Inner approximation of the maximal positively invariant set for polynomial dynamical systems

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Session: Optimization IV (ThA16)

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Outline

1. **Maximal positively invariant set**
2. **Modelling trajectories with measures**
3. **Linear programming on measures**

Stability region of polynomial differential systems

Autonomous ODE with polynomial dynamics and polynomial constraints:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) \tag{1a}$$

$$g_i(\mathbf{x}(t)) \geq 0 \qquad i = 1, \ldots, m \tag{1b}$$

$t \geq 0$, $\mathbf{x} \in \mathbb{R}^n, \mathbf{f} \in \mathbb{R}[\mathbf{x}]^n$, $\mathbf{g} := (g_1, \ldots, g_m) \in \mathbb{R}[\mathbf{x}]^m$.

Question

What are the initial conditions for (1a) s.t. (1b) is satisfied at all times $t \geq 0$?

Application Power systems transient stability analysis (see FrA25.5 tomorrow 11:20, room Athena).

Maximal positively invariant (MPI) set

- Admissible state set:

$$\mathbf{X} = \{\mathbf{x} \in \mathbb{R}^n : g_i(\mathbf{x}) \geq 0 \ \forall i = 1, \ldots, m\}$$

- MPI set:

$$\mathbf{X}_\infty = \{\mathbf{x}_0 \in \mathbb{R}^n : \mathbf{x}(t|\mathbf{x}_0) \in \mathbf{X} \ \forall t \geq 0\}$$

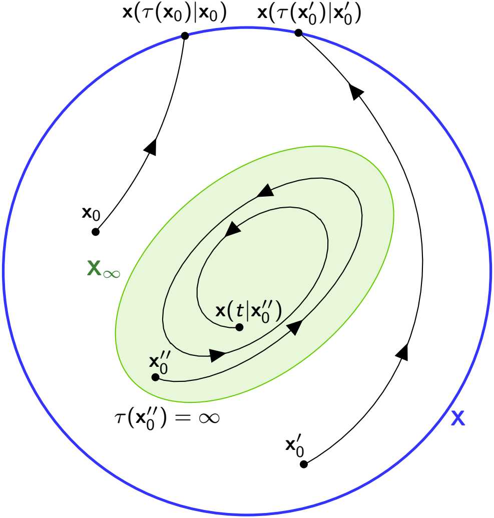
Exit time

$$\tau(\mathbf{x}_0) := \inf\{t \geq 0 : \mathbf{x}(t|\mathbf{x}_0) \notin \mathbf{X}\}$$

$\Rightarrow \mathbf{X}_\infty = \{\mathbf{x}_0 \in \mathbf{X} : \tau(\mathbf{x}_0) = \infty\}$
$\Rightarrow \mathbf{X} \setminus \mathbf{X}_\infty = \{\mathbf{x}_0 \in \mathbf{X} : \tau(\mathbf{x}_0) < \infty\}$

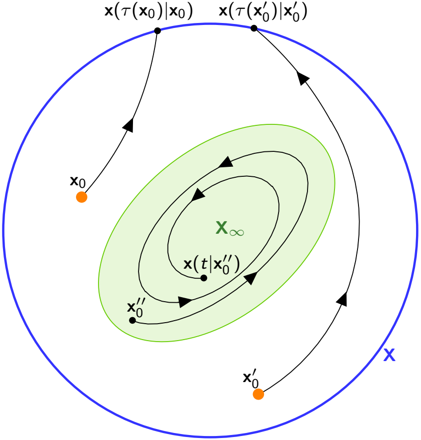
$\rightarrow$ Focus on $\mathbf{X}_\infty^c := \mathbf{X} \setminus \mathbf{X}_\infty$ (trajectories exist on finite time horizon).

MPI set: illustration



Occupation measures

- Initial condition $\mathbf{x}_0 \in \mathbf{X} \rightarrow$ Initial mass distribution $\mu_0 \in \mathcal{M}(\mathbf{X})_+$



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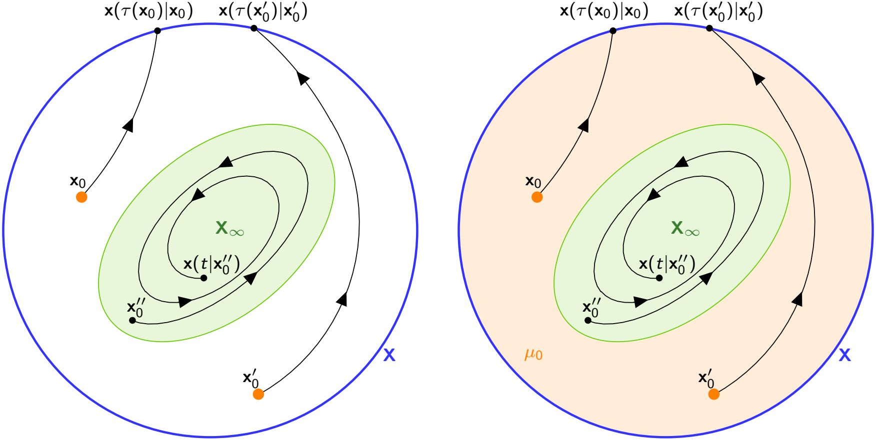
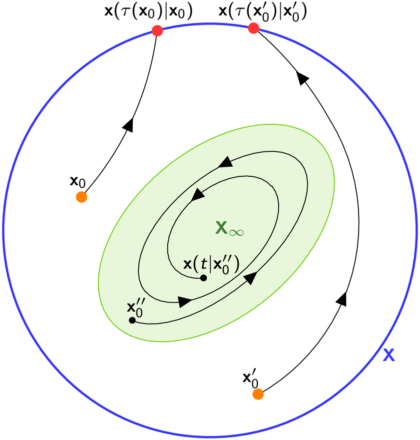


Occupation measures

- Initial condition $\mathbf{x}_0 \in \mathbf{X} \to$ Initial mass distribution $\mu_0 \in \mathcal{M}(\mathbf{X})_+$
- Exit point $\mathbf{x}(\tau(\mathbf{x}_0)|\mathbf{x}_0) \in \partial\mathbf{X}$ with $\mathbf{x}_0 \in \mathbf{X}_\infty^c \to$ Exit measure μ_∂



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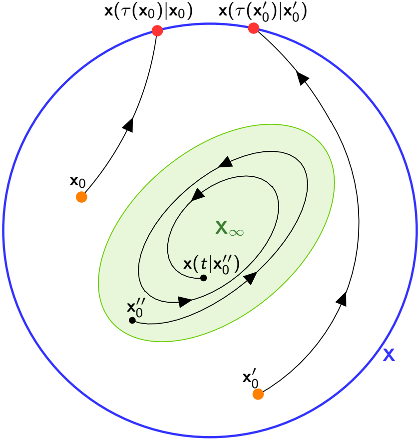

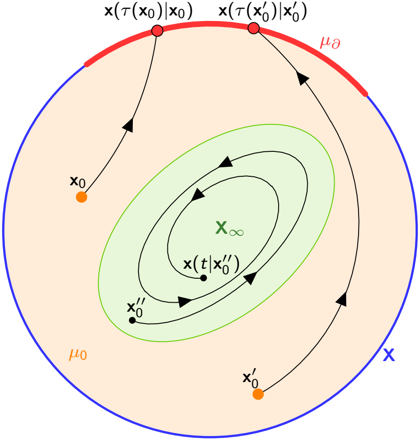

Occupation measures

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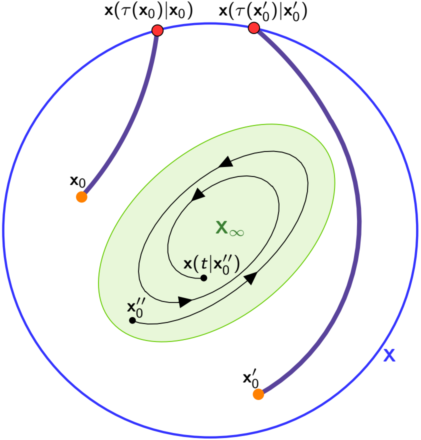
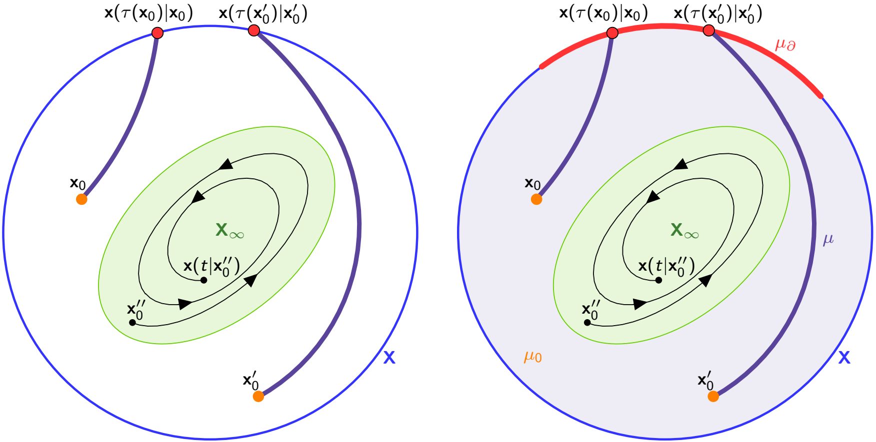
- Trajectory $t \mapsto \mathbf{x}(t|\mathbf{x}_0)$ with $\mathbf{x}_0 \in \mathbf{X}_\infty^c \to$ Occupation measure μ



Occupation measures

- Initial condition $\mathbf{x}_0 \in \mathbf{X}$ → Initial mass distribution $\mu_0 \in \mathcal{M}(\mathbf{X})_+$
- Exit point $\mathbf{x}(\tau(\mathbf{x}_0)|\mathbf{x}_0) \in \partial\mathbf{X}$ with $\mathbf{x}_0 \in \mathbf{X}_\infty^c$ → Exit measure μ_∂
- Trajectory $t \mapsto \mathbf{x}(t|\mathbf{x}_0)$ with $\mathbf{x}_0 \in \mathbf{X}_\infty^c$ → Occupation measure μ



A variant of Liouville's PDE

Easy result

If μ_0 is supported on $\mathbf{X}_\infty^c$ and μ_∂ and μ are the associated exit and occupation measures, then

$$\mathrm{div}(\mathbf{f}\mu) + \mu_\partial = \mu_0 \tag{2}$$

Difficult theorem (under assumptions)

If $\mu, \mu_0 \in \mathcal{M}(\mathbf{X})_+$, $\mu_\partial \in \mathcal{M}(\partial\mathbf{X})_+$ are generic and satisfy (2), then

$$\mathrm{spt}\ \ \mu_0 \subset \mathbf{X}_\infty^c$$

Proof intuition: $0 \leq \mu_0(\mathbf{X}_\infty) = \underbrace{\mathrm{div}(\mathbf{f}\mu)(\mathbf{X}_\infty)}_{\leq 0} + \underbrace{\mu_\partial(\mathbf{X}_\infty)}_{= 0} \leq 0$

$\Rightarrow$ maximize the support of μ_0 satisfying (2).

A linear program on measures

Primal LP

$$p = \sup_{\substack{\mu_0, \mu \in \mathcal{M}(\mathbf{X})_+ \\ \mu_\partial \in \mathcal{M}(\partial \mathbf{X})_+ \\ \text{s.t.}}} \begin{array}{l} \mu_0(\mathbf{X}) \\ \operatorname{div}(\mathbf{f}\mu) + \mu_\partial = \mu_0 \\ \mu_0 \leq \lambda \end{array} \tag{3}$$

Dual LP

$$d = \inf_{\substack{v \in \mathcal{C}^1(\mathbf{X}) \\ w \in \mathcal{C}^0(\mathbf{X})_+ \\ \text{s.t.}}} \begin{array}{l} \int_{\mathbf{X}} w(\mathbf{x}) d\mathbf{x} \\ \nabla v \cdot \mathbf{f}(\mathbf{x}) \leq 0, \ \forall \mathbf{x} \in \mathbf{X} \\ w(\mathbf{x}) \geq v(\mathbf{x}) + 1, \forall \mathbf{x} \in \mathbf{X} \\ v(\mathbf{x}) \geq 0, \forall \mathbf{x} \in \partial \mathbf{X}. \end{array} \tag{4}$$

$$\hat{\mathbf{X}}_\infty := \{\mathbf{x}_0 \in \mathbf{X} : v(\mathbf{x}) < 0\} \subset \mathbf{X}_\infty$$

Convergence & strong duality

LPs (3) & (4) approximated with LMIs using the Lasserre hierarchy.

Strong duality $\Rightarrow$ convergence in volume of the $\hat{\mathbf{X}}_\infty$ towards $\mathbf{X}_\infty$.

Sufficient condition for strong duality

The set of feasible measures should be bounded in mass

- $\mu_0(\mathbf{X}) \leq \lambda(\mathbf{X}) < \infty$ when $\mathbf{X}$ is compact
- $\mu_\partial(\partial\mathbf{X}) = \mu_0(\mathbf{X}) < \infty$

⚠ What about $\mu(\mathbf{X})$?

Two issues

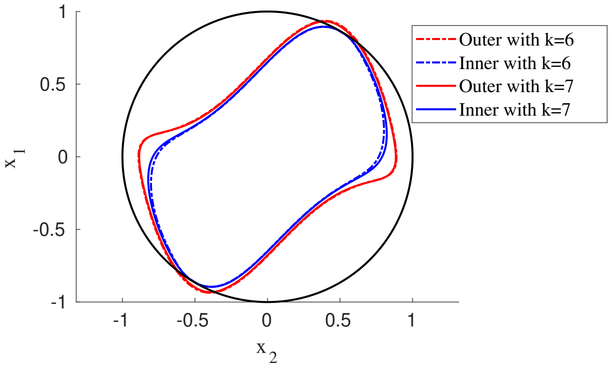
- $\mu_0 = \lambda_{\mathbf{X}_\infty^c} \Longrightarrow \mu(\mathbf{X}) = \int_{\mathbf{X}_\infty^c} \tau(\mathbf{x}_0)\, \mathrm{d}\mathbf{x}_0$: τ should be in $L^1(\mathbf{X}_\infty^c)$
- μ feasible $\Longrightarrow \mu + k\delta_{\bar{\mathbf{x}}}$ feasible for any $k \geq 0$, $\bar{\mathbf{x}}$ s.t. $\mathbf{f}(\bar{\mathbf{x}}) = 0$

$\Longrightarrow$ necessity to add a constraint $\mu(\mathbf{X}) \leq T$ with $\int_{\mathbf{X}_\infty^c} \tau(\mathbf{x}_0)\, \mathrm{d}\mathbf{x}_0 \leq T < \infty$

Numerical test on Van der Pol oscillator

$$\begin{cases} \dot{x}_1 = -2\ x_2 \\ \dot{x}_2 = 0.8\ x_1 + 10\ (1.02^2 x_1^2 - 0.2)\ x_2. \end{cases} \quad (5)$$

$\mathbf{X} = \{\mathbf{x} \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$ and $T = 100$. Degree $2k$ certificates v, w.



Take-home message

- Industrial security conditions $\rightsquigarrow$ trajectories should not leave a given secure zone.
- **Alternative stability assessment methods (nonlinear dynamics):**
  - Infinite time region of attraction (Anghel et al. 2013) $\rightarrow$ infinite time horizon, no proof of global convergence
  - Finite time region of attraction (Korda et al. 2013) $\rightarrow$ global convergence in volume, finite time horizon
  - Outer approximation of MPI (Korda et al. 2013) $\rightarrow$ converging MPI approximation, false negatives
- **Contribution: convex** programming yields **converging, conservative** approximations of an **infinite time** stability region.
- **Future work:** Removing constraint $\mu(\mathbf{X}) \leq T$, using general strong duality theorems.

Questions

Thank you for your attention!