of the American Mathematical Society, 2005.

D Henrion, JB Lasserre.
Detecting global optimality and extracting solutions in GloptiPoly.
Positive polynomials in control - Springer, 2005.

Lemma

Assume solution y of SPD relaxation of optimal value τ_k satisfies the flat-extension condition

$$\operatorname{rank}(M_k(y)) = \operatorname{rank}(M_{k-1}(y)).$$

Then $\tau_k = f^*$ there exist $\lambda_1, \ldots, \lambda_r \in$ and $x_1^*, \ldots, x_r^* \in \mathbb{R}^n$ such that y has the representing r-atomic measure

$$\mu = \sum_{i=1}^{r} \lambda_i \delta_{x_i^*}.$$

Moreover, $\operatorname{supp}(\mu) = \{x_i^* : i = 1, ..., r\}$, which belongs to the set of all minimizers of $f^* = \min\{f(x) : x \in S(g, h)\}$.

Algorithm (Henrion, Lasserre)

1. Find the Cholesky factorization VV^T of $M_d(y)$.
2. Reduce V to an echelon form U.
3. Extract from U the multiplication matrices N_i, $i = 1, ..., n$.
4. Compute $N := \sum_{i=1}^{n} \lambda_i N_i$ with randomly generated coefficients λ_i, and the Schur decomposition $N = QTQ^T$. Compute $Q = \begin{pmatrix} q_1 & q_2 & \ldots & q_r \end{pmatrix}$ and $\left(x_j^*\right)_i = q_j^T N_i q_j$, for each $i = 1, \ldots, n$, and each $j = 1, \ldots, r$.

Advantages of ASIC algorithm and Numerical experiments

Remark

It is the promising algorithm because of the following advantages:

Remark

It is the promising algorithm because of the following advantages:

1. solving the **systems of high dimensional varieties** by using hierachy SDP relaxations.

Advantages of ASIC algorithm and Numerical experiments

Remark

It is the promising algorithm because of the following advantages:

1. solving the **systems of high dimensional varieties** by using hierachy SDP relaxations.
2. giving an **approximate real solution** of the polynomial system at **every relaxation**. We can use this approximation as initial point for Newton method when the system is square.

Advantages of ASIC algorithm and Numerical experiments

Remark

It is the promising algorithm because of the following advantages:

1. solving the **systems of high dimensional varieties** by using hierachy SDP relaxations.
2. giving an **approximate real solution** of the polynomial system at **every relaxation**. We can use this approximation as initial point for Newton method when the system is square.
3. applying in **obtaining an approximation of a minimizer of POP** at every the new Lasserre's hierarchy relaxation via solving the system $S(g \cup \{\rho_k - f\}, h)$.

Advantages of ASIC algorithm and Numerical experiments

Remark

It is the promising algorithm because of the following advantages:

1. solving the **systems of high dimensional varieties** by using hierachy SDP relaxations.
2. giving an **approximate real solution** of the polynomial system at **every relaxation**. We can use this approximation as initial point for Newton method when the system is square.
3. applying in **obtaining an approximation of a minimizer of POP** at every the new Lasserre's hierarchy relaxation via solving the system $S(g \cup \{\rho_k - f\}, h)$.
4. **replacing the extraction of the flat-extension condition** in the case that the flat-extension condition is not satisfied (since the set of minimizers is high dimension varieties or the order relaxation k is too high to reach $\tau_k = f^*$).

Remark

It is the promising algorithm because of the following advantages:

1. solving the **systems of high dimensional varieties** by using hierachy SDP relaxations.
2. giving an **approximate real solution** of the polynomial system at **every relaxation**. We can use this approximation as initial point for Newton method when the system is square.
3. applying in **obtaining an approximation of a minimizer of POP** at every the new Lasserre's hierarchy relaxation via solving the system $S(g \cup \{\rho_k - f\}, h)$.
4. **replacing the extraction of the flat-extension condition** in the case that the flat-extension condition is not satisfied (since the set of minimizers is high dimension varieties or the order relaxation k is too high to reach $\tau_k = f^*$).
5. obtaining an approximation of a minimizer at the low order relaxations is suitable for **middle-scale POP** ($d, n \leq 20$).

Remark

It is the promising algorithm because of the following advantages:

1. solving the **systems of high dimensional varieties** by using hierachy SDP relaxations.
2. giving an **approximate real solution** of the polynomial system at **every relaxation**. We can use this approximation as initial point for Newton method when the system is square.
3. applying in **obtaining an approximation of a minimizer of POP** at every the new Lasserre's hierarchy relaxation via solving the system $S\left(g \cup \{\rho_k - f\}, h\right)$.
4. **replacing the extraction of the flat-extension condition** in the case that the flat-extension condition is not satisfied (since the set of minimizers is high dimension varieties or the order relaxation k is too high to reach $\tau_k = f^*$).
5. obtaining an approximation of a minimizer at the low order relaxations is suitable for **middle-scale POP** ($d, n \leq 20$).

Solving Dietmaier's system of **12 variable**, **12 equations** and **total degree 2**

Advantages of ASIC algorithm and Numerical experiments

Remark

It is the promising algorithm because of the following advantages:

1. solving the **systems of high dimensional varieties** by using hierachy SDP relaxations.
2. giving an **approximate real solution** of the polynomial system at **every relaxation**. We can use this approximation as initial point for Newton method when the system is square.
3. applying in **obtaining an approximation of a minimizer of POP** at every the new Lasserre's hierarchy relaxation via solving the system $S(g \cup \{\rho_k - f\}, h)$.
4. **replacing the extraction of the flat-extension condition** in the case that the flat-extension condition is not satisfied (since the set of minimizers is high dimension varieties or the order relaxation k is too high to reach $\tau_k = f^*$).
5. obtaining an approximation of a minimizer at the low order relaxations is suitable for **middle-scale POP** $(d, n \leq 20)$.

Solving Dietmaier's system of 12 variable, 12 equations and total degree 2

This algorithm takes $\sim$ **4** seconds while GloptiPoly takes $\sim$ **64** seconds.

Advantages of ASIC algorithm and Numerical experiments

Remark

It is the promising algorithm because of the following advantages:

1. solving the **systems of high dimensional varieties** by using hierachy SDP relaxations.
2. giving an **approximate real solution** of the polynomial system at **every relaxation**. We can use this approximation as initial point for Newton method when the system is square.
3. applying in **obtaining an approximation of a minimizer of POP** at every the new Lasserre's hierarchy relaxation via solving the system $S\left(g \cup \{\rho_k - f\}, h\right)$.
4. **replacing the extraction of the flat-extension condition** in the case that the flat-extension condition is not satisfied (since the set of minimizers is high dimension varieties or the order relaxation k is too high to reach $\tau_k = f^*$).
5. obtaining an approximation of a minimizer at the low order relaxations is suitable for **middle-scale POP** $(d, n \leq 20)$.

Solving Dietmaier's system of 12 variable, 12 equations and total degree 2

This algorithm takes $\sim$ **4** seconds while GloptiPoly takes $\sim$ **64** seconds.

Extracting a minimizer of unconstrained minimization problem for Mozkin polynomial $f = x_1^2 x_2^2 \left(x_1^2 + x_2^2 - 1\right)$

Remark

It is the promising algorithm because of the following advantages:

1. solving the **systems of high dimensional varieties** by using hierachy SDP relaxations.
2. giving an **approximate real solution** of the polynomial system at **every relaxation**. We can use this approximation as initial point for Newton method when the system is square.
3. applying in **obtaining an approximation of a minimizer of POP** at every the new Lasserre's hierarchy relaxation via solving the system $S\left(g \cup \{\rho_k - f\}, h\right)$.
4. **replacing the extraction of the flat-extension condition** in the case that the flat-extension condition is not satisfied (since the set of minimizers is high dimension varieties or the order relaxation k is too high to reach $\tau_k = f^*$).
5. obtaining an approximation of a minimizer at the low order relaxations is suitable for **middle-scale POP** $(d, n \leq 20)$.

Solving Dietmaier's system of 12 variable, 12 equations and total degree 2

This algorithm takes $\sim$ **4** seconds while GloptiPoly takes $\sim$ **64** seconds.

Extracting a minimizer of unconstrained minimization problem for Mozkin polynomial $f = x_1^2 x_2^2 \left(x_1^2 + x_2^2 - 1\right)$

$\rho_1 = -0.037037049511779$ with approximation of minimizer:

$$\begin{pmatrix} 0.577878471539354 \\ 0.576441444303031 \end{pmatrix}$$

Christoffel functions: Extracting information from moments combined with Newton's method

Chistoffel function and related works

Let μ be finite Borel measure.

Christoffel function for $M_d(\mu) \succ 0$

$$\Lambda_{\mu,d}(x) := \left[v_d(x)^T M_d(\mu)^{-1} v_d(x)\right]^{-1}, \forall x \in \mathbb{R}^n$$

Chistoffel function and related works

Let μ be finite Borel measure.

Christoffel function for $M_d(\mu) \succ 0$

$$\Lambda_{\mu,d}(x) := \left[v_d(x)^T M_d(\mu)^{-1} v_d(x) \right]^{-1}, \forall x \in \mathbb{R}^n$$

JB Lasserre and E Pauwels.
The empirical Christoffel function with applications in data analysis.
In *Advances in Computational Mathematics*, Springer, 2019.

E Pauwels, M Putinar and JB Lasserre.
Data analysis from empirical moments and the Christoffel function
In *arXiv preprint arXiv:1810.08480*, 2018.

Chistoffel function and related works

Let μ be finite Borel measure.

Christoffel function for $M_d(\mu) \succ 0$

$$\Lambda_{\mu,d}(x) := \left[v_d(x)^T M_d(\mu)^{-1} v_d(x) \right]^{-1}, \forall x \in \mathbb{R}^n$$

JB Lasserre and E Pauwels.
The empirical Christoffel function with applications in data analysis.
In *Advances in Computational Mathematics*, Springer, 2019.

E Pauwels, M Putinar and JB Lasserre.
Data analysis from empirical moments and the Christoffel function
In *arXiv preprint arXiv:1810.08480*, 2018.

Theorem for the Lebesgue measure μ supported on compact set S with some assumptions of d_k and γ_k (Lasserre, Pauwels)

For every $k \in \mathbb{N}$, $S_k := \left\{ x \in \mathbb{R}^n : \Lambda_{\mu,d_k}(x) \geq \gamma_k \right\}$.
Then as $k \to \infty$,
$d_H(S_k, S) \to 0$ and $d_H(\partial S_k, \partial S) \to 0$.

Chistoffel function and related works

Let μ be finite Borel measure.

Christoffel function for $M_d(\mu) \succ 0$

$$\Lambda_{\mu,d}(x) := \left[v_d(x)^T M_d(\mu)^{-1} v_d(x) \right]^{-1}, \forall x \in \mathbb{R}^n$$

JB Lasserre and E Pauwels.
The empirical Christoffel function with applications in data analysis.
In *Advances in Computational Mathematics*, Springer, 2019.

E Pauwels, M Putinar and JB Lasserre.
Data analysis from empirical moments and the Christoffel function
In *arXiv preprint arXiv:1810.08480*, 2018.

Theorem for the Lebesgue measure μ supported on compact set S with some assumptions of d_k and γ_k (Lasserre, Pauwels)

For every $k \in \mathbb{N}$, $S_k := \left\{ x \in \mathbb{R}^n : \Lambda_{\mu,d_k}(x) \geq \gamma_k \right\}$.
Then as $k \to \infty$,
$d_H(S_k, S) \to 0$ and $d_H(\partial S_k, \partial S) \to 0$.

S Marx, E Pauwels, T Weisser, D Henrion and J Lasserre.
Tractable semi-algebraic approximation using Christoffel-Darboux kernel.
hal.archives-ouvertes.fr, 2019.

Chistoffel function and related works

Let μ be finite Borel measure.

Christoffel function for $M_d(\mu) \succ 0$

$$\Lambda_{\mu,d}(x) := \left[v_d(x)^T M_d(\mu)^{-1} v_d(x) \right]^{-1}, \forall x \in \mathbb{R}^n$$

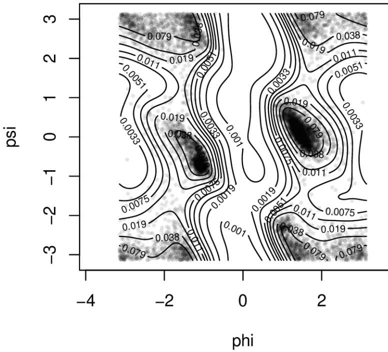
JB Lasserre and E Pauwels.
The empirical Christoffel function with applications in data analysis.
In *Advances in Computational Mathematics*, Springer, 2019.

E Pauwels, M Putinar and JB Lasserre.
Data analysis from empirical moments and the Christoffel function
In *arXiv preprint arXiv:1810.08480*, 2018.

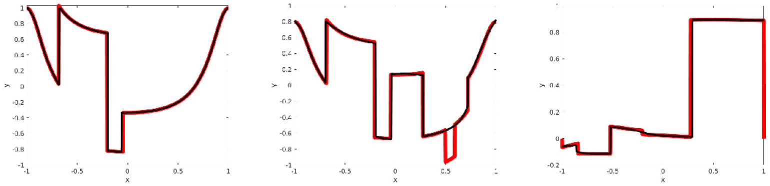
Theorem for the Lebesgue measure μ supported on compact set S with some assumptions of d_k and γ_k (Lasserre, Pauwels)

For every $k \in \mathbb{N}$, $S_k := \left\{ x \in \mathbb{R}^n : \Lambda_{\mu,d_k}(x) \geq \gamma_k \right\}$.
Then as $k \to \infty$,
$d_H(S_k, S) \to 0$ and $d_H(\partial S_k, \partial S) \to 0$.

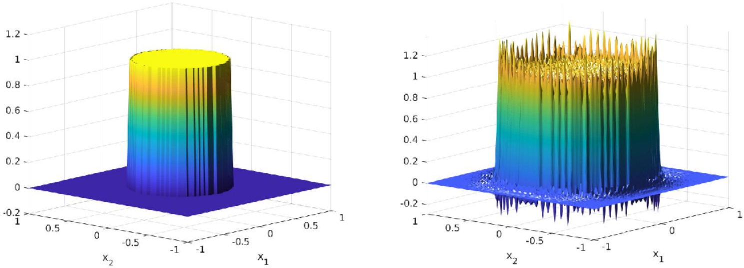
S Marx, E Pauwels, T Weisser, D Henrion and J Lasserre.
Tractable semi-algebraic approximation using Christoffel-Darboux kernel.
hal.archives-ouvertes.fr, 2019.



The level sets of the empirical Christoffel functions evaluated on the sphere in $\mathbb{R}^4$ by [Pauwels et al. 18]



(a) Degree 10 semialgebraic approximations (black) for the discontinuous univariate functions (red) of Examples 65 (left), 66 (middle) and 67 (right)



(b) Degree 4 (left) semialgebraic approximation, and Chebyshev polynomial approximation (right) of the indicator function of a disk

Semialgebraic approximations by [Marx et al. 19]

Convergence theorem

Convergence theorem

Problem

Find the support $\operatorname{supp}(\mu) := \{z_i\} \subset \mathbb{R}^n$ and the weights $\{\lambda_i\} \subset \mathbb{R}^*$ of signed atomic measure $\mu := \sum_{i=1}^{k} \lambda_i \delta_{z_i}$ from knowledge of moments of μ.

Convergence theorem

Problem

Find the support $\mathrm{supp}\,(\mu) := \{z_i\} \subset \mathbb{R}^n$ and the weights $\{\lambda_i\} \subset \mathbb{R}^*$ of signed atomic measure $\mu := \sum_{i=1}^{k} \lambda_i \delta_{z_i}$ from knowledge of moments of μ.

Let $N \in \mathbb{S}_{++}^{s(d)}$. Let $\varepsilon > 0$ such that $\varepsilon N + M_d(\mu)$ is full rank.

Convergence theorem

Problem

Find the support $\operatorname{supp}(\mu) := \{z_i\} \subset \mathbb{R}^n$ and the weights $\{\lambda_i\} \subset \mathbb{R}^*$ of signed atomic measure $\mu := \sum_{i=1}^{k} \lambda_i \delta_{z_i}$ from knowledge of moments of μ.

Let $N \in \mathbb{S}_{++}^{s(d)}$. Let $\varepsilon > 0$ such that $\varepsilon N + M_d(\mu)$ is full rank.

Perturbed Christoffel function

$\Lambda_{\mu,d,\varepsilon N} : \mathbb{R}^n \to \mathbb{R}$ defined by for $x \in \mathbb{R}^n$

$$\Lambda_{\mu,d,\varepsilon N}(x) := \left[v_d(x)^T (\varepsilon N + M_d(\mu))^{-1} v_d(x) \right]^{-1}.$$

Convergence theorem

Problem

Find the support $\operatorname{supp}(\mu) := \{z_i\} \subset \mathbb{R}^n$ and the weights $\{\lambda_i\} \subset \mathbb{R}^*$ of signed atomic measure $\mu := \sum_{i=1}^{k} \lambda_i \delta_{z_i}$ from knowledge of moments of μ.

Let $N \in \mathbb{S}_{++}^{s(d)}$. Let $\varepsilon > 0$ such that $\varepsilon N + M_d(\mu)$ is full rank.

Perturbed Christoffel function

$\Lambda_{\mu,d,\varepsilon N} : \mathbb{R}^n \to \mathbb{R}$ defined by for $x \in \mathbb{R}^n$

$$\Lambda_{\mu,d,\varepsilon N}(x) := \left[v_d(x)^T (\varepsilon N + M_d(\mu))^{-1} v_d(x)\right]^{-1}.$$

Theorem

Let $K \subset \mathbb{R}^n$ be compact set and $s(d) - r \geq n$. Let $\psi : \mathbb{R}_+^* \to \mathbb{R}_+^*$ be a function such that

$$\psi(\varepsilon)^{-1} \to \infty \text{ and } \varepsilon\psi(\varepsilon)^{-1} \to 0 \text{ as } \varepsilon \to 0^+.$$

Convergence theorem

Problem

Find the support $\operatorname{supp}(\mu) := \{z_i\} \subset \mathbb{R}^n$ and the weights $\{\lambda_i\} \subset \mathbb{R}^*$ of signed atomic measure $\mu := \sum_{i=1}^{k} \lambda_i \delta_{z_i}$ from knowledge of moments of μ.

Let $N \in \mathbb{S}_{++}^{s(d)}$. Let $\varepsilon > 0$ such that $\varepsilon N + M_d(\mu)$ is full rank.

Perturbed Christoffel function

$\Lambda_{\mu,d,\varepsilon N} : \mathbb{R}^n \to \mathbb{R}$ defined by for $x \in \mathbb{R}^n$

$$\Lambda_{\mu,d,\varepsilon N}(x) := \left[v_d(x)^T (\varepsilon N + M_d(\mu))^{-1} v_d(x)\right]^{-1}.$$

Theorem

Let $K \subset \mathbb{R}^n$ be compact set and $s(d) - r \geq n$. Let $\psi : \mathbb{R}_+^* \to \mathbb{R}_+^*$ be a function such that

$$\psi(\varepsilon)^{-1} \to \infty \text{ and } \varepsilon \psi(\varepsilon)^{-1} \to 0 \text{ as } \varepsilon \to 0^+.$$

For every $\varepsilon > 0$, denote the superlevel set of perturbed Christoffel function by

$$S_\varepsilon := \left\{x \in \mathbb{R}^n : \ \Lambda_{\mu,d,\varepsilon N}(x) \geq \psi(\varepsilon)\right\}.$$

Convergence theorem

Problem

Find the support $\operatorname{supp}(\mu) := \{z_i\} \subset \mathbb{R}^n$ and the weights $\{\lambda_i\} \subset \mathbb{R}^*$ of signed atomic measure $\mu := \sum_{i=1}^{k} \lambda_i \delta_{z_i}$ from knowledge of moments of μ.

Let $N \in \mathbb{S}_{++}^{s(d)}$. Let $\varepsilon > 0$ such that $\varepsilon N + M_d(\mu)$ is full rank.

Perturbed Christoffel function

$\Lambda_{\mu,d,\varepsilon N} : \mathbb{R}^n \to \mathbb{R}$ defined by for $x \in \mathbb{R}^n$

$$\Lambda_{\mu,d,\varepsilon N}(x) := \left[v_d(x)^T (\varepsilon N + M_d(\mu))^{-1} v_d(x) \right]^{-1}.$$

Theorem

Let $K \subset \mathbb{R}^n$ be compact set and $s(d) - r \geq n$. Let $\psi : \mathbb{R}_+^* \to \mathbb{R}_+^*$ be a function such that

$$\psi(\varepsilon)^{-1} \to \infty \text{ and } \varepsilon \psi(\varepsilon)^{-1} \to 0 \text{ as } \varepsilon \to 0^+.$$

For every $\varepsilon > 0$, denote the superlevel set of perturbed Christoffel function by

$$S_\varepsilon := \left\{ x \in \mathbb{R}^n : \ \Lambda_{\mu,d,\varepsilon N}(x) \geq \psi(\varepsilon) \right\}.$$

Then

Convergence theorem

Problem

Find the support $\operatorname{supp}(\mu) := \{z_i\} \subset \mathbb{R}^n$ and the weights $\{\lambda_i\} \subset \mathbb{R}^*$ of signed atomic measure $\mu := \sum_{i=1}^{k} \lambda_i \delta_{z_i}$ from knowledge of moments of μ.

Let $N \in \mathbb{S}_{++}^{s(d)}$. Let $\varepsilon > 0$ such that $\varepsilon N + M_d(\mu)$ is full rank.

Perturbed Christoffel function

$\Lambda_{\mu,d,\varepsilon N} : \mathbb{R}^n \to \mathbb{R}$ defined by for $x \in \mathbb{R}^n$

$$\Lambda_{\mu,d,\varepsilon N}(x) := \left[v_d(x)^T (\varepsilon N + M_d(\mu))^{-1} v_d(x) \right]^{-1}.$$

Theorem

Let $K \subset \mathbb{R}^n$ be compact set and $s(d) - r \geq n$. Let $\psi : \mathbb{R}_+^* \to \mathbb{R}_+^*$ be a function such that

$$\psi(\varepsilon)^{-1} \to \infty \text{ and } \varepsilon \psi(\varepsilon)^{-1} \to 0 \text{ as } \varepsilon \to 0^+.$$

For every $\varepsilon > 0$, denote the superlevel set of perturbed Christoffel function by

$$S_\varepsilon := \left\{ x \in \mathbb{R}^n : \Lambda_{\mu,d,\varepsilon N}(x) \geq \psi(\varepsilon) \right\}.$$

Then

1. $\operatorname{int}(S_\varepsilon) \supset v_d^{-1}(\operatorname{Im}(M_d(\mu))) \cap K$.

Convergence theorem

Problem

Find the support $\operatorname{supp}(\mu) := \{z_i\} \subset \mathbb{R}^n$ and the weights $\{\lambda_i\} \subset \mathbb{R}^*$ of signed atomic measure $\mu := \sum_{i=1}^{k} \lambda_i \delta_{z_i}$ from knowledge of moments of μ.

Let $N \in \mathbb{S}_{++}^{s(d)}$. Let $\varepsilon > 0$ such that $\varepsilon N + M_d(\mu)$ is full rank.

Perturbed Christoffel function

$\Lambda_{\mu,d,\varepsilon N} : \mathbb{R}^n \to \mathbb{R}$ defined by for $x \in \mathbb{R}^n$

$$\Lambda_{\mu,d,\varepsilon N}(x) := \left[v_d(x)^T (\varepsilon N + M_d(\mu))^{-1} v_d(x) \right]^{-1}.$$

Theorem

Let $K \subset \mathbb{R}^n$ be compact set and $s(d) - r \geq n$. Let $\psi : \mathbb{R}_+^* \to \mathbb{R}_+^*$ be a function such that

$$\psi(\varepsilon)^{-1} \to \infty \text{ and } \varepsilon\psi(\varepsilon)^{-1} \to 0 \text{ as } \varepsilon \to 0^+.$$

For every $\varepsilon > 0$, denote the superlevel set of perturbed Christoffel function by

$$S_\varepsilon := \left\{ x \in \mathbb{R}^n : \ \Lambda_{\mu,d,\varepsilon N}(x) \geq \psi(\varepsilon) \right\}.$$

Then

1. $\operatorname{int}(S_\varepsilon) \supset v_d^{-1}(\operatorname{Im}(M_d(\mu))) \cap K$.
2. $d_H\left(S_\varepsilon \cap K, v_d^{-1}(\operatorname{Im}(M_d(\mu))) \cap K \right) \to 0$ *as* $\varepsilon \to 0^+$.

Convergence theorem

Problem

Find the support $\operatorname{supp}(\mu) := \{z_i\} \subset \mathbb{R}^n$ and the weights $\{\lambda_i\} \subset \mathbb{R}^*$ of signed atomic measure $\mu := \sum_{i=1}^{k} \lambda_i \delta_{z_i}$ from knowledge of moments of μ.

Let $N \in \mathbb{S}_{++}^{s(d)}$. Let $\varepsilon > 0$ such that $\varepsilon N + M_d(\mu)$ is full rank.

Perturbed Christoffel function

$\Lambda_{\mu,d,\varepsilon N} : \mathbb{R}^n \to \mathbb{R}$ defined by for $x \in \mathbb{R}^n$

$$\Lambda_{\mu,d,\varepsilon N}(x) := \left[v_d(x)^T (\varepsilon N + M_d(\mu))^{-1} v_d(x) \right]^{-1}.$$

Theorem

Let $K \subset \mathbb{R}^n$ be compact set and $s(d) - r \geq n$. Let $\psi : \mathbb{R}_+^* \to \mathbb{R}_+^*$ be a function such that

$$\psi(\varepsilon)^{-1} \to \infty \text{ and } \varepsilon \psi(\varepsilon)^{-1} \to 0 \text{ as } \varepsilon \to 0^+.$$

For every $\varepsilon > 0$, denote the superlevel set of perturbed Christoffel function by

$$S_\varepsilon := \left\{ x \in \mathbb{R}^n : \ \Lambda_{\mu,d,\varepsilon N}(x) \geq \psi(\varepsilon) \right\}.$$

Then

1. $\operatorname{int}(S_\varepsilon) \supset v_d^{-1}(\operatorname{Im}(M_d(\mu))) \cap K$.
2. $d_H\left(S_\varepsilon \cap K, v_d^{-1}(\operatorname{Im}(M_d(\mu))) \cap K\right) \to 0$ *as* $\varepsilon \to 0^+$.
3. $v_d^{-1}(\operatorname{Im}(M_d(\mu))) \supset \operatorname{supp}(\mu)$. *If $d \geq r$, then* $v_d^{-1}(\operatorname{Im}(M_d(\mu))) = \operatorname{supp}(\mu)$.

Convergence theorem

Problem

Find the support $\operatorname{supp}(\mu) := \{z_i\} \subset \mathbb{R}^n$ and the weights $\{\lambda_i\} \subset \mathbb{R}^*$ of signed atomic measure $\mu := \sum_{i=1}^{k} \lambda_i \delta_{z_i}$ from knowledge of moments of μ.

Let $N \in \mathbb{S}_{++}^{s(d)}$. Let $\varepsilon > 0$ such that $\varepsilon N + M_d(\mu)$ is full rank.

Perturbed Christoffel function

$\Lambda_{\mu,d,\varepsilon N} : \mathbb{R}^n \to \mathbb{R}$ defined by for $x \in \mathbb{R}^n$

$$\Lambda_{\mu,d,\varepsilon N}(x) := \left[v_d(x)^T (\varepsilon N + M_d(\mu))^{-1} v_d(x) \right]^{-1}.$$

Theorem

Let $K \subset \mathbb{R}^n$ be compact set and $s(d) - r \geq n$. Let $\psi : \mathbb{R}_+^* \to \mathbb{R}_+^*$ be a function such that

$$\psi(\varepsilon)^{-1} \to \infty \text{ and } \varepsilon\psi(\varepsilon)^{-1} \to 0 \text{ as } \varepsilon \to 0^+.$$

For every $\varepsilon > 0$, denote the superlevel set of perturbed Christoffel function by

$$S_\varepsilon := \left\{ x \in \mathbb{R}^n : \ \Lambda_{\mu,d,\varepsilon N}(x) \geq \psi(\varepsilon) \right\}.$$

Then

1. $\operatorname{int}(S_\varepsilon) \supset v_d^{-1}(\operatorname{Im}(M_d(\mu))) \cap K$.
2. $d_H\left(S_\varepsilon \cap K, v_d^{-1}(\operatorname{Im}(M_d(\mu))) \cap K\right) \to 0$ *as* $\varepsilon \to 0^+$.
3. $v_d^{-1}(\operatorname{Im}(M_d(\mu))) \supset \operatorname{supp}(\mu)$. *If* $d \geq r$, *then* $v_d^{-1}(\operatorname{Im}(M_d(\mu))) = \operatorname{supp}(\mu)$.
4. *Let* $b_1, \ldots, b_{s(d)-r}$ *be a basis of the linear subspace* $\operatorname{Ker}(M_d(\mu))$. *Let* $G(x) = \left(b_1^T v_d(x), \ldots, b_{s(d)-r}^T v_d(x)\right), \ \forall x \in \mathbb{R}^n$ *and* $F(x) = \left(b_1^T v_d(x), \ldots, b_n^T v_d(x)\right), \ \forall x \in \mathbb{R}^n$. *Then* $G^{-1}(0) = v_d^{-1}(\operatorname{Im}(M_d(\mu))) \subset F^{-1}(0)$.

Building algorithm

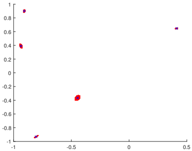
Building algorithm

Idea of algorithm

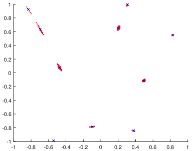
Using the superlevel set S_ε as the **initial points** when solving the square system $F(x) = 0$ by **Newton method**.

Idea of algorithm

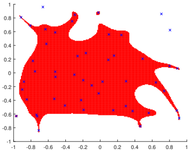
Using the superlevel set S_ε as the **initial points** when solving the square system $F(x) = 0$ by **Newton method**.



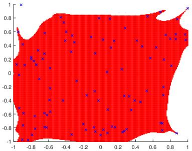
(a) Case $r = 5$ with $\varepsilon = 10^{-7}$ and $d = 3$.



(b) Case $r = 10$ with $\varepsilon = 10^{-9}$ and $d = 4$.



(c) Case $r = 50$ with $\varepsilon = 10^{-12}$ and $d = 9$.



(d) Case $r = 100$ with $\varepsilon = 10^{-12}$ and $d = 13$.

Illustration of the superlevel set of perturbed Christoffel function (red domain) and the approximate atoms (blue x)

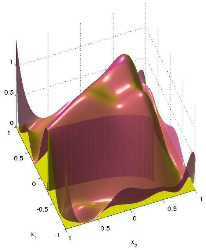
All other applications of Christoffel functions (intended PhD topics)

All other applications of Christoffel functions (intended PhD topics)

- Extraction of global optimizers in **polynomial optimization** problem
- Extraction of real root of **polynomial systems**.
- **Graph approximation** of non-polynomial function.
- **Approximation of "density"** for signed measure
- Applications to **D-finite functions**.
- **Optimal control**, occupation measures, **PDEs**.

All other applications of Christoffel functions (intended PhD topics)

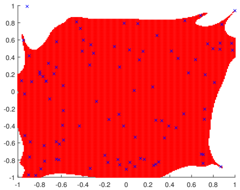
- Extraction of global optimizers in **polynomial optimization** problem
- Extraction of real root of **polynomial systems**.
- **Graph approximation** of non-polynomial function.
- **Approximation of "density"** for signed measure
- Applications to **D-finite functions**.
- **Optimal control**, occupation measures, **PDEs**.



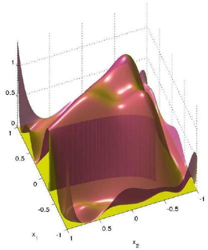
Graph approximation: Degree 10 polynomial approximation surface in pink of the indicator function of a curve

All other applications of Christoffel functions (intended PhD topics)

- Extraction of global optimizers in **polynomial optimization** problem
- Extraction of real root of **polynomial systems**.
- **Graph approximation** of non-polynomial function.
- **Approximation of "density"** for signed measure
- Applications to **D-finite functions**.
- **Optimal control**, occupation measures, **PDEs**.

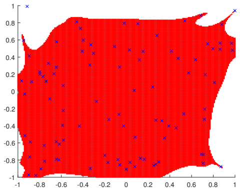


Approximation of cloud by supper level set of perturbed Christoffel function.

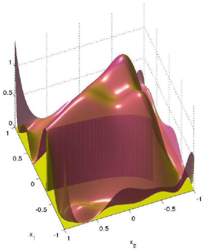


Graph approximation: Degree 10 polynomial approximation surface in pink of the indicator function of a curve

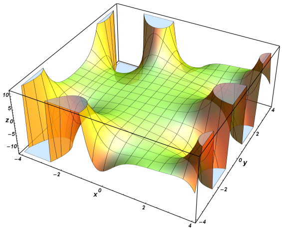
- Extraction of global optimizers in **polynomial optimization** problem
- Extraction of real root of **polynomial systems**.
- **Graph approximation** of non-polynomial function.
- **Approximation of "density"** for signed measure
- Applications to **D-finite functions**.
- **Optimal control**, occupation measures, **PDEs**.



Approximation of cloud by supper level set of perturbed Christoffel function.



Graph approximation: Degree 10 polynomial approximation surface in pink of the indicator function of a curve



D-finite functions: Plot of the real value of the Airy function.

End

Thank for your attention!