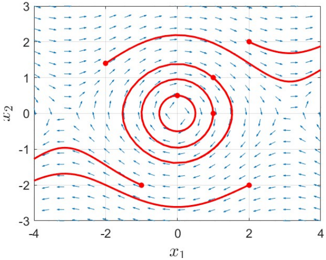
Chapitre 2 : Phase Plane

Yassine ARIBA



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Sommaire

Sommaire

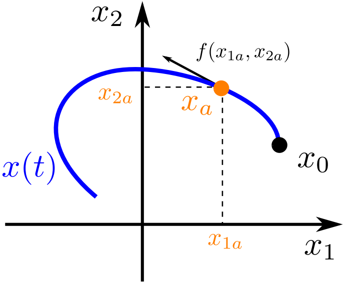
Second-order systems

In general, one can not find solution $x(t)$ of a nonlinear system

Some techniques exist to draw $x(t)$ for second-order system in a plane

$$\dot{x} = f(x) \quad \equiv \quad \begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases} \qquad \text{with } x(0) = x_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$$

$\hookrightarrow$ **Phase plane**



Definitions

Trajectory or orbit

The curve of $x(t)$ in the $x_1 - x_2$ plane is called a *trajectory* or *orbit* of the system from the point x_0.

Phase portrait

The *phase portrait* of the system is the set of all trajectories for different initial conditions x_0.

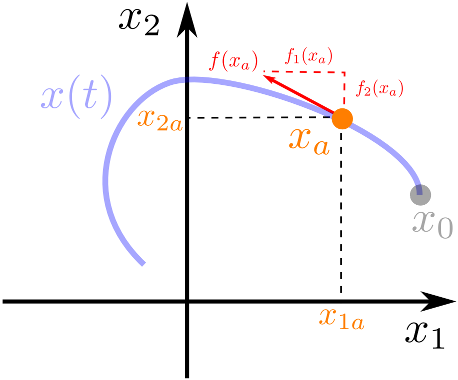
Vector field

The *vector field* is the representation, in the $x_1 - x_2$ plane, of the vector $f(x) = \Big(f_1(x_1, x_2)\ ,\ f_2(x_1, x_2)\Big)$. It is drawn with arrows.

Vector field

The vector $f(x) = \Big(f_1(x)\ ,\ f_2(x)\Big)$ is tangent to the trajectory at point x

$$\frac{dx_2}{dx_1} = \frac{f_2(x)}{f_1(x)}$$



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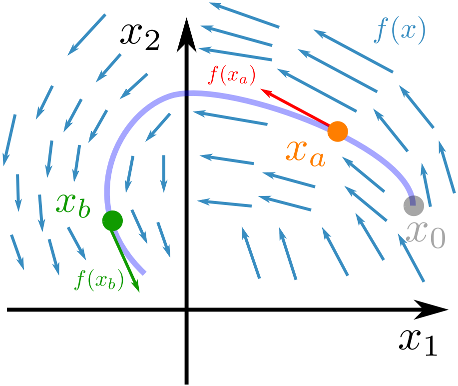
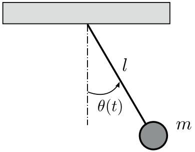
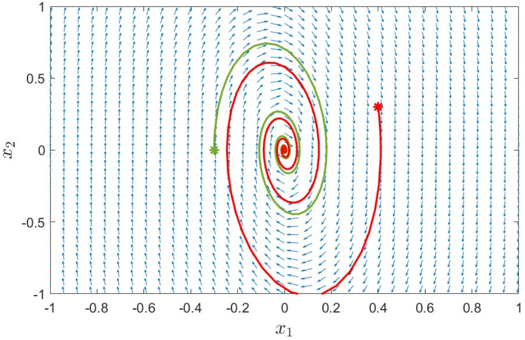


Pendulum example

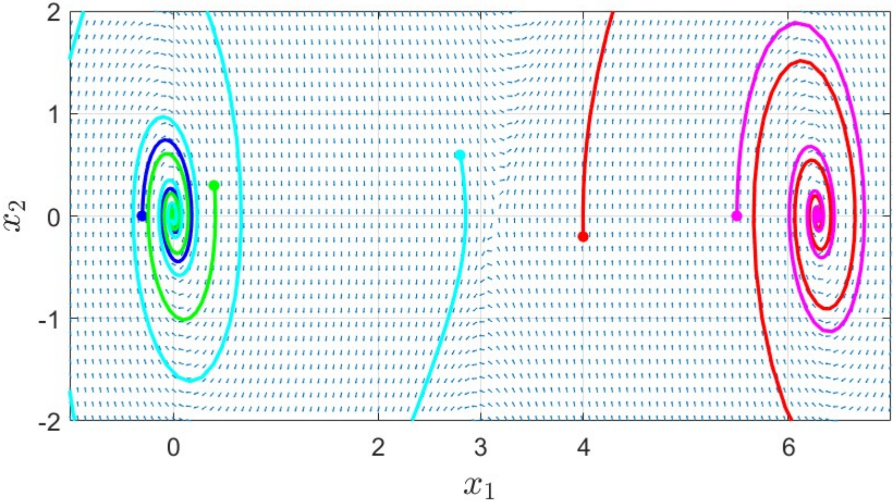
Variables : $x_1 = \theta$ and $x_2 = \dot{\theta}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 \end{bmatrix}$$





Pendulum example



Sommaire

Construction of phase portrait

Several techniques exist to draw trajectories on the phase plane

Two will be presented here :

- analytical method - solve the differential equations
- isoclines method - graphical method

◇ But nowadays numerical computing softwares are used (MATLAB, Scilab, Python)

Analytical method

The objective is to get a relationship between x_1 and x_2

$$g(x_1, x_2) = 0$$

- First approach : solve the state equation

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases} \Rightarrow \begin{cases} x_1 = g_1(t) \\ x_2 = g_2(t) \end{cases}$$

  Eliminate the time t between the two parametric curves

- Second approach : Eliminate the time t first

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}$$

  Solve the new differential equation (with separated variables)

◇ Theses methods are restricted to quite simple/particular nonlinearities

Example

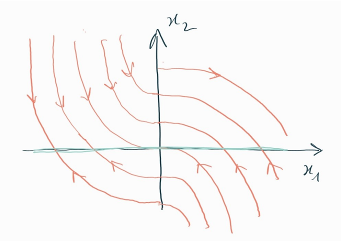
Consider the system

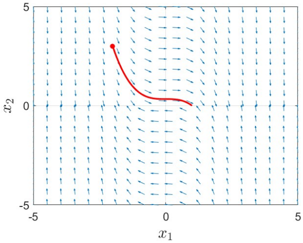
$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_2 x_1^2 \end{cases} \qquad \text{with} \qquad x_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$$

- Equilibrium points : $x_1^* \in \mathbb{R}$ and $x_2^* = 0 \quad \Rightarrow \quad x_1$-axis
- Analytical resolution :

$$x_2 = -\frac{1}{3}x_1^3 + \underbrace{x_{20} + \frac{1}{3}x_{10}^3}_{\text{cst}}$$

- Sketch and simulation





Isoclines method

Isocline = locus in the phase plane of trajectory's points of given slope α

$$s(x_1, x_2) = \alpha = \frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}$$

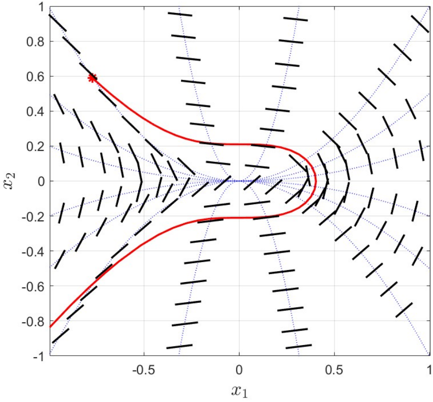
Step :

- For a given α, draw the curve such that $s(x_1, x_2) = \alpha$
- Along the curve, draw small segments of slope α
- Each segment is tangent to a trajectory, the direction s given by sign of $f_1(x)$ and $f_2(x)$
- Repeat from first step to draw several isoclines, for different α
- Then, from a given initial condition x_0, sketch a solution joining segments

◇ Also restricted to quite simple/particular nonlinearities

Example

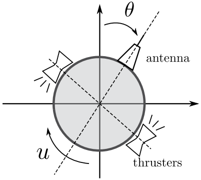
$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1^2 \end{cases} \qquad \text{slope :} \quad \alpha = \frac{f_2(x)}{f_1(x)} = \frac{-x_1^2}{x_2} \quad \Leftrightarrow \quad x_2 = -\frac{1}{\alpha} x_1^2$$

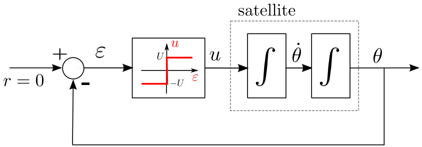


Plot for $\alpha = \{-5, -2, -1, -0.1, 0.1, 1, 2, 5\}$

Exercise (analytical method)

Consider the simple control of a simple satellite model





- Write the state space model
- What is (are) the the equilibrium point(s) ?
- Express x_1 as a function of x_2
- Draw a sketch of the phase portrait.

Solution :

Exercise (isocline method)

Consider the previous (controlled) system

- Apply the isocline method to retrieve the phase portrait

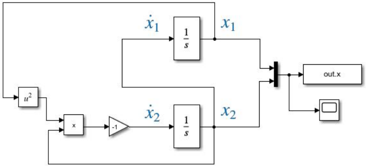
Solution :

Numerical simulations

General steps with MATLAB

- Define the system (function f) with a MATLAB function or Simulink

```
% anonymous functions
f = @(t,x) [x(2); -x(2)*x(1)^2];
```



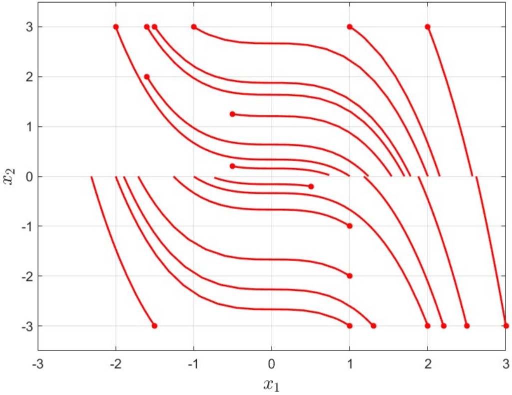
- Select an initial point x_0
- Solve the differential equation $\dot{x} = f(x)$

```
x0 = [-2;3];
[t,x] = ode45(f,[0 20],x0);
x1 = x(:,1);
x2 = x(:,2);
plot(x1,x2);
plot(x1(1),x2(2),'*');
```

- Repeat from step 2

Numerical simulations

Resulting plot for several x_0



In MATLAB, the instruction `quiver` plots the vector field

Sommaire

What about linear systems ?

Autonomous linear system :

$$\begin{cases} \dot{x}_1 = a_{11}\, x_1 + a_{12}\, x_2 \\ \dot{x}_2 = a_{21}\, x_1 + a_{22}\, x_2 \end{cases} \qquad \Leftrightarrow \qquad \dot{x} = Ax$$

- Solution : $x(t) = e^{At} x_0$
- Jordan canonical form with a change of basis : $Mz = x$

  Simpler system : $\dot{z} = \underbrace{M^{-1}AM}_{J}\, z \quad \Rightarrow \quad$ Solution : $z(t) = e^{Jt} z_0$
- According to eigenvalues of $A \rightarrow$ different forms for J

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad \begin{bmatrix} \lambda & k \\ 0 & \lambda \end{bmatrix} \qquad \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

(k = 0 or 1) / (if an eigenvalue = 0 $\rightarrow$ specific study)

Case 1 : real distinct eigenvalues

Two eigenvalues : $\lambda_1 \neq \lambda_2 \neq 0$

- Change of basis matrix $M = [v_1, v_2]$ made of the eigenvectors
- Give two decoupled first-order differential equation

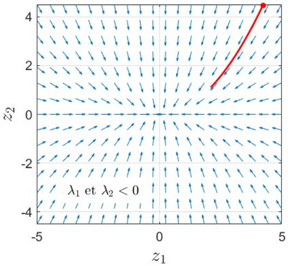
$$\left\{ \begin{array}{l} \dot{z}_1 = \lambda_1 \, z_1 \\ \dot{z}_2 = \lambda_2 \, z_2 \end{array} \right. \quad \Rightarrow \quad \left\{ \begin{array}{l} z_1(t) = z_{10} \, e^{\lambda_1 t} \\ z_2(t) = z_{20} \, e^{\lambda_2 t} \end{array} \right.$$

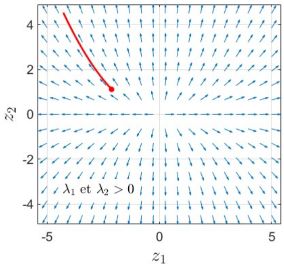
- Eliminate the time t

$$z_2 = c \, z_1^{\lambda_2/\lambda_1} \qquad \text{with } c = \frac{z_{20}}{z_{10}^{\lambda_2/\lambda_1}}$$

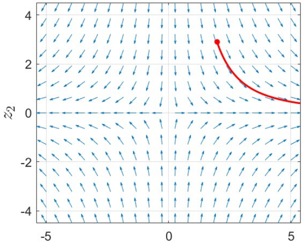
The shape of the curves depends on signs of λ_1 and λ_2

- Same signs $\Rightarrow$ the equilibrium point is a **stable** or **unstable node**



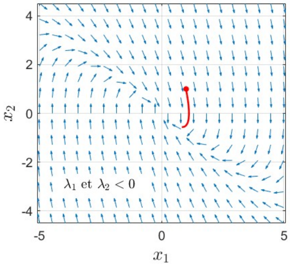


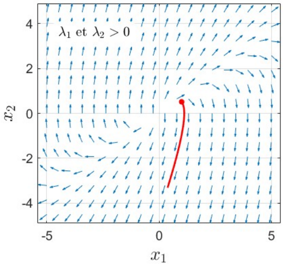
- Opposite signs $\Rightarrow$ the equilibrium point is a **saddle point**



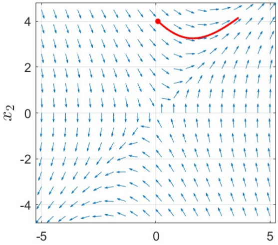
Back in the x-coordinates basis : $x = Mz$

- Same signs $\Rightarrow$ the equilibrium point is a **stable** or **unstable node**





- Opposite signs $\Rightarrow$ the equilibrium point is a **saddle point**



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Case 2 : real identical eigenvalues

Two eigenvalues : $\lambda_1 = \lambda_2 = \lambda \neq 0$

- Change of basis matrix $x = Mz$ (eigenvectors or chain of eigenvect.)
- Give two first-order differential equation

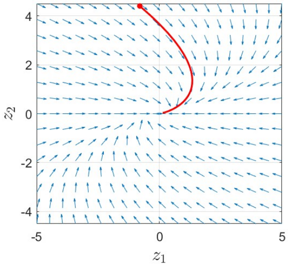
$$\begin{cases} \dot{z}_1 = \lambda\, z_1 + k\, z_2 \\ \dot{z}_2 = \lambda\, z_2 \end{cases} \quad \Rightarrow \quad \begin{cases} z_1(t) = \left(z_{10} + k z_{20} t\right)\, e^{\lambda t} \\ z_2(t) = z_{20}\, e^{\lambda t} \end{cases}$$

- If $k = 0$, particular case of the previous one
- Eliminate the time t

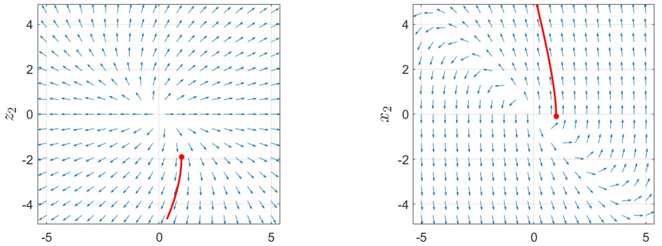
$$z_1 = z_2 \left(\frac{z_{10}}{z_{20}} + \frac{k}{\lambda} \ln\left(\frac{z_2}{z_{20}}\right)\right) \qquad \text{and also} \qquad \frac{dz_2}{dz_1} = \frac{\lambda z_2}{\lambda z_1 + k z_2}$$

Again, the shape of the curves depends on sign of λ

- negative $\Rightarrow$ the equilibrium point is a **stable node**



- positive $\Rightarrow$ the equilibrium point is an **unstable node**



Case 3 : complex conjugate eigenvalues

Two eigenvalues : $\lambda_{1,2} = \alpha \pm j\beta$

$\rightarrow$ Two complex conj. eigenvectors v_1 and $v_2 = \bar{v}_1$

- Change of basis matrix with $M = \begin{bmatrix} \mathrm{R}_e[v_1] \; , \; \mathrm{I}_m[v_1] \end{bmatrix}$

$$\left\{ \begin{array}{l} \dot{z}_1 = \alpha\, z_1 + \beta\, z_2 \\ \dot{z}_2 = -\beta\, z_1 + \alpha\, z_2 \end{array} \right.$$

- Change of variable $\rightarrow$ polar coordinates : $z_1 = r\cos\theta$ and $z_2 = r\sin\theta$

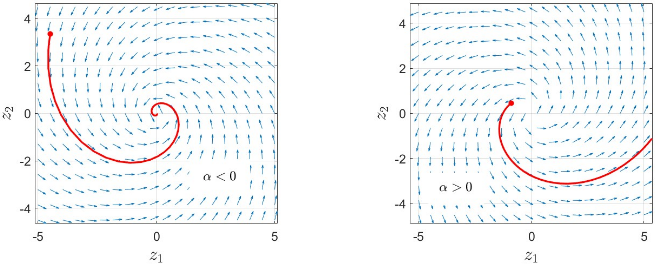
$$\left\{ \begin{array}{l} \dot{r} = \alpha\, r \\ \dot{\theta} = -\beta \end{array} \right.$$

- that has for solution :

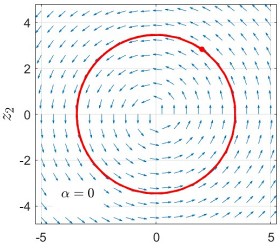
$$\left\{ \begin{array}{l} r(t) = r_0\, e^{\alpha\, t} \\ \theta(t) = -\beta\, t + \theta_0 \end{array} \right. \qquad \text{with} \qquad \left\{ \begin{array}{l} r_0 = \sqrt{z_{10}^2 + z_{20}^2} \\ \theta_0 = \arctan \frac{z_{20}}{z_{10}} \end{array} \right.$$

The shape of the curves depends on signs of $\alpha = \mathrm{R}_e[\lambda]$

- negative or positive real part $\Rightarrow$ the equ. pt is a **stable** or **unstable focus**

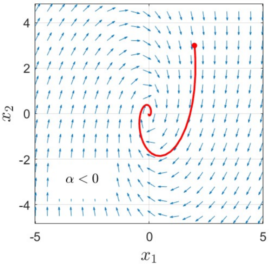


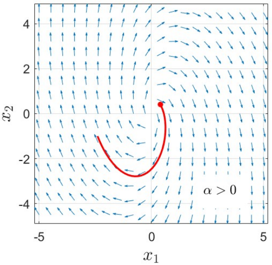
- Pure imaginary $\Rightarrow$ the equilibrium point is a **center** (circle of radius r_0)



Back in the x-coordinates basis : $x = Mz$

- negative or positive real part $\Rightarrow$ the equ. pt is a **stable** or **unstable focus**





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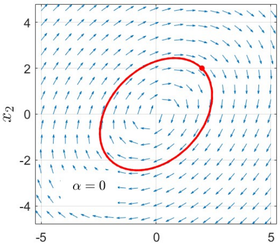


Case 4 (degenerate) : one or both eigenvalues are zero

Matrix A is singular $\rightarrow$ an equilibrium subspace (infinitely many points)

First case : $\lambda_1 = 0$ and $\lambda_2 \neq 0$

- Change of basis gives

$$\left\{ \begin{array}{l} \dot{z}_1 = 0 \\ \dot{z}_2 = \lambda_2\, z_2 \end{array} \right. \quad \Rightarrow \quad \left\{ \begin{array}{l} z_1(t) = z_{10} \\ z_2(t) = z_{20}\, e^{\lambda_2 t} \end{array} \right.$$

- if $\lambda_2 < 0$, trajectories converge, and if $\lambda_2 > 0$, they diverge

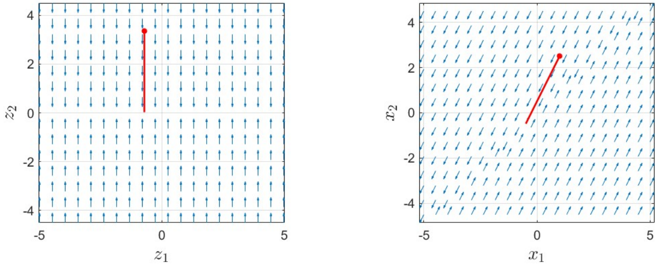
Second case : $\lambda_1 = \lambda_2 = 0$

- Change of basis gives

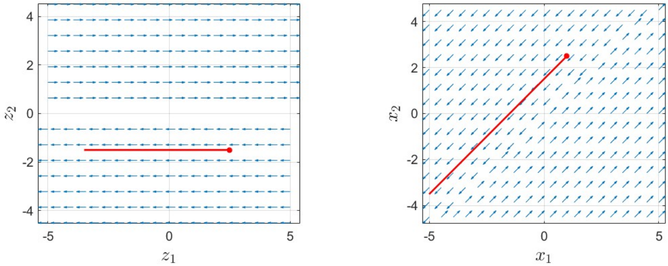
$$\left\{ \begin{array}{l} \dot{z}_1 = z_2 \\ \dot{z}_2 = 0 \end{array} \right. \quad \Rightarrow \quad \left\{ \begin{array}{l} z_1(t) = z_{10} + z_{20}\, t \\ z_2(t) = z_{20} \end{array} \right.$$

- z_1 increases or decreases depending on the sign of z_{20}

- First case, $\lambda_1 = 0$ and $\lambda_2 \neq 0$ (below $\lambda_2 < 0$)



- Second case, $\lambda_1 = \lambda_2 = 0$



Recap

Qualitative behavior for linear systems around the isolated equilibrium $x = 0$

- Real eigenvalues
  - λ_1 and λ_2 positive $\Rightarrow$ **unstable node**
  - λ_1 and λ_2 negative $\Rightarrow$ **stable node**
  - λ_1 and λ_2 opposite $\Rightarrow$ **saddle point**
- Complex conjugate eigenvalues
  - real part $\alpha > 0 \Rightarrow$ **unstable focus**
  - real part $\alpha < 0 \Rightarrow$ **stable focus**
  - real part $\alpha = 0 \Rightarrow$ **center**

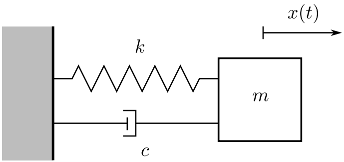
Behavior determined by the eigenvalues of A

- Determined for the whole plane (**global**), characteristic of linear systems
- For nonlinear systems, study interesting to get the **local** behavior around an equilibrium point

Example 1 : simple mass-spring system

Equation of motion :

mass ($m = 1\ kg$)
spring (stiffness : $k = 1\ N/m$)
damper (viscous coefficient : $c\ N/m/s$)

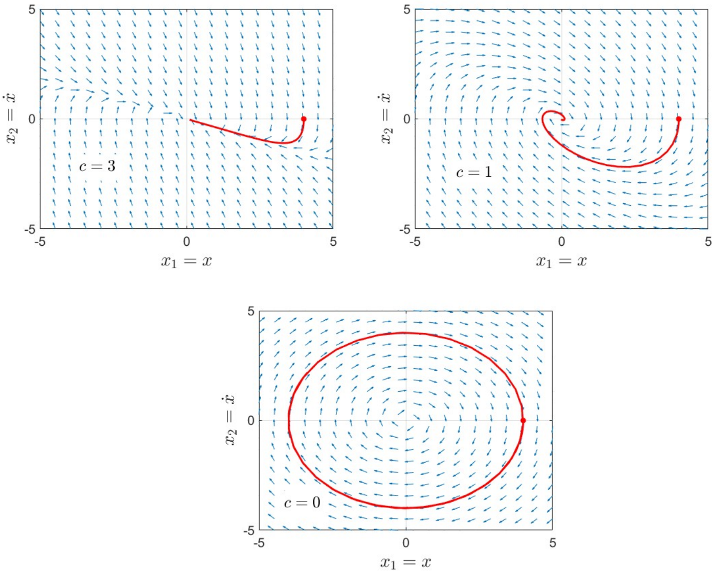


$$\ddot{x} + c\dot{x} + x = 0 \quad \Rightarrow \quad \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \quad \text{with} \begin{cases} x(0) = x_0 \\ \dot{x}(0) = 0 \end{cases}$$

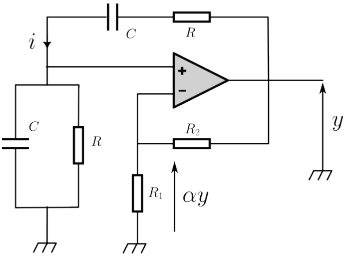
Eigenvalues of the dynamic matrix

$c \geq 2$	$0 < c < 2$	$c = 0$
$\lambda_{1/2} = \frac{-c \pm \sqrt{c^2 - 4}}{2}$	$\lambda_{1/2} = -\frac{c}{2} \pm i \frac{\sqrt{\lvert c^2 - 4 \rvert}}{2}$	$\lambda_{1/2} = \pm i$
noeud stable	foyer stable	centre

Simulation of the mass-spring system



Example 2 : Wien bridge oscillator



Kirchhoff's circuit laws :

with $\alpha = \dfrac{R_1}{R_1 + R_2}$

$$R^2C^2\ddot{y} + RC\frac{3\alpha - 1}{\alpha}\dot{y} + y = 0 \quad \Rightarrow \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{R^2C^2} & -\frac{1}{RC}\frac{3\alpha-1}{\alpha} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

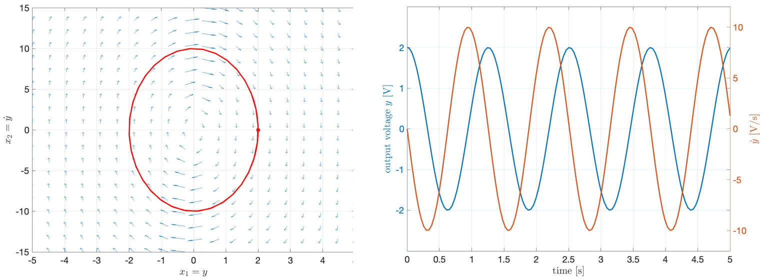
Charac. pol. $= \lambda^2 + \frac{1}{RC}\frac{3\alpha-1}{\alpha}\lambda + \frac{1}{R^2C^2}$

$$\begin{cases} \zeta = \dfrac{3\alpha - 1}{2\alpha} \\ \omega_n = \dfrac{1}{RC} \end{cases} \Rightarrow \begin{cases} \zeta > 1 & \text{si} & \alpha > 1 & \rightarrow \text{noeud stable} \\ \zeta < 0 & \text{si} & \alpha < 1/3 & \rightarrow \text{instable} \\ 0 < \zeta < 1 & \text{si} & 1/3 < \alpha < 1 & \rightarrow \text{foyer stable} \\ \zeta = 0 & \text{si} & \alpha = 1/3 & \rightarrow \text{centre} \end{cases}$$

Simulation of the Wien bridge oscillator

with

- circuit parameters : $RC = 0.2\,\Omega F \rightarrow \omega_n = 5\,rd/s \rightarrow T = 1.25\,s$
- $\alpha = 1/3 \qquad (\Leftrightarrow R_2 = 2R_1)$
- initial condition $x_0 = [2\,,\,0]^T$



Exercise 1

Consider the system

$$\dot{x} = \begin{bmatrix} -2 & 2 \\ 1 & -3 \end{bmatrix} x \qquad \text{with} \qquad x_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$$

- What is the qualitative behavior of the equilibrium point 0 ?
- What is the representation of the system in the z-coordinates ?
- Draw a sketch of the phase portrait in z and x-coordinates.

Solution :

Exercise 2

Consider the system

$$\dot{x} = \begin{bmatrix} 1 & -1 \\ 9 & 1 \end{bmatrix} x \qquad \text{with} \qquad x_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$$

- What is the qualitative behavior of the equilibrium point 0 ?
- What is the representation of the system in the z-coordinates ?
- Draw a sketch of the phase portrait in z-coordinates.

Solution :

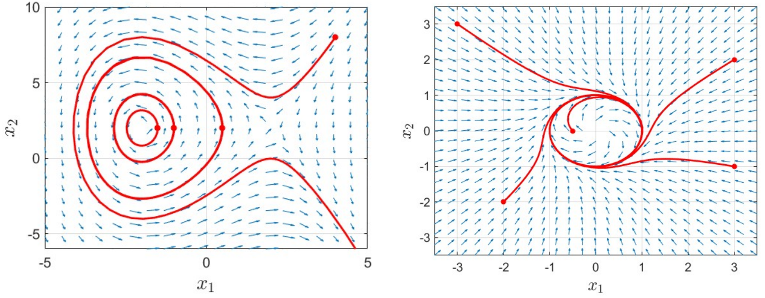
Sommaire

Closed orbits

A closed orbit is a periodic trajectory

Two cases can be distinguished :

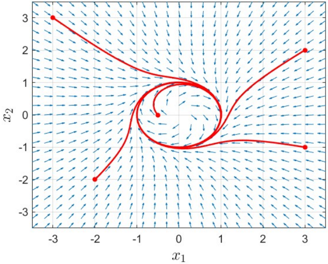
- **non-isolated** : there are other closed curves in the neighborhood, depend on initial conditions (left)
- **isolated** : from initial conditions in the neighborhood, trajectories converge or diverge from it $\rightarrow$ limit cycle (right)



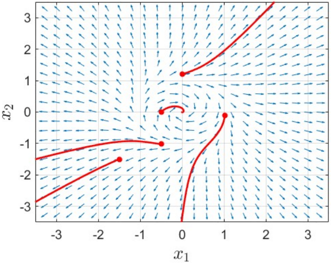
Limit cycles

Three kinds of limit cycle can be observed

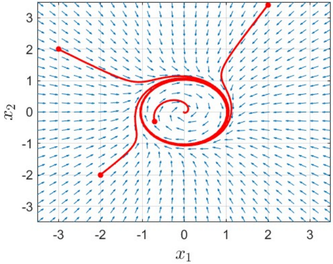
Stable limit cycle



Unstable limit cycle



Semi-stable limit cycle



Existence of limit cycles

Can we predict the existence of a limit cycle ?

3 theorems are stated that may help (valid only for 2^{nd} order autonomous systems)

Theorem (Poincaré)

If a closed orbit exists, then $N = S + 1$, with

- N, the number of nodes/centers/foci enclosed by the closed orbit
- S, the number of saddle points enclosed by the closed orbit

$\hookrightarrow$ A closed orbit must enclose at least one equilibrium point

Theorem (Poincaré-Bendixson)

If a trajectory remains in a closed bounded region $\mathcal{D}$ in the phase plane, then one of the following is true :

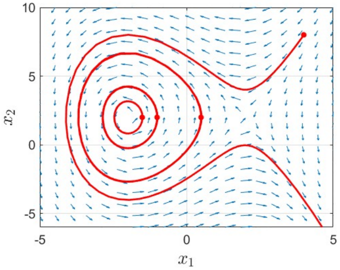
- the trajectory goes to an equilibrium
- the trajectory tends to a closed orbit
- the trajectory is itself a closed orbit

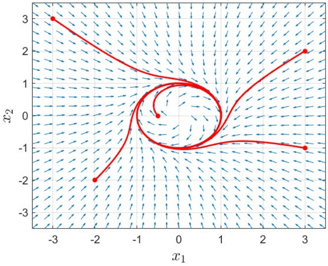
$\hookrightarrow$ Asymptotic properties of trajectories

These results can be easily verified on previous examples

$$\begin{cases} \dot{x}_1 = 4 - 2x_2 \\ \dot{x}_2 = 12 - 3x_1^2 \end{cases}$$

$$\begin{cases} \dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2 - 1) \\ \dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2 - 1) \end{cases}$$





Non-existence condition

This last theorem provides a sufficient condition for the non-existence of a limit cycle

Theorem (Bendixson)

No limit cycle can exist in a region $\mathcal{D}$ of the phase plane in which

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$$

does not vanish and does not change sign

Example :

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -ax_1(1 - bx_1^2) - cx_2 \end{cases}$$

with positive paramters $a, \ b, \ c \ > 0$

Let's apply formula

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 0 - c$$

$\hookrightarrow \neq 0$ and no change of sign $\Rightarrow$ no limit cycle

Sommaire

Case study

Prey-Predator model (or Lotka-Volterra model)

study the evolution of two populations x_1 (preys) and x_2 (predators)

$$\begin{cases} \dot{x}_1 = \alpha\, x_1 - \beta\, x_1 x_2 \\ \dot{x}_2 = \gamma\, x_2 x_1 - \delta\, x_2 \end{cases}$$

α, β, γ and δ are positive constant parameters

- $\alpha\, x_1$ is the growth rate of preys if there is no predators
- $\beta\, x_1 x_2$ is the death rate of preys because of predators
- $\gamma\, x_2 x_1$ is the growth rate of predators with x_1 preys available
- $\delta\, x_2$ is the death rate of predators

To simplify, let's set $\alpha = \beta = \gamma = \delta = 1$

Model :

$$\begin{cases} \dot{x}_1 = x_1(1 - x_2) \\ \dot{x}_2 = x_2(x_1 - 1) \end{cases}$$

- What is (are) the equilibrium point(s) ?
- Calculate the linearized model around it (them).
- What is (are) their nature ? Then, how heights will evolve ?
- Simulate the system to draw the phase portrait.

Solution

Solution

INSA TOULOUSE

In short

- Phase plane : study of the time evolution of the state for second order systems

  $\hookrightarrow$ trajectories of $x = (x_1, x_2)$ in the place and vector field

- Usually, numerical software are used to simulate system responses

  $\hookrightarrow$ with MATLAB, Scilab, Python... or your own program implementing numerical methods

- In the linear case, analytical solutions can be found and the nature of equilibrium point can be derived from eigenvalues

  $\hookrightarrow$ node, saddle point, focus, center, stable/unstable

- Useful when linearizing nonlinear systems to have the local behavior (around an equilibrium point)