Stakes and techniques of stability set approximation for power systems

MAC team PhD seminar

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Toulouse, January 24th 2019

Plan

Stability assessment

RTE needs to

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Nowadays

Security risk = identified by operators (consumption peak)

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Consequence

Difficult to identify & secure unstable operating points

The challenge and a perspective

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- Tools:
  - Lyapunov-LaSalle stability theory [Anghel et al. 2013]

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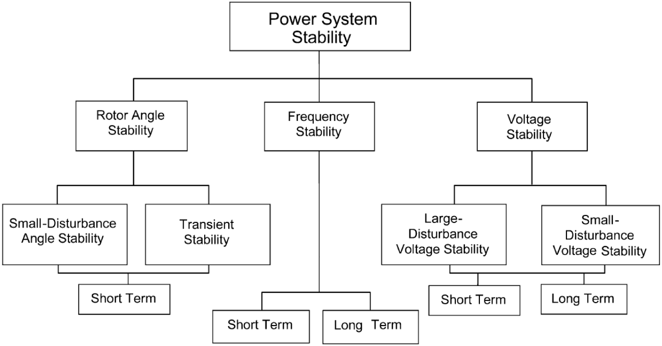
- Idea: Inner estimate of stability regions of power systems
- Tools:
  - Lyapunov-LaSalle stability theory [Anghel et al. 2013]
  - Moment approach for set approximation [Korda et al. 2013.]

Plan

Stability in electrical engineering



Stability notions depend on modelling & approximation hypothesis[1]

[1] Kundur et al. Definition and Classification of Power System Stability. 2004.

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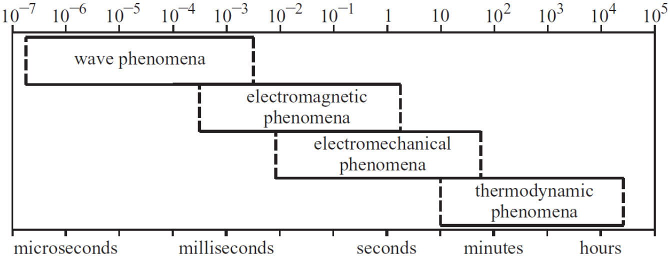

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- Existing algorithms resort to BMIs

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$$\begin{cases} \dot{\theta}_k = \omega_k & k = 1, 2 \\ \dot{\omega}_k = -\lambda_k \omega_k + \frac{1}{M_k}\left(P_k^m - P_k^e(\theta_1, \theta_2)\right) \end{cases}$$

3 bus 2^{nd} order model

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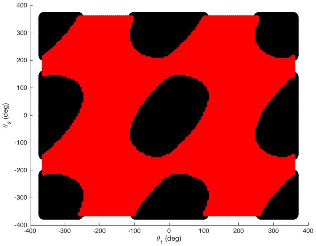

AI set - $\tau = 8$s, $\mathbf{X}_T = B(\mathbf{0}, 0.1)$

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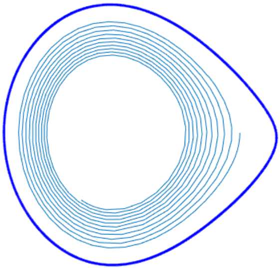

A trajectory in the MPI set

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- MPI set may include non-converging limit cycles

Plan

The volume problem

Problem statement

Given $\mathbf{g}, \mathbf{h} \in \mathbb{R}[\mathbf{x}]^m$, compute the Lebesgue volume of

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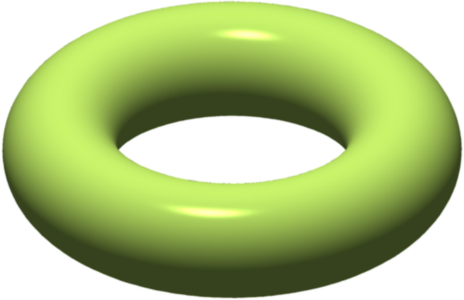

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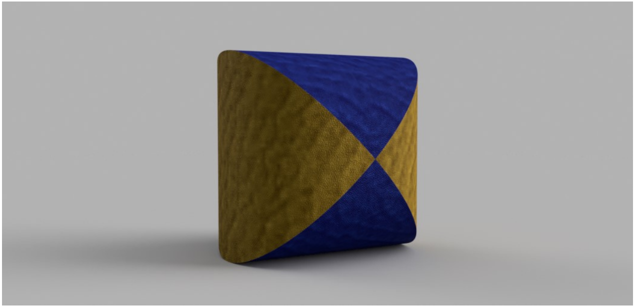

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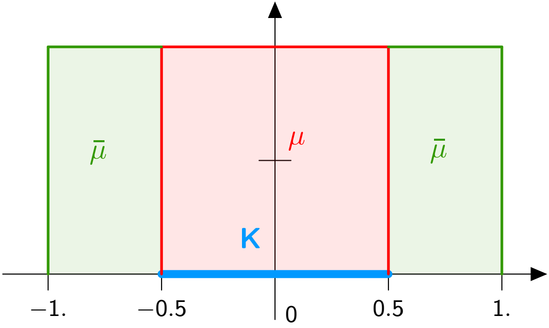

A dual approach

The dual of (5) on continuous functions

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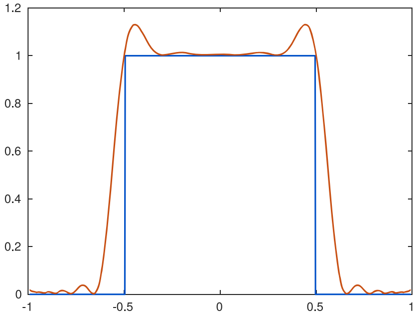

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Some useful definitions

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Memo: $\mathbf{K} = \{\mathbf{x} \in \mathbb{R}^n \,;\, \mathbf{g}(\mathbf{x}) \geq 0\}, \mathbf{X} = \{\mathbf{x} \in \mathbb{R}^n \,;\, \mathbf{h}(\mathbf{x}) \geq 0\}$.

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- $q(x) = 1 + x^2 + (1 + x^4) \, x \, (1 - x) \in \Sigma([0, 1])$

Writing the SOS problem

Putinar's Positivstellensatz

Notation: $\mathcal{P}(\mathbf{X})_{++} := \{p \in \mathbb{R}[\mathbf{x}] \; ; \forall \mathbf{x} \in \mathbf{X}, p(\mathbf{x}) > 0\}$

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The reinforced SOS problem

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Some useful definitions

Functionals on moments

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Writing the moment problem

Memo: $\mathbf{K} = \{\mathbf{x} \in \mathbb{R}^n \,;\, \mathbf{g}(\mathbf{x}) \geq 0\}, \mathbf{X} = \{\mathbf{x} \in \mathbb{R}^n \,;\, \mathbf{h}(\mathbf{x}) \geq 0\}$.

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Examples

first moment matrix: $n = 2, d = 1 \implies$ basis $(1, x_1, x_2)$

$$\mathbf{M}^1_z = \begin{bmatrix} z_{00} & z_{10} & z_{01} \\ z_{10} & z_{20} & z_{11} \\ z_{01} & z_{11} & z_{02} \end{bmatrix}$$

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a localizing matrix: $n = 1, d = 2, \chi = 1 + x \Longrightarrow$ basis $(1, x, x^2)$

$$\mathbf{M}^2_{1+x\, z} = \begin{bmatrix} z_0 + z_1 & z_1 + z_2 & z_2 + z_3 \\ z_1 + z_2 & z_2 + z_3 & z_3 + z_4 \\ z_2 + z_3 & z_3 + z_4 & z_4 + z_5 \end{bmatrix}$$

The Lasserre relaxation

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The relaxed problem on moments

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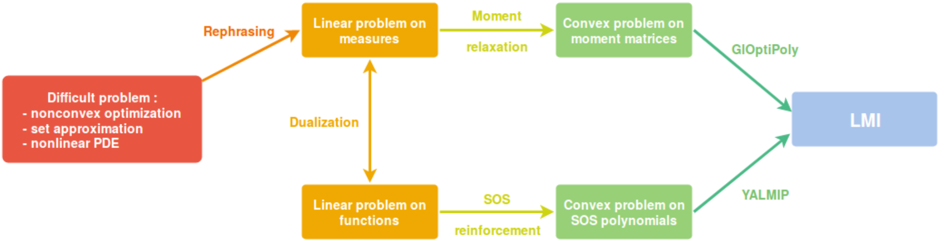
Summary: the Lasserre hierarchy framework



Plan

Liouville's formalism

Probabilistic heuristic

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Probabilistic heuristic

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$\Longrightarrow \mathbf{x}(t \,|\, X_0)$ is a random variable $X_t \sim \mathbb{P}_t$.

N.B.: $\mathbb{P}_0 = \delta_{\mathbf{x}_0} \Longrightarrow \mathbb{P}_t = \delta_{\mathbf{x}(t \,|\, \mathbf{x}_0)}$ (deterministic case)

Theorem (Liouville)

$$\frac{\partial}{\partial t}\mathbb{P}_t + \mathrm{div}(\mathbf{f}\, \mathbb{P}_t) = 0 \tag{8}$$

Integral Liouville PDE

$\mathbb{P}_0 \longleftrightarrow \mu_0$, $\mathbb{P}_T \longleftrightarrow \mu_T$, $\mathbb{P}_t(\mathrm{d}\mathbf{x})\,\mathrm{d}t \longleftrightarrow \mu(\mathrm{d}t, \mathrm{d}\mathbf{x})$

Liouville's formalism

Probabilistic heuristic

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \quad ; \quad \mathbf{x} \in \mathbf{X} \Subset \mathbb{R}^n \quad ; \quad \mathbf{x}(0) = X_0 \sim \mathbb{P}_0$$

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Integral Liouville PDE

$\mathbb{P}_0 \longleftrightarrow \mu_0$, $\mathbb{P}_\tau \longleftrightarrow \mu_T$, $\mathbb{P}_t(\mathrm{d}\mathbf{x})\,\mathrm{d}t \longleftrightarrow \mu(\mathrm{d}t, \mathrm{d}\mathbf{x})$

$$\frac{\partial \mu}{\partial t} + \operatorname{div}(\mathbf{f}\,\mu) = \mu_0 \otimes \delta_{t=0} - \mu_T \otimes \delta_{t=\tau} \tag{9}$$

GMP for AI set estimation

The problem on measures

GMP for AI set estimation

The problem on measures

$$\begin{aligned}
\mathrm{vol}(\mathbf{X}_0) &= \max \mu_0(\mathbf{X}) && (10)\\
\text{s.t.} \quad & \mu \in \mathcal{M}([0,\tau] \times \mathbf{X})_+ && (11)\\
& \mu_0, \bar{\mu}_0 \in \mathcal{M}(\mathbf{X})_+ && (12)\\
& \mu_T \in \mathcal{M}(\mathbf{X}_T)_+ \subset \mathcal{M}(\mathbf{X})_+ && (13)\\
& \mu_0 + \bar{\mu}_0 = \lambda_{\mathbf{X}} && (14)\\
& \frac{\partial \mu}{\partial t} + \mathrm{div}(\mathbf{f}\,\mu) = \mu_0 \otimes \delta_{t=0} - \mu_T \otimes \delta_{t=\tau} && (15)
\end{aligned}$$

GMP for AI set estimation

The problem on measures

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$$\frac{\partial \mu}{\partial t} + \mathrm{div}(\mathbf{f}\,\mu) = \mu_0 \otimes \delta_{t=0} - \mu_T \otimes \delta_{t=\tau} \tag{15}$$

Statistical physics interpretation

Maximize (10) density $\rho_0 \leq 1$ (12),(14) of particles transported by equation (15) that end up in $\mathbf{X}_T$ (13) in time τ (11).

Dual GMP for AI set estimation

The problem on functions

Dual GMP for AI set estimation

The problem on functions

$$\mathrm{vol}(\mathbf{X}_0) = \inf \int_{\mathbf{X}} w(\mathbf{x})\mathrm{d}\mathbf{x} \tag{16}$$

$$\begin{aligned} \text{s.t.} \quad & \forall \mathbf{x} \in \mathbf{X}, w(\mathbf{x}) \geq 0 \\ & \forall \mathbf{x} \in \mathbf{X}, w(\mathbf{x}) \geq v(0, \mathbf{x}) + 1 \\ & \forall \mathbf{x} \in \mathbf{X}_T, v(\tau, \mathbf{x}) \geq 0 \\ & \forall t \in [0, \tau], \mathbf{x} \in \mathbf{X}, \frac{\partial v}{\partial t}(t, \mathbf{x}) + \nabla v(t, \mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) \leq 0 \end{aligned} \tag{17}$$

Dual GMP for AI set estimation

The problem on functions

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Remark

Constraint (17) reminds of a Lyapunov inequality.

Dual GMP for AI set estimation

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Remark

Constraint (17) reminds of a Lyapunov inequality.

Set approximation

$\hat{\mathbf{X}}_0^d := \{\mathbf{x} \in \mathbf{X} \; ; \; w_d(\mathbf{x}) \geq 1\}$ is an outer approximation of $\mathbf{X}_0$!

Plan

Parsimonious ROA estimation

Parsimonious ROA estimation

Problem statement

Assess the stability of the system

$$\forall i, j \in \{1, \ldots, m\}, \begin{cases} \dot{\mathbf{x}}_i = f_i(\mathbf{x}_i, \mathbf{y}_i) \\ g_{ij}(\mathbf{x}_i, \mathbf{y}_i, \mathbf{x}_j, \mathbf{y}_j) = 0 \quad ; \quad (g_{ij})_{ij} \text{ “sparse”} \end{cases} \tag{18}$$

Parsimonious ROA estimation

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Adapting parsimony to algebraically coupled dynamics

- Inner ROA estimation $\simeq$ lower estimation of ROA volume

Parsimonious ROA estimation

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Parsimonious ROA estimation

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Adapting parsimony to algebraically coupled dynamics

- Inner ROA estimation $\simeq$ lower estimation of ROA volume
- We developed parsimony for volume computation

Combine both $\rightarrow$ parsimonious algo. for ROA/MPI/IAS estimation

Question Time