Almost Everywhere Conditions for Hybrid Lipschitz Lyapunov Functions

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Preliminaries

Hybrid System

We consider the system

$$\mathcal{H} : \begin{cases} \dot{x} \in F(x), & x \in \mathcal{C}, \\ x^+ \in G(x), & x \in \mathcal{D}, \end{cases}$$

such that $\mathcal{H} = (\mathcal{C}, \mathcal{D}, F, G)$ satisfies the hybrid basic conditions.

We want to analyze the stability of $\mathcal{H}$ via **locally Lipschitz** Lyapunov functions.
Main Question: Can we check the flow Lyapunov inequality only almost everywhere (i.e. only where the candidate Lyapunov function is differentiable)?

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Main Question: Can we check the flow Lyapunov inequality only almost everywhere (i.e. only where the candidate Lyapunov function is differentiable)?
We ask that

- $\text{cl}(\text{int}(\mathcal{C})) = \mathcal{C}$, (t.i. $\mathcal{C}$ regular-closed set),
- $F : \mathcal{C} \rightrightarrows \mathbb{R}^n$ inner semicontinuous.

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A remarkable "precedent" of Smooth vs Non-Smooth:

- For a LDI of the form

$$\dot{x} \in \mathrm{co}\left\{A_i x \mid i \in \{1, \dots, K\}\right\},$$

  GAS is **equivalent** to the existence of a smooth Lyapunov function homogeneous of degree 2 (but not necessary quadratic)[Dayawansa and Martin. '99];
- We can approximate this function using max of quadratics [Goebel, Teel, Hu, Lin. '06], or polyhedral functions [Molchanov and Pyatnitskiy. '89].

$\mathrm{cl}(\mathrm{int}(\mathcal{C})) \neq \mathcal{C}$.

Counterexample 1

$\mathcal{H} = (\mathcal{C}, \mathcal{D}, F, G)$ with

$$\mathcal{C} := \{x \in \mathbb{R}^2 \mid x_2 = 0\},\ \mathcal{D} := \emptyset,$$
$$F(x) := \left\{ \begin{pmatrix} x_1^2 \\ 0 \end{pmatrix} \right\}, \qquad G(x) := \emptyset.$$

$\mathcal{H}$ satisfies the basic Assumption and F is continuous.

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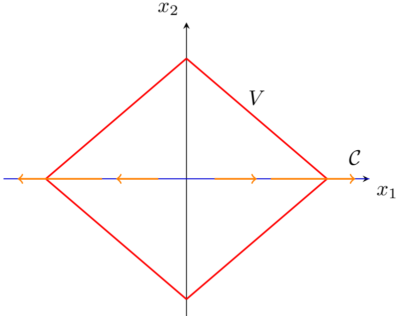
Candidate Lyapunov function

$$V(x) = |x_1| + |x_2|.$$

Problem: V is locally Lipschitz but the set $\mathcal{N}_V$ where it is not differentiable covers the flow sets $\mathcal{C}$, the system is unstable.



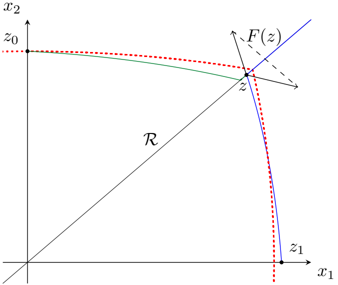
Non Continuity: Switching systems

Counterexample 2
$\mathcal{H} = (\mathcal{C}, \mathcal{D}, F, G)$ with $\mathcal{C} = \mathbb{R}^2$, $\mathcal{D} = G = \emptyset$ and $F : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ defined as the Filippov regularization of:

$$\dot{x} = \begin{cases} A_1 x, & \text{if } x^\top Q x \geq 0, \\ A_2 x, & \text{if } x^\top Q x < 0. \end{cases}$$

Candidate Lyapunov Function:

- Max of Quadratics $V(x) = \max\{x^\top P_1 x, x^\top P_2 x\}$,
- The Lyapunov inequality is satisfied almost everywhere in $\mathbb{R}^2$,
- We have diverging sliding motion (solutions flowing along the discontinuity surface).



A class of locally Lipschitz Functions

Properly Piecewise $\mathcal{C}^1$ functions

Let $V : \mathcal{C} \cup \mathcal{D} \cup G(\mathcal{D}) \to \mathbb{R}$ be a continuous function. Given $I = \{1, \dots, K\}$, let us suppose there exist $\{\mathcal{X}_i\}_{i \in I}$ closed sets and functions $V_i \in \mathcal{C}^1(\mathcal{X}_i + \varepsilon\mathbb{B}, \mathbb{R})$, such that:

- $\overline{\mathrm{int}(\mathcal{X}_i)} = \mathcal{X}_i$, (regular-closed) for all $i \in I$,
- $\mathrm{int}(\mathcal{X}_i) \cap \mathrm{int}(\mathcal{X}_j) = \emptyset$, for all $i \neq j$,
- $\mathcal{C} \subset \bigcup_{i \in I} \mathcal{X}_i$,
- $V(x) = V_i(x), \quad$ if $x \in \mathcal{X}_i$.

Then the function V is called a *properly piecewise $\mathcal{C}^1$ function*.

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Intuitively, the flow set $\mathcal{C}$ is covered by a finite number of sets $\mathcal{X}_i$ and V on $\mathcal{C}$ is obtained "gluing" together some functions $V_i \in \mathcal{C}^1$.
Locally Lipschitz; we consider $\mathcal{N}_V$ the null measure set where V is not differentiable (Rademacher Theorem)

Example: Max/Min Functions

Given a family $\mathcal{V} = \{V_1, \dots, V_K\} \subset \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$, the functions

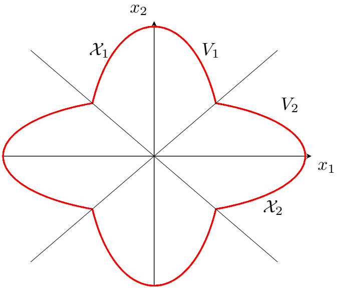
$$V_M(x) := \max_{i \in I} V_i(x) \text{ and } V_m(x) := \min_{i \in I} V_i(x)$$

are properly piecewise $\mathcal{C}^1$ on $\mathbb{R}^n$. For the V_M we have

$$\mathcal{X}_i = \overline{\{x \in \mathbb{R}^n \mid V_i(x) > V_j(x),\ \forall j \in I,\ j \neq i\}}.$$

Example:

- $V_m(x) = \min\{x^\top P_1 x, x^\top P_2 x\}$,
- Min of 2 quadratics, non-convex,
- Homogeneous of degree 2,
- $\mathcal{X}_1, \mathcal{X}_2$ are symmetric cones,
- $\mathcal{N}_V = \{x \in \mathbb{R}^n \mid V_1(x) = V_2(x)\}$.



Stability Conditions

Main Theorem: "Almost everywhere" conditions

Consider a closed set $\mathcal{A}$. Given a properly piecewise $\mathcal{C}^1$ function $V : \mathcal{C} \cup \mathcal{D} \cup G(\mathcal{D}) \to \mathbb{R}$, suppose that there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\rho \in \mathcal{PD}$ such that

$$\alpha_1(|x|_\mathcal{A}) \leq V(x) \leq \alpha_2(|x|_\mathcal{A}) \quad \forall x \in \mathcal{C} \cup \mathcal{D},$$

$$\langle \nabla V_i(x), f \rangle \leq -\rho(|x|_\mathcal{A}), \quad \begin{array}{l} \forall x \in \mathrm{int}(\mathcal{X}_i) \cap \mathrm{int}(\mathcal{C}), \\ \forall f \in F(x), \, \forall i \in I. \end{array}$$

$$V(g) - V(x) \leq -\rho(|x|_\mathcal{A}), \quad \forall x \in \mathcal{D}, \forall g \in G(x).$$

Then $\mathcal{A}$ is UGpAS for $\mathcal{H}$.

Proof Sketch:

- **Main Idea**: $\mathcal{C} \subset \bigcup_{i\in I} \mathcal{X}_i$ and $\bigcup_i \mathrm{bd}(\mathcal{X}_i)$ has zero measure and contains the points at which V is not differentiable.

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Proof Sketch:

- **Main Idea**: $\mathcal{C} \subset \bigcup_{i \in I} \mathcal{X}_i$ and $\bigcup_i \operatorname{bd}(\mathcal{X}_i)$ has zero measure and contains the points at which V is not differentiable.
- Considering a flowing solution $\phi : [0, T_\phi) \to \mathcal{C}$, we show that

$$\frac{d}{dt} V(\phi(\cdot))(t) \leq -\rho(|\phi(t)|_\mathcal{A}) \text{ for a.e.} t \in [0, T_\phi).$$

Clegg Integrator System

Let us consider the hybrid system

$$\begin{cases} \dot{x} = A_F x, & x \in \mathcal{C} = \{x \in \mathbb{R}^2 \,|\, x^\top Q x \geq 0\}, \\ x^+ = A_J x, & x \in \mathcal{D} = \{x \in \mathbb{R}^2 \,|\, x^\top Q x \leq 0\}, \end{cases}$$

with

$$A_F = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ A_J = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ Q = \begin{bmatrix} 1 & -\frac{1}{2\varepsilon} \\ -\frac{1}{2\varepsilon} & 0 \end{bmatrix},$$

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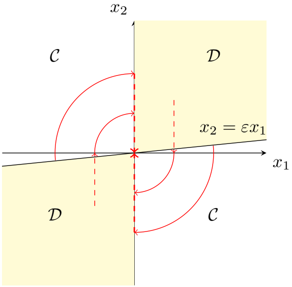
Linear hybrid system, “intuitively” UGpAS, **but** there does not exist a *quadratic* Lyapunov function. We construct 3 properly piecewise $\mathcal{C}^1$ Lyapunov functions, homogeneous of degree 2.



Max of 2 sign-indefinite quadratics

Lyapunov function

$$V_M(x) := \max\{x^\top P_1 x, x^\top P_2 x\},$$

with $P_1 = \begin{bmatrix} 1 & -0.1 \\ -0.1 & 0.5 \end{bmatrix}, \ P_2 = \begin{bmatrix} 2.5 & 1.4 \\ 1.4 & 0.5 \end{bmatrix},$

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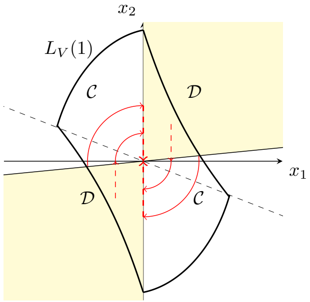
- Easy to check the Lyapunov inequalities,
- $P_2 \not> 0$,
- It is positive when "active",
- Homogeneous of degree 2,
- Non convex



Mid of quadratics

Lyapunov function

$$V_{\mathsf{mid}}(x) = \mathsf{mid}\{V_1, V_2, V_3\} = \max\{\min\{V_1, V_2\}, \min\{V_2, V_3\}, \min\{V_1, V_3\}\}$$

with

$$P_1 = \begin{bmatrix} 1 & 0.25 \\ 0.25 & 0.7 \end{bmatrix}, P_2 = \begin{bmatrix} 0.55 & -0.2 \\ -0.2 & 0.25 \end{bmatrix}, P_3 = \begin{bmatrix} \frac{25}{16} & \frac{49}{160} \\ \star & 0.25 \end{bmatrix}$$

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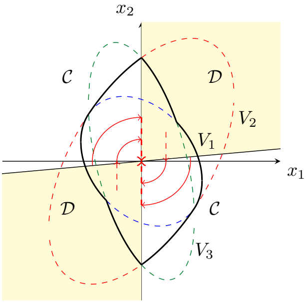
- The mid "choose" the quadratics between the other 2,
- C^1 inside $\mathcal{C}$,
- Homogeneous of degree 2,
- Non convex.



Convex Lyapunov Function

Lyapunov function

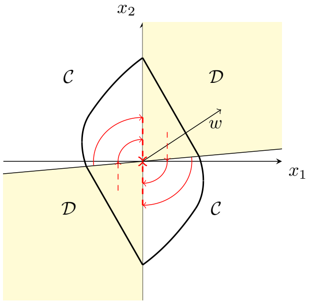
$$V_{\mathsf{conv}}(x) = \begin{cases} V_{\mathsf{mid}}(x), & \text{if } x \in \mathcal{C}, \\ \langle w, x \rangle^2, & \text{if } x \in \mathcal{D}, \end{cases}$$

Convex Lyapunov Function

Lyapunov function

$$V_{\text{conv}}(x) = \begin{cases} V_{\text{mid}}(x), & \text{if } x \in \mathcal{C}, \\ \langle w, x\rangle^2, & \text{if } x \in \mathcal{D}, \end{cases}$$

- "Convexification" of the Mid function,
- Homogeneous of degree 2,
- $\mathcal{C}^1$ inside $\mathcal{C}$,
- Convex.



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- A class of locally Lipschitz functions for Hybrid Systems that contains pointwise max and min of $\mathcal{C}^1$ functions;
- Conditions under which it suffices to check the Lyapunov inequality almost everywhere in the flow set $\mathcal{C}$;
- Application to a linear reset system.

Thank you !!

Questions ??