Chapitre 1 : Introduction

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$$\dot{x} = f(x)$$

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1. Nonlinear models ?
2. Existence of a solution
3. Equilibrium point
4. Linearization
5. Case study

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1. Nonlinear models ?
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Linear systems

What you have seen so far... Models of the form

Linear ordinary différential equ.

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = u(t)$$

Transfer function

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2 + 3s + 2}$$

Linear state space

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

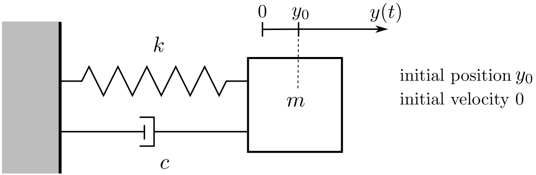
Fundamental property : superposition principle

$$\begin{cases} u_1(t) \xrightarrow{\text{lin. sys.}} y_1(t) \\ u_2(t) \xrightarrow{\text{lin. sys.}} y_2(t) \end{cases} \Rightarrow \quad a\,u_1(t) + b\,u_2(t) \xrightarrow{\text{lin. sys.}} a\,y_1(t) + b\,y_2(t)$$

↪ Allows to establish very strong and generic results

Example

Consider a mass spring damper system



A simple model is obtained from Newton's law

$$m\ddot{y} + c\dot{y} + ky = 0$$

One can derive a Laplace domain or a state space representation

$$Y(s) = \frac{y_0(ms + c)}{ms^2 + cs + k}$$

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} x$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

Nonlinear systems

More realistic models

$$\dot{x} = f(t, x, u)$$

where x is the state vector, u the input vector, $f(\cdot)$ a **nonlinear** function.

Other cases :

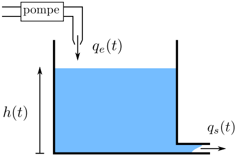
- Unforced system : $\dot{x} = f(t, x)$
- Autonomous system : $\dot{x} = f(x)$ **(case considered in the following)**
- Affine in u : $\dot{x} = f(x) + g(x)u$

Such a general modeling enables to better capture features of physical systems

$\hookrightarrow$ However, there is no general methods to deal with all nonlinear systems

Example 1

Liquid level control



S_1 : section réservoir
S_2 : section sortie
ρ : densité fluide
P_1 : pression en haut réservoir
P_2 : pression en sortie
v_1 : vitesse en haut réservoir
v_2 : vitesse en sortie

The change in mass in the tank is

$$\dot{m}(t) = \rho S_1 \dot{h}(t) = q_e(t) - \underbrace{\rho S_2 v_2(t)}_{q_s(t)}$$

Using the Bernouilli's equation : $\frac{1}{2}\rho v_1^2(t) + P_1 + \rho g h(t) = \frac{1}{2}\rho v_2^2(t) + P_2$

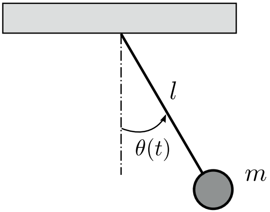
$$\hookrightarrow v_2(t) = \sqrt{2gh(t)}$$

Let us define the state variable $x = h$, we get :

$$\dot{x}(t) = -a\sqrt{x(t)} + \frac{1}{\rho} q_e(t), \qquad \text{with } a = \frac{S_2}{S_1}\sqrt{2g}$$

Example 2

A simple free pendulum



Applying the Newton's second law, the equation of motion is obtained :

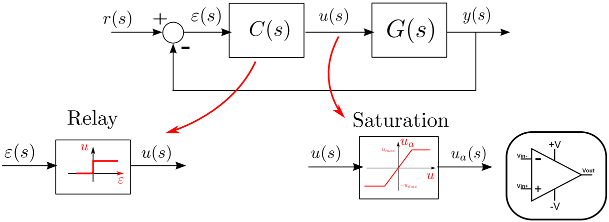
$$ml\,\ddot{\theta}(t) = -mg\sin\theta(t) - kl\,\dot{\theta}(t)$$

Let us define the state variables $x_1 = \theta$ and $x_2 = \dot{\theta}$, we get :

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2\end{bmatrix}$$

Origins of nonlinearities

- **Physical modeling**. Inherent to laws of Physics as in previous examples
- **Engineering design**. Inherent to how the system work, introduced by the engineer, technological aspect...



Nonlinear phenomena

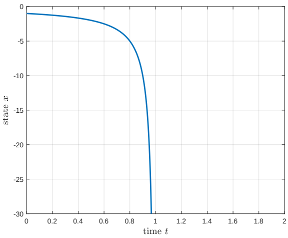
...that do not exist with linear modeling.

- **Multiple isolated equilibria**. Pendulum example
- **Finite escape time**. The state goes to infinity when time approaches a finite value. Example :

$$\dot{x} = -x^2, \qquad \text{with the initial condition } x(0) = -1$$

$\Rightarrow$ The solution is

$$x(t) = \frac{1}{t-1}$$



Nonlinear phenomena

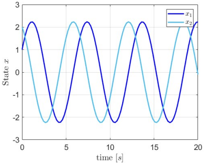
- **Limit cycles.**

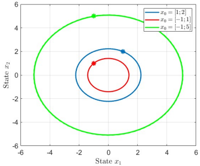
Linear case LTI systems oscillate if they have pure imaginary poles.

↪ It is a critical stability and nonrobust condition

↪ Oscillation amplitude depends on initial condition

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 \end{cases}$$



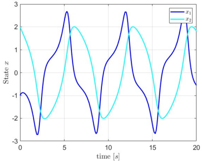


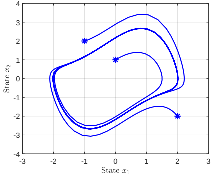
Nonlinear case Can produce stable oscillations

↪ with fixed amplitude and frequency independently from initial conditions

Van der Pol equation

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + (1 - x_1^2)x_2 \end{cases}$$





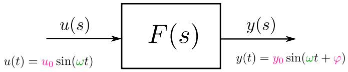
Nonlinear phenomena

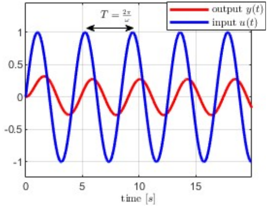
- **Frequency response**

Linear case The response to a sine function is also a sine function (at steady state)

↪ with the same frequency ω

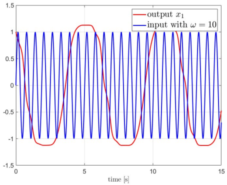
↪ and different amplitude and phase shift w.r.t. ω

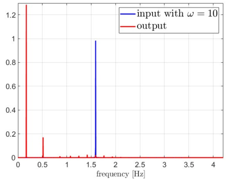




Nonlinear case Can produce harmonics, subharmonics, and even almost-periodic output

$$\begin{cases} \dot{x}_1 = x_2^3 \\ \dot{x}_2 = -x_1 + (1 - x_2)u \end{cases}$$





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Existence of a solution

Question : (Cauchy problem)

Let be the system

$$\dot{x} = f(t, x), \qquad \text{with the initial condition } x(t_0) = x_0 \in \mathbb{R}^n$$

Does a solution $x(t)$ exist for $t > t_0$? Is it unique ? dependence on init. cond. ?

Theorem : local existence and uniqueness

If $f(t, x)$ is piecewise continuous in t and satisfy the **Lipschitz condition**, that is, there exists a constant $L > 0$ such that $\forall x_1, x_2 \in B = \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq r\}$, and $\forall t \in [t_0, t_1]$

$$\|f(t, x_2) - f(t, x_1)\| < L\|x_2 - x_1\|$$

then, there exists some $\delta > 0$ such that the above system has a **unique solution** over $[t_0, t_0 + \delta]$.

Example 1

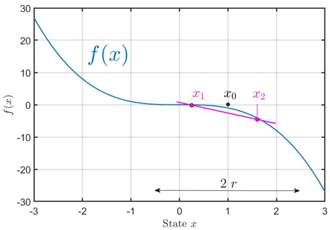
$$\dot{x} = -x^3 \qquad \text{with } x(0) = 1$$

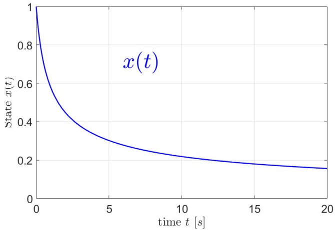
$-x^3$ is Lipschitz for all x such that $|x - x_0| \leq r = 1.5$

$$\frac{|-x_2^3 - (-x_1^3)|}{|x_2 - x_1|} \leq L$$

(but not true $\forall x \in \mathbb{R}$)

$\Rightarrow$ It exists a unique solution : $x(t) = \dfrac{1}{\sqrt{1+2t}}$





Example 2

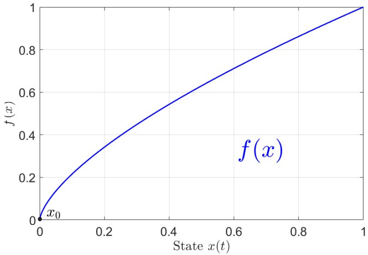
$$\dot{x} = x^{2/3} \qquad \text{with } x(0) = 0$$

has two solutions (non unicity) : $x(t) = 0$ and $x(t) = \frac{1}{27}t^3$.

$\Rightarrow$ actually, $x^{2/3}$ not Lipschitz around 0

$$\frac{|x^{2/3} - 0|}{|x - 0|} = |x^{-1/3}|$$

(not bounded when $x \to 0$)



Example 3

$$\dot{x} = -x^2, \qquad \text{with } x(0) = -1$$

$$\Rightarrow -x^2 \text{ is Lipschitz for } \forall x_1, x_2 \in B = \{x \in \mathbb{R} \mid |x - x_0| \leq r\}$$

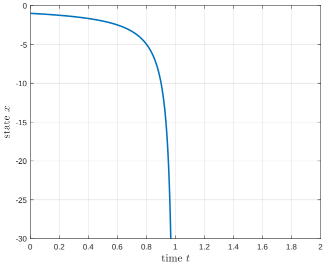
$$\frac{|-x_2^2 - (-x_1^2)|}{|x_2 - x_1|} \leq L$$

(locally Lipschitz $\forall x \in \mathbb{R}$)

$\Rightarrow$ a unique solution for $t \in [0, \delta]$

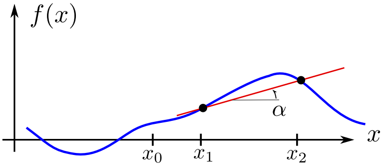
$$x(t) = \frac{1}{t - 1}$$

but $\delta < 1$



Lipschitz condition and derivative of f

Scalar and autonomous example : $\dot{x} = f(x)$ with $x \in \mathbb{R}$



A unique solution exists if

$$\frac{|f(x_2) - f(x_1)|}{|x_2 - x_1|} = \alpha \leq L \qquad \forall x_1, x_2 \in B = \{x \in \mathbb{R} \mid |x - x_0| \leq r\}$$

$\hookrightarrow$ then $f(x)$ is Lipschitz if $|f'(x)|$ is bounded by L

Lipschitz condition and derivative of f

This observation extends to vector-valued functions

$$\left\| \frac{\partial f}{\partial x}(t,x) \right\| \leq L \qquad f \text{ is Lipschitz} \qquad \text{(for some domain)}$$

Lemma : Locally Lipschitz

If $f(t,x)$ and $\frac{\partial f}{\partial x}(t,x)$ are continuous on $[t_0, t_1] \times D$, for some domain $D \subset \mathbb{R}^n$, then f is locally Lipschitz on $[t_0, t_1] \times D$.

Lemma : Globally Lipschitz

If $f(t,x)$ and $\frac{\partial f}{\partial x}(t,x)$ are continuous on $[t_0, t_1] \times \mathbb{R}^n$, then f is globally Lipschitz on $[t_0, t_1] \times \mathbb{R}^n$ if and only if $\frac{\partial f}{\partial x}$ is uniformly bounded on $[t_0, t_1] \times \mathbb{R}^n$.

Back on previous examples

Example 2 : $\dot{x} = x^{2/3}$, with $x(0) = 0$

$$\left(x^{2/3}\right)' = \frac{2}{3}x^{-1/3}$$

Hence, $|f(x)'|$ unbounded at 0 $\Rightarrow f$ not Lipschitz around 0

Example 3 : $\dot{x} = -x^2$, with $x(0) = -1$

$$\left(-x^2\right)' = -2x$$

Hence, $|f(x)'|$ bounded for any x in some domain $D \Rightarrow f$ locally Lipschitz $\forall x \in \mathbb{R}$

Exercise

Consider system

$$\dot{x} = f(x) = -x^2 + a\sin(x)$$

Is $f(x)$ Lipschitz (locally or globally) or not ?

Exercise

Consider system

$$\dot{x} = \underbrace{\begin{bmatrix} -x_1 + x_1 x_2 \\ x_2 - x_1 x_2 \end{bmatrix}}_{f(x)}$$

Is $f(x)$ Lipschitz (locally or globally) or not ?

Exercise

Consider system

$$\dot{x} = f(x) = -x + a\sin(x)$$

Is $f(x)$ Lipschitz (locally or globally) or not ?

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Equilibrium point

Definition

A point $x^\star$ is an **equilibrium point** if when the current state $x = x^\star$, the system remains at this point ($\rightarrow \dot{x} = 0$). The equilibrium points are given by the roots of

$$f(x) = 0$$

For the pendulum example, equilibrium points are characterized by

$$\left\{ \begin{array}{l} 0 = x_2 \\ 0 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 \end{array} \right. \Rightarrow \left\{ \begin{array}{ll} x_2^\star = 0 & \\ x_1^\star = 0 \pm \ n\pi, & n = 0, 1, 2, \ldots \end{array} \right.$$

$\hookrightarrow$ mathematically infinitely many points, physically two positions

Exercise

Consider system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1(1 - a^2 x_1^2) - x_2 \end{cases}$$

where $a > 0$ is a constant parameter.

Calculate the equilibrium point(s) ?

Reminder : for linear systems

$$\dot{x} = Ax + Bu \qquad \text{(}A\text{ being non-singular)}$$

there can be only one isolated equilibrium point $x^\star = -A^{-1}Bu^\star$.

- This equilibrium point is 0 in the case of an unforced system $\dot{x} = Ax$.
- If A is singular, there are infinitely many continuous equilibrium points (not isolated), this set is a subspace in the state-space.

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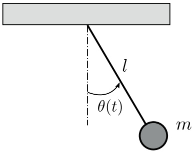
Linearization

Linear approximation of a nonlinear model around an equilibrium point

Pendulum example :

$$ml\,\ddot{\theta} = -mg\sin\theta - kl\,\dot{\theta}$$



around $\theta = 0$, $\quad \sin(\theta) \sim \theta$

A **linear model** is obtained :

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2\end{bmatrix} \xrightarrow{\text{around } \theta=0} \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} \simeq \begin{bmatrix} x_2 \\ -\frac{g}{l}x_1 - \frac{k}{m}x_2\end{bmatrix}$$

$$\simeq \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{k}{m}\end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

More generally

Let's consider an equilibrium point $x^\star$ for system

$$\dot{x} = f(x), \qquad \text{with } x(0) = x_0$$

and define the deviation variable : $\tilde{x} = x - x^\star$

Its dynamic is

$$\dot{\tilde{x}} = \dot{x} = f(x) = f(x^\star + \tilde{x}), \qquad \text{with } \tilde{x}(0) = x_0 - x^\star$$

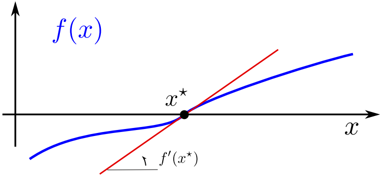
Use **Taylor series** around $x^\star$

$$f(x^\star + h) = f(x^\star) + f'(x^\star)h + \frac{1}{2!}f''(x^\star)h^2 + \frac{1}{3!}f^{(3)}(x^\star)h^3 + \cdots$$

valid if h $(= \tilde{x})$ small enough

Linear approximation (scalar case)

$$f(x^\star + h) \simeq \underbrace{f(x^\star)}_{=0} + f'(x^\star)h + \cancel{\frac{1}{2!}f''(x^\star)h^2} + \cancel{\frac{1}{3!}f^{(3)}(x^\star)h^3} + \cancel{\cdots}$$



For our system

$$\begin{aligned}\dot{\tilde{x}} &= f(x^\star + \tilde{x}) \\ &\simeq f'(x^\star)\,\tilde{x}\end{aligned}$$

$\Rightarrow$ **linear model** of the form : $\dot{\tilde{x}} \simeq a\,\tilde{x},$ with $\tilde{x}(0) = x_0 - x^\star$

Linear approximation (general case)

Let $x^\star$ be an equilibrium point for system $\dot{x} = f(x)$, a linear model around that point is given by :

$$\dot{\tilde{x}} \simeq \underbrace{\frac{\partial f}{\partial x}(x^\star)}_{A} \ \tilde{x} \qquad \text{with } \tilde{x} = x - x^\star$$

and $\frac{\partial f}{\partial x}(\cdot)$ the Jacobian matrix of the vector-valued function f at the equ. pt.

- Reminder, Jacobian matrix :

$$\frac{\partial f}{\partial x}(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

- One could also linearize around an operating point or a trajectory

Back on the liquid level example

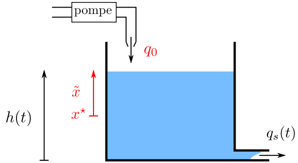
Nonlinear model :

$$\dot{x}(t) = -a\sqrt{x(t)} + \frac{1}{\rho}q_e(t), \qquad \text{with } x(0) = 0.5\ m$$

For a constant input mass flow rate $q_e(t) = q_0\ kg/s \Rightarrow$ equilibrium pt $x^\star = \left(\frac{q_0}{a\rho}\right)^2$

Linearization :

$$\dot{\tilde{x}} = -\frac{a}{2\sqrt{x^\star}}\ \tilde{x}$$



Back on the pendulum example

Nonlinear model :

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2 \end{bmatrix}$$

Consider the equilibrium pt $x^\star = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$

Linearization :

Jacobian matrix

$$\frac{\partial f}{\partial x}(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} \cos x_1 & -\frac{k}{m} \end{bmatrix}$$

Linear model

$$\dot{\tilde{x}} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{k}{m} \end{bmatrix} \tilde{x}$$

Exercise

Consider system

$$\begin{cases} \dot{x}_1 = -x_1 + x_1 x_2 \\ \dot{x}_2 = x_1 + x_2 - 2x_1 x_2 \end{cases}$$

Calculate the equilibrium point(s) ? Linearize the system around $(1, 1)$

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Case study

Population dynamics study the evolution of the size $N(t)$ of a population

First simple model : *Malthus model*

$$\dot{N}(t) = \alpha N(t) - \beta N(t)$$

α is the birth rate and β the death rate

- Model is linear are nonlinear ?
- What is (are) the equilibrium point(s) ?
- Existence and unicity of the solution ?

Solution

Second case

Second model : *Verhulst (or logistic) model*

$$\dot{N}(t) = rN(t)\left(1 - \frac{N(t)}{K}\right)$$

that takes into account a maximal critical size of the population K (carrying capacity). r is the growth rate.

- Model is linear are nonlinear ?
- What is (are) the equilibrium point(s) ?
- Existence and unicity of the solution ?

Solution

In short

- Nonlinear model are very general model

$$\dot{x} = f(x)$$

  Results for linear model $\dot{x} = Ax$ not applicable

- A solution exists and is unique if a Lipschitz condition is satisfied.

- The equilibrium points x^* are given by the roots of

$$f(x) = 0$$

- A nonlinear system may be approximated by a linear system around an equilibrium point

$$\dot{x} = f(x) \qquad \xrightarrow{approx} \qquad \delta\dot{x} = A\,\delta x$$

  with $x = x^* + \delta x$ and $A = \frac{\partial f}{\partial x}(x^*)$