Small-time global stabilization of the KdV equation with three scalar controls.

Florent Koudohode

MAC team PhD seminar

Laboratoire d'analyse et d'architecture des systèmes (LAAS)

Institut de Mathématiques de Toulouse (IMT)

Advisors:

Sylvain Ervedoza(IMT) et Alexandre Seuret(LAAS)

July 1, 2019

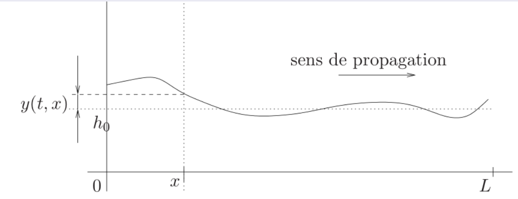
Outline

Modelization of the KdV equation

Modelization

- Description of long waves in water of relatively shallow depth: $y_t + y_x + y_{xxx} + yy_x = 0$,
- $=$**Korteweg-de Vries equation (KdV)**: 1895

$y(t,x)$



Our KdV control system

We consider the following Korteweg-de Vries controlled system:

$$
(1) \quad \begin{cases} y_t + y_x + y_{xxx} + yy_x = v(t) & \text{in } (s,+\infty)\times(0,L), \\ y(t,0) = w(t) & \text{in } (s,+\infty), \\ y(t,L) = h(t) & \text{in } (s,+\infty), \\ y_x(t,L) = 0 & \text{in } (s,+\infty), \\ y(0,\cdot) = y_0(\cdot). \end{cases}
$$

Control objective: Stabilize (1) globally in small time:

$y(T) = 0$ with the controls v, w, h given by a feedback law.

The main result

Theorem

Let $T_0 > 0$. There exists a $2T_0-$periodic time-varying feedback law for the system (1) *such that*

(i) *The closed-loop system is well-posed, in particular the flow φ is defined in $\Delta := \{(t,s); t > s\}$.*

(ii) $\varphi(4T_0 + t, t, y_0) = 0$, $\forall t \in \mathbb{R}$, $\forall y_0 \in L^2(0, L)$.

Strategy to prove the main result

To prove this Theorem we split the small-time global stabilization into two steps:

1. **Step 1**: Global approximate (pratical) stabilization

   $\rightsquigarrow \forall \varepsilon, T_0 > 0$, for all arbitrary $y_0 \in L^2(0,L)$ one has $\|y(T_0)\|_{L^2(0,L)} < \varepsilon$.
2. **Step 2**: Small-time local stabilization

   $\rightsquigarrow$ Based on the study of the linearized control system
   - $\|y_0\|_{L^2(0,L)} < \eta$ then $\|\varphi(t,t',y_0)\|_{L^2(0,L)} \leq \delta \ \forall t \geq t'$ and
   - $\varphi(T_0, 0, y_0) = 0$ if $\|y_0\| \leq \varepsilon$

Small-time global stabilization of Viscous Burgers equation [Coron-Xiang (2018)]

The Viscous Burgers control system

$$\begin{cases} y_t - y_{xx} + yy_x = \alpha(t) & \text{in } (0,+\infty)\times(0,1), \\ y(t,0) = u_1(t) & \text{in } (0,+\infty), \\ y(t,L) = u_2(t) & \text{in } (0,+\infty), \\ y(0,\cdot) = y_0(\cdot). \end{cases}$$

Heuristic description of the two steps

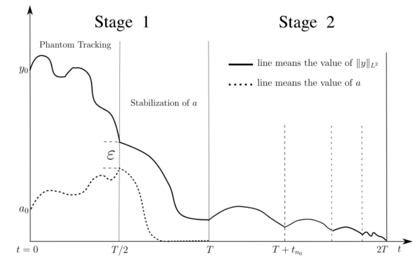


Figure 1: Small-time global stabilization of (y,a).

Global approximate stabilization of Viscous Burgers equation

1. In $(0, T/2) \rightsquigarrow$ Phantom tracking method

   Principle: Follow the trajectory $\boxed{\bar{y}(t,x) = \bar{a}}$ with $\bar{a}$ large.

   - Setting $z = y - \bar{a} \rightsquigarrow z_t - z_{xx} + \bar{y} z_x = 0 \rightsquigarrow \boxed{z_t + \bar{y} z_x = 0}$ if $\bar{y}$ large.
   - For a global argument one constructs a time-varying feedback such that suitable **Lyapunov functional** decays.
   - In fact $\bar{a} = \bar{a}(t)$, so that it can vary and get small at the end.

   $$\boxed{\bar{a}' = \alpha}$$

2. In $(T/2, T)$: stabilization on a in other to get $z = y$

Global approximate stabilization of our KdV control system

We perform the following change of variables $z := y - u$
Then (1) becomes:

$$\begin{cases} z_t + z_x + z_{xxx} + zz_x + u(t)z_x = 0 & \text{for (t,x)} \in (s, +\infty) \times (0, L), \\ z(t,0) = w(t) - u(t) & \text{for t} \in (s, +\infty), \\ z(t,L) = h(t) - u(t) & \text{for t} \in (s, +\infty), \\ z_x(t,L) = 0 & \text{for t} \in (s, +\infty), \\ z(0,\cdot) = z_0. & \end{cases}$$

Now set: $w(t) = h(t) = u(t)$ Then we obtain $\frac{d}{dt}\|z\|_{L^2}^2 \leq 0$

Dynamical extension (adding integrator)

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = v(t) & \text{in } (0,T)\times(0,L), \\ y(t,0) = u(t) & \text{in } (0,T), \\ y(t,L) = u(t) & \text{in } (0,T), \\ y_x(t,L) = 0 & \text{in } (0,T), \\ u_t = v(t) & \text{in } (0,T), \\ y(0,\cdot) = y_0(\cdot) \\ u(0) = u_0. \end{cases}$$

State: $(y,u) = (y(t),u(t)) \rightsquigarrow$ Phantom trajectory
Control: $v \in \mathbb{R}$. Then

$$\begin{cases} z_t + z_x + z_{xxx} + zz_x + u(t)z_x = 0 & \text{for (t,x)} \in (0,T)\times(0,L), \\ z(t,0) = 0 & \text{for t} \in (0,T), \\ z(t,L) = 0 & \text{for t} \in (0,T), \\ z_x(t,L) = 0 & \text{for t} \in (0,T), \\ u_t = v(t) & \text{for t} \in (0,T), \\ z(0,\cdot) = z_0. \end{cases}$$

State: $(z,u) \in L^2(0,L)\times\mathbb{R}$
Control:$v \in \mathbb{R}$.

We introduce the weighted energy $V_1(z) := \int_0^L |z(t,x)|^2 e^x dx.$

One has $\frac{d}{dt}V_1 \leq (2+u)V_1 + V_1^2$

Setting $u(t) = -\left[2 + (k+1)V_1(z)\right] \Longrightarrow \frac{d}{dt}V_1 \leq -kV_1^2.$

$$\text{Then } V_1(T_0) \leq \frac{1}{kT_0}.$$

$$\text{Choose } k = \frac{1}{\varepsilon T_0} \rightsquigarrow V_1(T_0) \leq \varepsilon$$

For $T_0 > 0$ given and $\forall\ \varepsilon > 0, \exists \lambda > 0$ such that by considering the following Lyapunov functional generated from the phantom tracking idea :

$$V_2(z,u) := V_1(z) + (u(t) - \lambda V_1(z))^2.$$

and chosing

$$v(t) = \lambda \left[(2+u)V_1(z) - 3\int_0^L z_x^2 e^x dx + \frac{2}{3}\int_0^L z^3 e^x dx \right] - \frac{V_1(z)}{2} + \frac{1}{2}\lambda \left(u - \lambda V_1(z)\right)^3$$

we get

$$|V_2[z(T_0/2)] \leq \varepsilon.$$

Small-time global stabilization of the variable u

We only need to find a feedback law which stabilizes u. For that it suffices to define on $\left(\frac{T_0}{2}, T_0\right)$, v by

$$v(u) := -\mu(u^2 + \sqrt{|u|}).sgn(u)$$

Indeed, with this v, there exists $\mu_{T_0} > 0$ such that, whatever is $u(T_0/2)$, if $\mu \geq \mu_{T_0}$ and $\dot{u} = v(u)$ then $u(T_0) = 0$.

Small-time local stabilization

The idea

1. Local $\Longrightarrow$ based on the linearized control system
2. Small-time stabilization: bring exactly to 0
   - Contruction of a feedback F_λ such that the associated semigroup (trajectory) decays in $e^{-\lambda t}$
   - Concatenation of the estimates

Small-time local stabilization of KdV

1. Construction of feedback via Backstepping approach
    - **Step 1**: Backstepping approach

Linear system

$$\begin{cases} y_t + y_x + y_{xxx} = 0, \\ y(t,L) = y_x(t,L) = 0, \\ y(t,0) = w(t), \end{cases}$$

$\xrightarrow{\Pi_\lambda}$

Target system

$$\begin{cases} z_t + z_x + z_{xxx} + \lambda z = 0, \\ z(t,L) = z_x(t,L) = 0, \\ z(t,0) = 0, \end{cases}$$

$$\|z(t,\cdot)\|_{L^2(0,L)} \leq \|z(0,\cdot)\|_{L^2(0,L)} e^{-\lambda t}$$

- Proposition: **Voltera operator**[Coron-Cerpa(2013)]
  $z(x) = \Pi_\lambda(y) := y(x) - \int_x^L k(x,x')y(x')dx',$

Small-time local stabilization

Kernel equation

$$\begin{cases} k_{xxx}(x,x') + k_{yyy}(x,x') + k_x(x,x') + k_y(x,x') = -\lambda k(x,x') & \text{in } \mathcal{T}, \\ k(x,L) = 0, & \text{in } [0,L], \\ k(x,x) = 0, & \text{in } [0,L], \\ k_x(x,x) = \dfrac{\lambda}{3}(L-x), & \text{in } [0,L], \end{cases}$$

where $\mathcal{T} = \{(x,x')/x \in [0,L], x' \in [x,L]\}$

- Difficulty : Find the kernel k_λ
  Is it exists? Yes! How? $\rightsquigarrow$ (Successive approximation) [**Coron-Cerpa**(2013)]

Step 2: Backstepping approach applied to the linearized of our system [**S.Xiang** (2018)]

Concatenation of the estimates

- $(0,T) \rightsquigarrow 0 < t_1 < \cdots < t_n < \cdots < T$
  on each $[t_n, t_{n+1}]$,construction of F_{λ_n}, where $(\lambda_n) \underset{n\to+\infty}{\longrightarrow} +\infty$
- Finally the flow between 0 and T

$$= \cdots e^{-(t_{i+1}-t_i)(A+F_{\lambda_i})} \dots e^{-(t_{i+1}-t_i)(A+F_{\lambda_i})} y_0$$
$$= 0 \text{ for suitable choices of } t_{i+1}-t_i \text{ and } \lambda_i$$

$\rightsquigarrow$ In the spirit of the construction of **Lebeau-Robbiano** (1995) for the control of the heat equation.
$\rightsquigarrow$ **Coron-Nguyen**(2017), **Xiang** (2018)

Future works

1. Well-posedness of the closed-loop system for the constructed time-varying feedback laws.
2. Exponentional stabilization of cascade **ODE-PDE**
   - ODE-linearized KdV [**H.Ayadi** (2018)]
   - ODE-Heat equation
   - ODE-Kuramoto-Sivashinsky equation

Merci pour votre aimable attention.