L'art et la manière de faire monde

Digital Control

Yassine ARIBA

version 3.0

Sommaire

Introduction
- Review
- General points
- Analog to digital converter
- Digital to analog converter

Modelling
- Discrete-time systems
- Sampled systems

Feeback system analysis
- Stability
- Tracking error

Controller design
- Principle
- Discretization methods
- Implementation

Sommaire

Introduction

- Review
- General points
- Analog to digital converter
- Digital to analog converter

Modelling

- Discrete-time systems
- Sampled systems

Feeback system analysis

- Stability
- Tracking error

Controller design

- Principle
- Discretization methods
- Implementation

Introduction

Review on classical automatic control

These last years, you have learned about “Automatic Control”

- It is a distinct scientific discipline
- Many areas of applications : aeronautic and space, robotic, automotive, process control,...
- Several connections with other engineering disciplines : mechanical, electrical, thermodynamics, chemical, computer science,...
- Based on a system approach, general and theoretical approach but for concrete problems

What is “Control” ?

What is "Control" ?

Make some object - called system - behaves as desired

system = aircraft, car, satellite, machine tool,
chemical reaction, communication network,...

What is “Control” ?

Make some object - called system - behaves as desired

system = aircraft, car, satellite, machine tool,
chemical reaction, communication network,...

A **system** is a collection of components which are coordinated together to perform a function.
Systems interact with their environment, those interactions are **signals** :

- inputs
- outputs
- disturbances

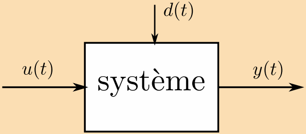
What is “Control” ?

Make some object - called system - behaves as desired

system = aircraft, car, satellite, machine tool,
chemical reaction, communication network,...

A **system** is a collection of components which are coordinated together to perform a function.
Systems interact with their environment, those interactions are **signals** :

- inputs
- outputs
- disturbances

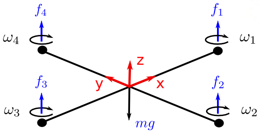


What is “Control” ?

Balance control of a drone

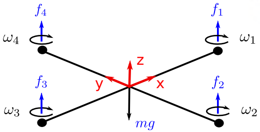
What is “Control” ?

Balance control of a drone



What is "Control" ?

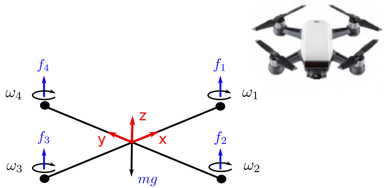
Balance control of a drone

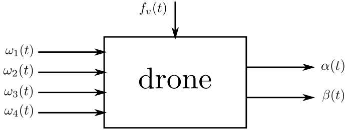


- inputs = rotation speed of blades - $\omega_i(t)$
- outputs = pitch and roll angles - $\alpha(t)$, $\beta(t)$
- disturbance = wind force - $f_v(t)$

What is "Control" ?

Balance control of a drone

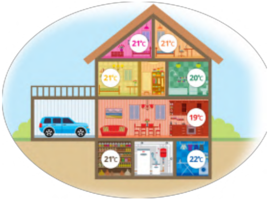




- inputs = rotation speed of blades - $\omega_i(t)$
- outputs = pitch and roll angles - $\alpha(t)$, $\beta(t)$
- disturbance = wind force - $f_v(t)$

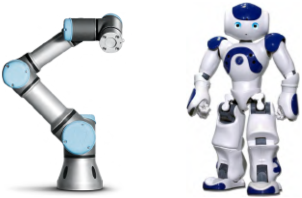
What is “Control” ?

Temperature control

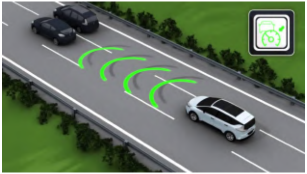


Attitude and orbit control

Motion control



Speed control



What is “Automatic” ?

What is “Automatic” ?

- Not manual !
- No human operator
- The system works in an autonomous way

What is “Automatic” ?

- Not manual !
- No human operator
- The system works in an autonomous way

Why ?

What is “Automatic” ?

- Not manual !
- No human operator
- The system works in an autonomous way

Why ?

- convenient (just give setpoints : temperature, speed, position... ; repetitive tasks)
- hostile environment (space, hot/cold places, dangerous places...)
- impossible for human (small scale, strict requirements : precision, rapidity, minimal energy consumption...)

What is “Automatic” ?

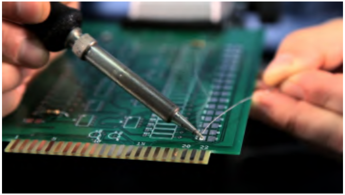
Example : cruise control with a car

Manual

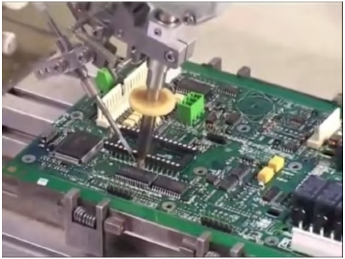
Automatic

What is “Automatic” ?

Example : soldering of electronic components on circuit boards



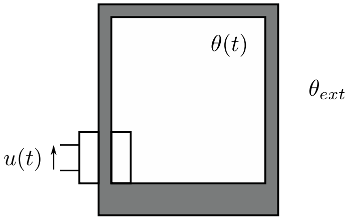
Manual



Automatic

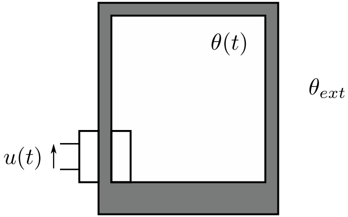
Example

Let us control the temperature in an industrial oven



Example

Let us control the temperature in an industrial oven



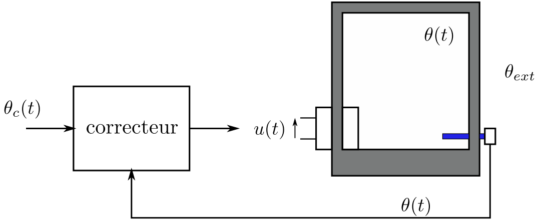
- A heating system is controlled with the voltage $u(t)$.
- The temperature inside the oven is $\theta(t)$.
- The ambiant temperature is θ_{ext}.
- Desired temperature : 40°

Example

What do we do to control the temperature $\theta(t)$?

Example

What do we do to control the temperature $\theta(t)$?

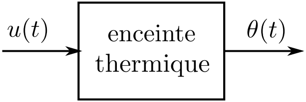


- A sensor measures the output $\theta(t)$
- A control law is computed as a function of the output and the reference
- The computed value drives the system input $u(t)$

$\Rightarrow$ Closed-loop control

Example

Theoretical study

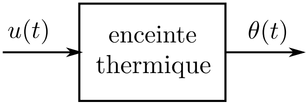


First, a model is required between the input $u(t)$ and the output $\theta(t)$

$$240\ \dot{\theta}(t) + \theta(t) = 4\ u(t)$$

Example

Theoretical study



First, a model is required between the input $u(t)$ and the output $\theta(t)$

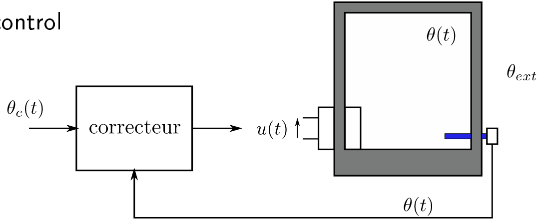
$$240\ \dot{\theta}(t) + \theta(t) = 4\ u(t)$$

we deduce the transfer function of the system :

$$\frac{\Theta(p)}{U(p)} = \frac{4}{240p + 1}$$

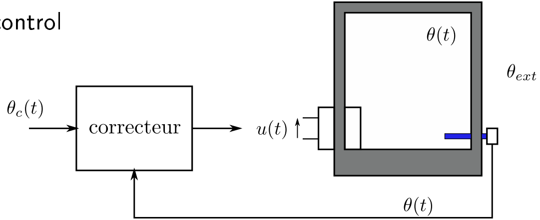
Example

Closed-loop control

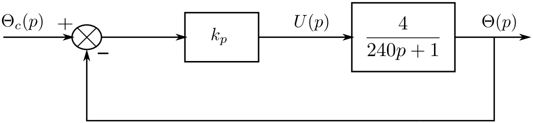


Example

Closed-loop control



Proportional controller



Example

Proportional control law : $$U(p) = k_p\Big(\Theta_c(p) - \Theta(p)\Big)$$

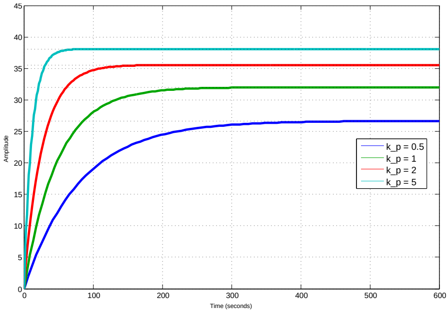
that is in time domain : $$u(t) = k_p\Big(\theta_c(t) - \theta(t)\Big)$$

Example

Proportional control law : $U(p) = k_p\Big(\Theta_c(p) - \Theta(p)\Big)$

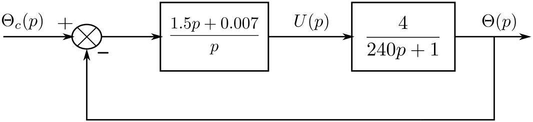
that is in time domain : $u(t) = k_p\Big(\theta_c(t) - \theta(t)\Big)$

Step response with $\theta_c(t) = 40°, \forall t \geq 0$



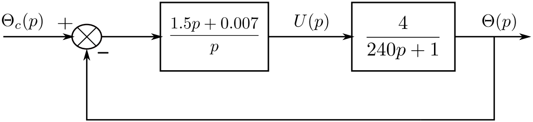
Example

PI controller

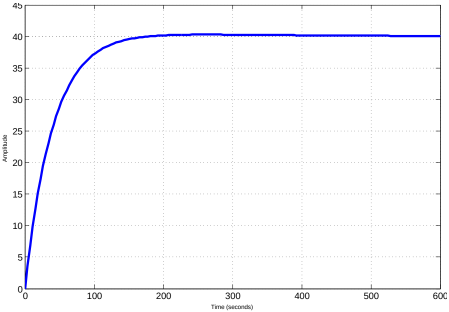


Example

PI controller



Step response with $\theta_c(t) = 40°$, $\forall t \geq 0$



Various theoretical concepts

Example presented through simulations...

Various theoretical concepts

Example presented through simulations...

... but several calculation need to be carried out :

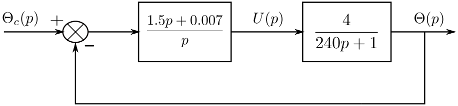
Various theoretical concepts

Example presented through simulations...

... but several calculation need to be carried out :

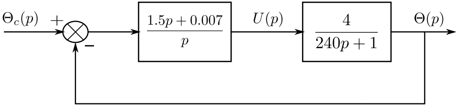
- time response (solving the ODE)
- frequency response (Bode or Nyquist diagrams)
- stability (poles, Routh or Nyquist criteria)
- static and tracking error (transfer function and final value theorem)
- stability margin (gain and phase margins with Bode/Nyquist diagrams)
- design of a controller (P, PID, phase lead...)

Example

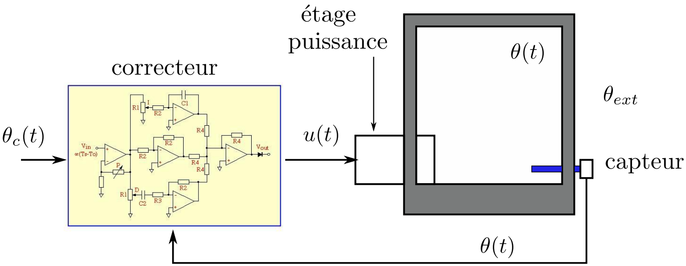


Implementation ?

Example



Implementation ?

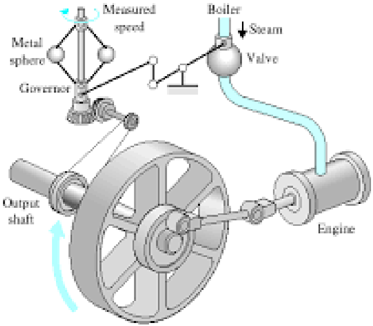


$\Rightarrow$ **analog** control system

Another example (historical)

Speed control of a steam engine

James Watt's flyball governor in 1788 during the first industrial revolution



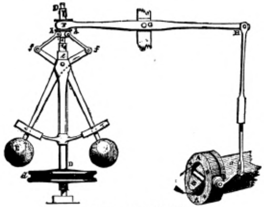


FIG. 4.—Governor and Throttle-Valve.

Sommaire

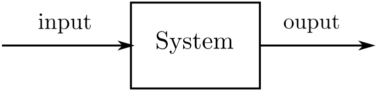
Introduction
- Review
- General points
- Analog to digital converter
- Digital to analog converter

Modelling
- Discrete-time systems
- Sampled systems

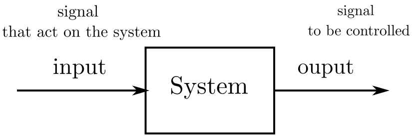
Feeback system analysis
- Stability
- Tracking error

Controller design
- Principle
- Discretization methods
- Implementation

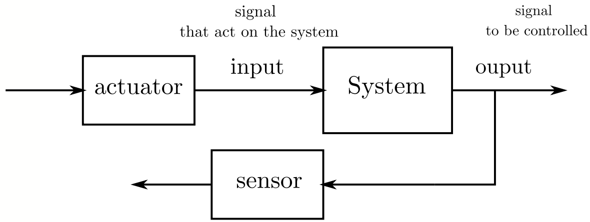
Stucture of a feedback control system



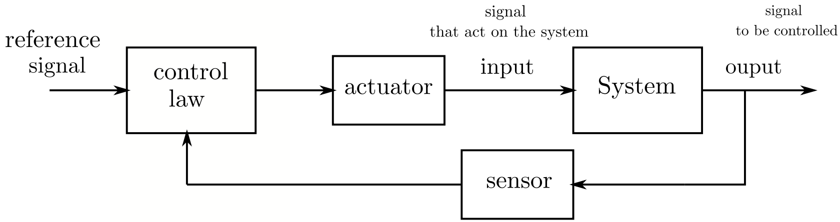
Stucture of a feedback control system



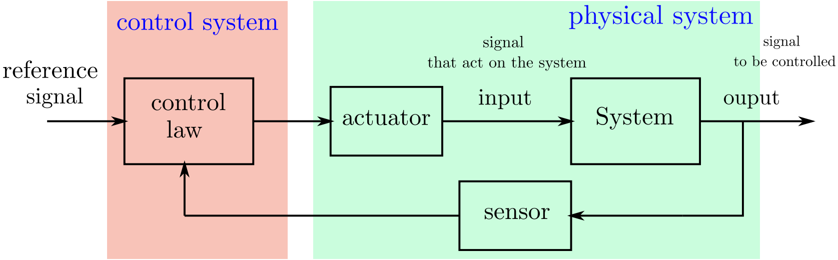
Stucture of a feedback control system



Stucture of a feedback control system



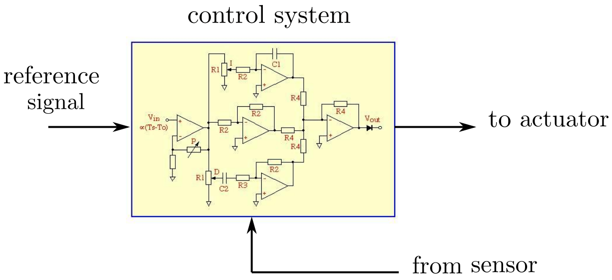
Stucture of a feedback control system



★ Objective : design the "control system"
to make the output signal converge to the reference signal

Implementation

control law implementation : analog



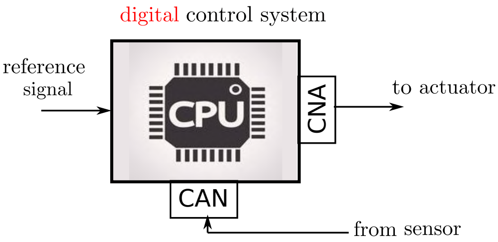
Controller implemented in analog electronic circuit

⇒ with resistors, capacitors, inductors and operational amplifier...

(could be also a mechanical system, mechanism...)

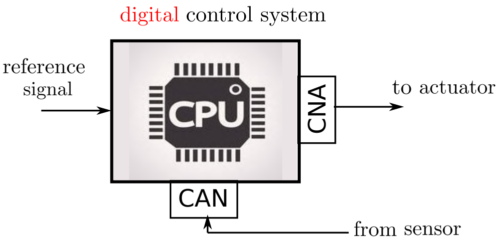
Implementation

Nowadays, digital implementation is used



Implementation

Nowadays, digital implementation is used



★ Analog and digital signals are of different nature

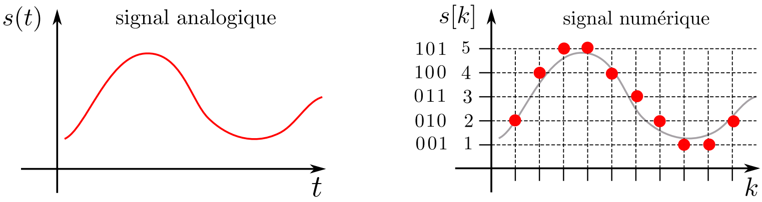
- ADC → analog to digital converter
- DAC → digital to analog converter

Analog signals

In the analog "world", we think in terms of physical quantities. An analog signal is a continuous signal that varies in continuous way over time.

Analog signals

In the analog "world", we think in terms of physical quantities. An analog signal is a continuous signal that varies in continuous way over time.



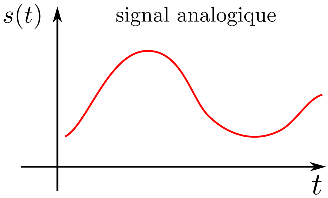
Examples of sensors : thermocouple, potentiometer, mercury thermometer, speedometer (with a needle)

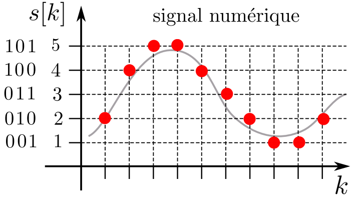
Digital signals

In the digital "world", information is encoded as a set of 0 and 1. A computer works with binary codes with a specific pace (clock). A digital signal is a discrete signal that varies in discrete way over time.

Digital signals

In the digital “world”, information is encoded as a set of 0 and 1. A computer works with binary codes with a specific pace (clock). A digital signal is a discrete signal that varies in discrete way over time.

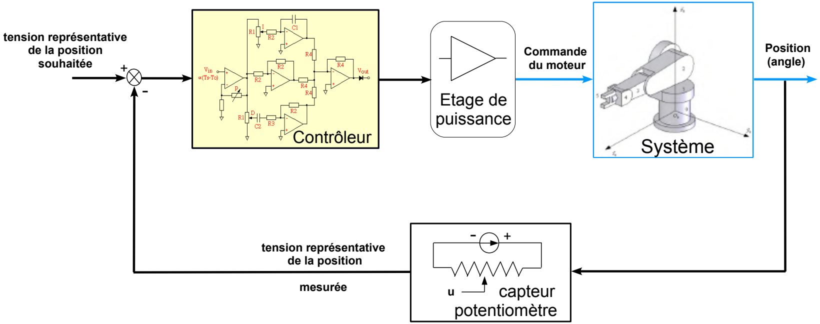




Examples :

Example

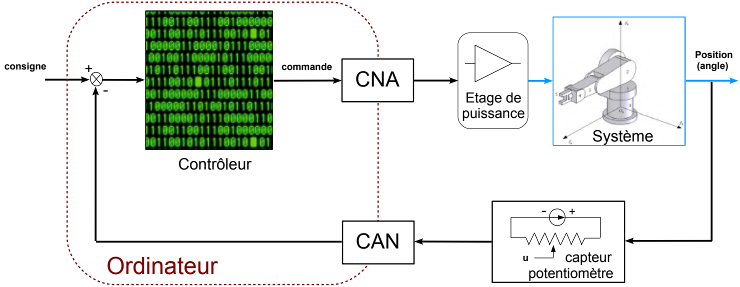
Analog control of a robotic arm



- the potentiometer measures the angular position and deliver a voltage
- the user specifies the desired position with an analog device (e.g. potentiometer of joystick)

Example

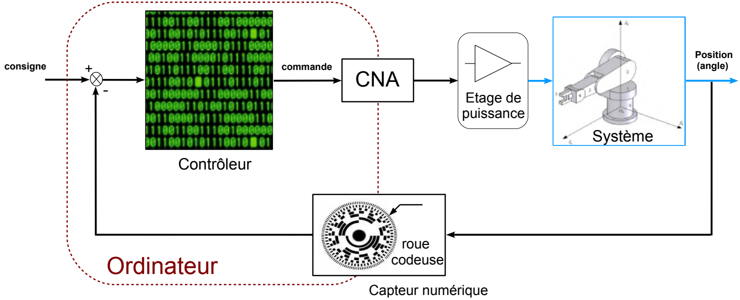
Digital control of a robotic arm



- ADC → analog to digital converter
- DAC → digital to analog converter
- Possibility to use digital HMI

Example

Digital control of a robotic arm



- ADC → analog to digital converter
- DAC → digital to analog converter
- Possibility to use digital HMI
- Possibility to use digital sensors, the measure is a binary code.

Basically, control systems are implemented on a computing system

- computers with data acquisition devices
- electronic boards with microcontrollers
- FPGA

Basically, control systems are implemented on a computing system

- computers with data acquisition devices
- electronic boards with microcontrollers
- FPGA





pros/cons

Digital control, why ?

Avantages

- Flexibility for implentation and change of the control law.
- Easier to implement (programming) fancy control law / algorithms / calculation.
- Insensitive to noise and reliability (no drift).
- Possibility to connect to other digital devices : computer, network...

Drawbacks

- Combination of analog and digital signals, require conversions.
- Imply a new theoretic framework.
- Loss of information because of sampling and quantization.

Sommaire

Introduction

Review

General points

Analog to digital converter

Digital to analog converter

Modelling

Discrete-time systems

Sampled systems

Feeback system analysis

Stability

Tracking error

Controller design

Principle

Discretization methods

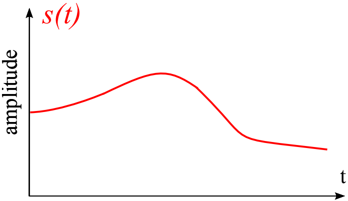
Implementation

Analog signal to digital signal

You are used to using Analog (or continuous) signals

Continuous signal

$$\begin{cases} \mathbb{R}^+ & \longrightarrow & \mathbb{R} \\ t & \longrightarrow & s(t) \end{cases}$$

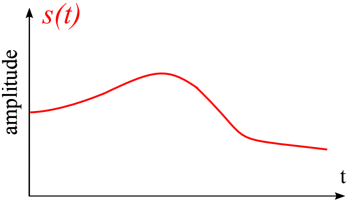


Analog signal to digital signal

You are used to using Analog (or continuous) signals

Continuous signal

$$\begin{cases} \mathbb{R}^+ & \longrightarrow & \mathbb{R} \\ t & \longrightarrow & s(t) \end{cases}$$



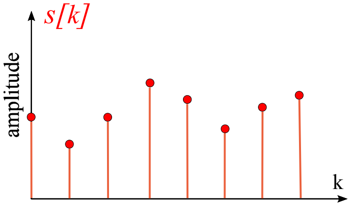
- ...

Analog signal to digital signal

In digital systems, digital (or discrete) signals are considered

Discrete signal

$$\begin{cases} \mathbb{N}^+ & \longrightarrow & \mathbb{R} \\ k & \longrightarrow & s_k \qquad \text{(ou } s[k]\text{)} \end{cases}$$

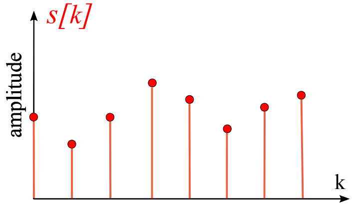


Analog signal to digital signal

In digital systems, digital (or discrete) signals are considered

Discrete signal

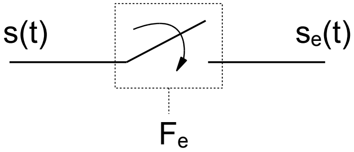
$$\begin{cases} \mathbb{N}^+ & \longrightarrow & \mathbb{R} \\ k & \longrightarrow & s_k & \qquad \text{(ou } s[k]) \end{cases}$$



- Quantization not considered in this course
- Only discretization in time is considered $\Rightarrow$ Sampling

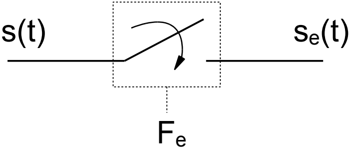
Signal sampling

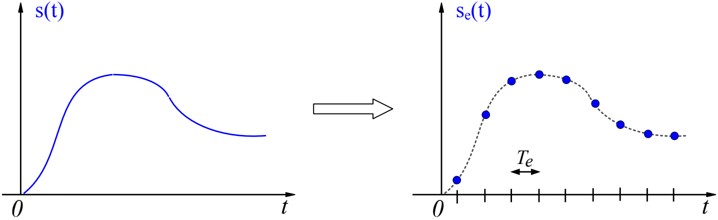
Sampling consists in getting values from an analog signal at specific instants



Signal sampling

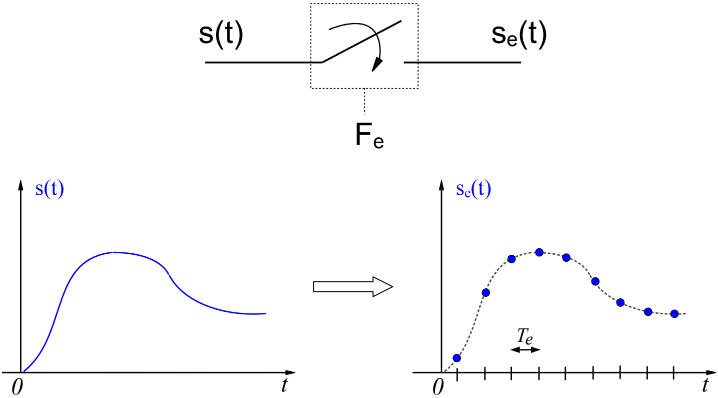
Sampling consists in getting values from an analog signal at specific instants





Signal sampling

Sampling consists in getting values from an analog signal at specific instants

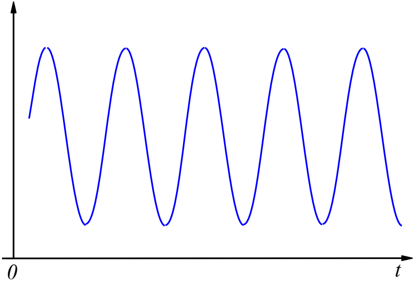


★ How many times per second should we sample the signal to have a correct information ?

$\Rightarrow$ sampling frequency ($F_e = \frac{1}{T_e}$) ?

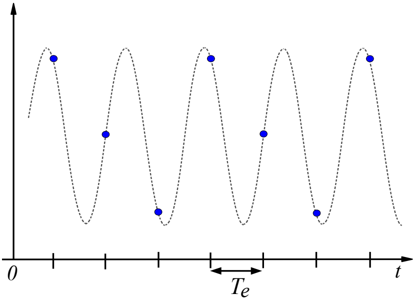
Example

Let sample a sine wave of frequency f_0 with a sampling frequence $F_e = \frac{3}{2} f_0$.



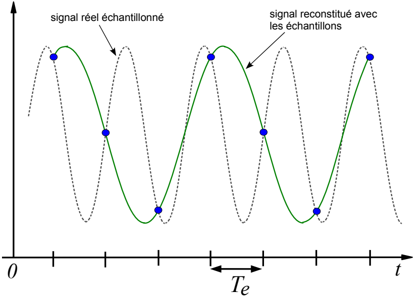
Example

Let sample a sine wave of frequency f_0 with a sampling frequence $F_e = \frac{3}{2} f_0$.



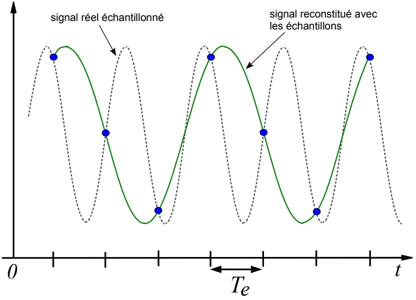
Example

Let sample a sine wave of frequency f_0 with a sampling frequence $F_e = \frac{3}{2} f_0$.



Example

Let sample a sine wave of frequency f_0 with a sampling frequence $F_e = \frac{3}{2} f_0$.



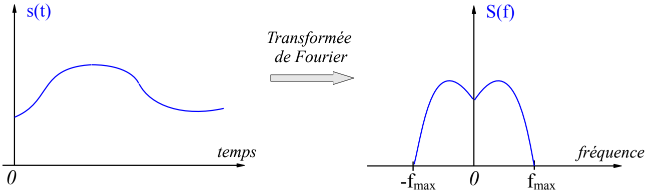
Basically, tuning F_e :

- high enough to have a reliable representation of the original signal,
- low enough to limit the number of samples (memory), the acquisition speed (cpu time), energy

Shannon theorem

(fundamental result, not detailed in this course)

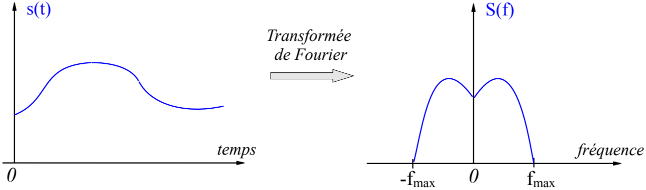
For a signal $x(t)$ having a spectrum defined over a bounded interval ($X(f) = 0$ for $f > f_{\text{max}}$)



Shannon theorem

(fundamental result, not detailed in this course)

For a signal $x(t)$ having a spectrum defined over a bounded interval ($X(f) = 0$ for $f > f_{\max}$)



If a function $x(t)$ contains no frequencies higher than $f_{\max}$ hertz, then it is completely determined for a sampling frequency

$$F_e \geq 2f_{\max}$$

$f_{\max}$ is the maximum frequency of the relevant part of the spectrum of x (that includes the information of the original signal x).

Example 1

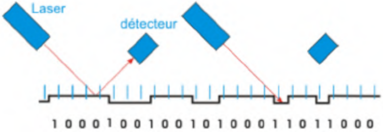
Vinyl disc : analog recording (mechanical) of music

Example 1

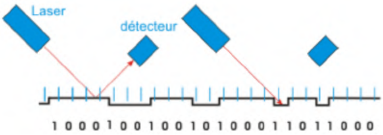
Vinyl disc : analog recording (mechanical) of music

- the needle is placed on the groove
- the arm moves along the groove structure
- a sensor converts the vibrations into an electrical signal before going to the amplifier

Compact disc (CD) : digital recording of music

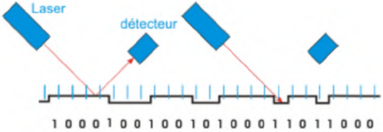


Compact disc (CD) : digital recording of music



- a pit is $0.12\mu m$ deep, the head reads the surface every $0.278\mu m$
- "1" if transition ; "0" if not
- the CD capacity is about 700MB

Compact disc (CD) : digital recording of music



- a pit is $0.12\mu m$ deep, the head reads the surface every $0.278\mu m$
- "1" if transition ; "0" if not
- the CD capacity is about 700MB

★ The sampling frequency on CD is 44.1 kHz, why this frequency is sufficient ?

Example 2

Let's consider an arcade game

equipped with two analog joysticks

Example 2

Let's consider an arcade game

equipped with two analog joysticks

Advantech
USB-4702-AE
* 8 analog input channels
* 12-bit resolution AI
* Sampling rate up to 10 kS/s
* 8-ch DI/DO, 2-ch AO
and one 32-bit counter
Price : 90€

National Instruments
USB-6002
* 8 analog input channels
* 16-bit resolution AI
* Sampling rate up to 50 kS/s
* 13-ch DI/DO, 2-ch AO
and one 32-bit counter
Price : 445€

National Instruments
PCI-6111
* 2 analog input channels
* 12-bit resolution AI
* Sampling rate up to 5 MS/s
* 8-ch DI/DO, 2-ch AO
and two 24-bit counter
Price : 3250€

Example 2

Let's consider an arcade game

equipped with two analog joysticks

Advantech
USB-4702-AE
* 8 analog input channels
* 12-bit resolution AI
* Sampling rate up to 10 kS/s
* 8-ch DI/DO, 2-ch AO
and one 32-bit counter
Price : 90€

National Instruments
USB-6002
* 8 analog input channels
* 16-bit resolution AI
* Sampling rate up to 50 kS/s
* 13-ch DI/DO, 2-ch AO
and one 32-bit counter
Price : 445€

National Instruments
PCI-6111
* 2 analog input channels
* 12-bit resolution AI
* Sampling rate up to 5 MS/s
* 8-ch DI/DO, 2-ch AO
and two 24-bit counter
Price : 3250€

★ Which data acquisition card should we pick ?

Sommaire

Introduction
- Review
- General points
- Analog to digital converter
- Digital to analog converter

Modelling
- Discrete-time systems
- Sampled systems

Feeback system analysis
- Stability
- Tracking error

Controller design
- Principle
- Discretization methods
- Implementation

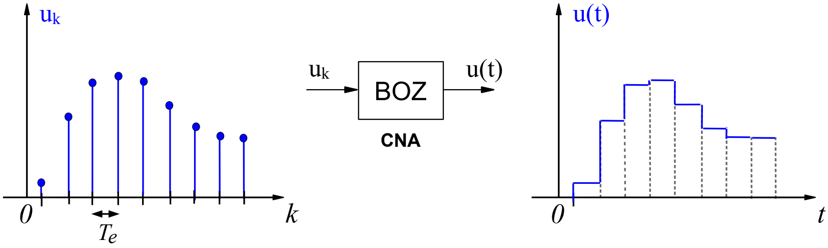
Digital signal to analog signal

Having discrete values, how can we generate a signal to drive an analog system ?

Digital signal to analog signal

Having discrete values, how can we generate a signal to drive an analog system ?

⇒ simplest way : hold each value over an interval T_e.



Zero Order Hold : polynomial interpolation of order 0 between 2 samples.

Sommaire

Introduction

- Review
- General points
- Analog to digital converter
- Digital to analog converter

Modelling

- Discrete-time systems
- Sampled systems

Feeback system analysis

- Stability
- Tracking error

Controller design

- Principle
- Discretization methods
- Implementation

Modélisation

Sommaire

Introduction
- Review
- General points
- Analog to digital converter
- Digital to analog converter

Modelling
- Discrete-time systems
- Sampled systems

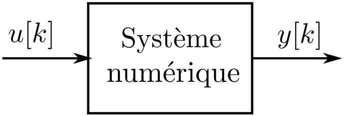
Feeback system analysis
- Stability
- Tracking error

Controller design
- Principle
- Discretization methods
- Implementation

Discrete-time systems

A discrete-time system is a device or process that

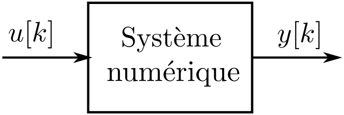
- operates on discrete-time signal - the input
- produce another discrete-time signal - the output



Discrete-time systems

A discrete-time system is a device or process that

- operates on discrete-time signal - the input
- produce another discrete-time signal - the output



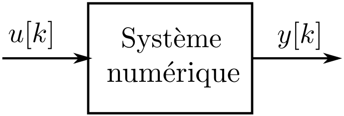
General model for linear discrete-time systems are recurrence equations

$$a_n y[k]+a_{n-1}y[k-1]+\ldots+a_0 y[k-n] = b_m u[k]+b_{m-1}u[k-1]+\ldots+b_0 u[k-m]$$

Discrete-time systems

A discrete-time system is a device or process that

- operates on discrete-time signal - the input
- produce another discrete-time signal - the output



General model for linear discrete-time systems are recurrence equations

$$a_n y[k]+a_{n-1}y[k-1]+\ldots+a_0 y[k-n] = b_m u[k]+b_{m-1}u[k-1]+\ldots+b_0 u[k-m]$$

example :

$$2y[k]+4y[k-1]+y[k-2]=u[k]+0.5u[k-1]$$

Values of the output can be directly computed :

$$y[k] = \frac{1}{a_n}\Big(b_m u[k] + b_{m-1} u[k-1] + \ldots + b_0 u[k-m] - a_{n-1} y[k-1] - \ldots - a_0 y[k-n]\Big)$$

At time instant k, the output value is expressed as a function of past samples of both the input and the output

Values of the output can be directly computed :

$$y[k] = \frac{1}{a_n}\Big(b_m u[k]+b_{m-1}u[k-1]+\ldots+b_0 u[k-m]-a_{n-1}y[k-1]-\ldots-a_0 y[k-n]\Big)$$

At time instant k, the output value is expressed as a function of past samples of both the input and the output

example :

$$y[k] = \frac{1}{2}\Big(u[k] + 0.5u[k-1] - 4y[k-1] - y[k-2]\Big)$$

The system is causal if $y[k]$ do not depend on future input samples $u[k+i]$

Example 1 : amount of money in a saving account

for each year k, we define :

$y[k]$: amount of money $\qquad r$: interest rate $\qquad u[k]$: money saved

Example 1 : amount of money in a saving account

for each year k, we define :

$y[k]$: amount of money r : interest rate $u[k]$: money saved

Discrete-time model :

$$y[k+1] = (1+r)y[k] + u[k]$$

Example 1 : amount of money in a saving account

for each year k, we define :

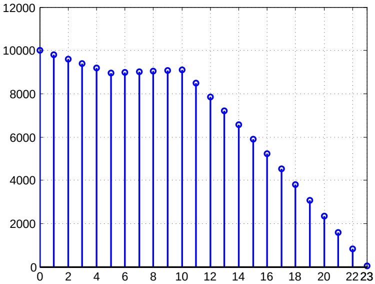
$y[k]$: amount of money r : interest rate $u[k]$: money saved

Discrete-time model :

$$y[k+1] = (1+r)y[k] + u[k]$$

Simulation

$$y[0] = 10k€$$
$$r = 5\%$$
$$u[k] = 1k€$$



Exemple 2 population dynamique

Population = a set of individuals of the same living species in a given space, reproducing among themselves

$$N_{t+1} - N_t = \text{ birth } - \text{ death } + \text{ migrations}$$

Exemple 2 population dynamique

Population = a set of individuals of the same living species in a given space, reproducing among themselves

$$N_{t+1} - N_t = \text{ birth } - \text{ death } + \text{ migrations}$$

Exponential model :

- isolated population,
- no migrations,
- unlimited resources.

$$N[k+1] = N[k] + r\,N[k]$$

r : rate of growth

Exemple 2 population dynamique

Population = a set of individuals of the same living species in a given space, reproducing among themselves

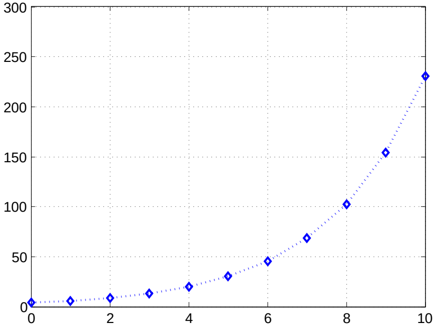
$$N_{t+1} - N_t = \text{birth} - \text{death} + \text{migrations}$$

Exponential model :

- isolated population,
- no migrations,
- unlimited resources.

$$N[k+1] = N[k] + r\,N[k]$$

r : rate of growth



example with $N[0] = 4$ and $r = 0.5$

Continuous-time systems case

- mathematical model between the input and the output

  $\Rightarrow$ differential equation

- essential tool for system analysis

  $\Rightarrow$ Laplace transform

$$x(t) \xrightarrow{\mathcal{L}} X(p)$$

Continuous-time systems case

- mathematical model between the input and the output
  $\Rightarrow$ differential equation
- essential tool for system analysis
  $\Rightarrow$ Laplace transform

$$x(t) \xrightarrow{\mathcal{L}} X(p)$$

Properties that makes calculation easier :

linearity	$a\,x_1(t) + b\,x_2(t)$	$\xrightarrow{\mathcal{L}}$	$a\,X_1(p) + b\,X_2(p)$
derivation	$\dot{x}(t)$	$\xrightarrow{\mathcal{L}}$	$p\,X(p) - x(0)$
integration	$\int_0^t x(\theta)\,d\theta$	$\xrightarrow{\mathcal{L}}$	$\frac{1}{p}X(p)$
final value	$\lim_{t\to\infty} x(t)$	$\xrightarrow{\mathcal{L}}$	$\lim_{p\to 0} pX(p)$

Discrete-time systems case

The Z *transform* of a discrete signal $x[k]$ is the sequence :

$$X(z) = \sum_{k=0}^{+\infty} x[k]\, z^{-k}$$

z is a complex variable.

Discrete-time systems case

The *Z transform* of a discrete signal $x[k]$ is the sequence :

$$X(z) = \sum_{k=0}^{+\infty} x[k]\, z^{-k}$$

z is a complex variable.

Properties :

linearity	$a\, x_1[k] + b\, x_2[k]$	$\xrightarrow{\mathcal{Z}}$	$a\, X_1(z) + b\, X_2(z)$
delay	$x[k-n]$	$\xrightarrow{\mathcal{Z}}$	$z^{-n}\, X(z)$
ahead	$x[k+n]$	$\xrightarrow{\mathcal{Z}}$	$z^n X(z) - \sum_{i=0}^{n-1} x[i] z^{n-i}$
final value	$\lim_{k \to \infty} x[k]$	$\xrightarrow{\mathcal{Z}}$	$\lim_{z \to 1} (1 - z^{-1}) X(z)$

Table of Z transforms

Continuous sig. $x(t)$	Sampled sig. $x(kT_e)$	Z transform : $X(z)$
1	1	$\frac{z}{z-1}$
t	kT_e	$\frac{T_e z}{(z-1)^2}$
	a^k	$\frac{z}{z-a}$
e^{-at}	e^{-akT_e}	$\frac{z}{z-e^{-aT_e}}$
$\sin(\omega t)$	$\sin(\omega k T_e)$	$\frac{z\sin(\omega T_e)}{z^2-2z\cos(\omega T_e)+1}$
$\cos(\omega t)$	$\cos(\omega k T_e)$	$\frac{z^2-z\cos(\omega T_e)}{z^2-2z\cos(\omega T_e)+1}$

$x(t) = 0$ for $t < 0$

Let's apply the Z transform to a recurrence equation

$$y[k] + 3y[k-1] + 6y[k-2] = u[k]$$

Let's apply the Z transform to a recurrence equation

$$y[k] + 3y[k-1] + 6y[k-2] = u[k]$$

$$\Big\downarrow \; \mathcal{Z}$$

$$Y(z) + 3z^{-1}Y(z) + 6z^{-2}Y(z) = U(z)$$

Let's apply the Z transform to a recurrence equation

$$y[k] + 3y[k-1] + 6y[k-2] = u[k]$$

$$\downarrow \mathcal{Z}$$

$$Y(z) + 3z^{-1}Y(z) + 6z^{-2}Y(z) = U(z)$$

$$\downarrow$$

Z transfer function : $$\boxed{F(z) = \frac{Y(z)}{U(z)} = \frac{1}{1 + 3z^{-1} + 6z^{-2}}}$$

Let's apply the Z transform to a recurrence equation

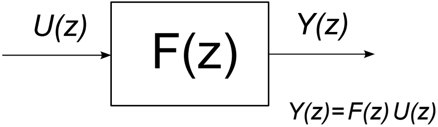
$$y[k] + 3y[k-1] + 6y[k-2] = u[k]$$

$$\Big\downarrow \quad \mathcal{Z}$$

$$Y(z) + 3z^{-1}Y(z) + 6z^{-2}Y(z) = U(z)$$

$$\Big\downarrow$$

Z transfer function : $$\boxed{F(z) = \frac{Y(z)}{U(z)} = \frac{1}{1 + 3z^{-1} + 6z^{-2}}}$$



Z transfer function

General case :

$$a_n y[k] + a_{n-1} y[k-1] + \ldots + a_0 y[k-n] \; = \; b_m u[k] + b_{m-1} u[k-1] + \ldots + b_0 u[k-m]$$

Z transfer function

General case :

$$a_n y[k] + a_{n-1} y[k-1] + \ldots + a_0 y[k-n] \; = \; b_m u[k] + b_{m-1} u[k-1] + \ldots + b_0 u[k-m]$$

$$\Big\downarrow \; \mathcal{Z}$$

$$a_n Y(z) + a_{n-1} z^{-1} Y(z) + \ldots + a_0 z^{-n} Y(z) = b_m U(z) + b_{m-1} z^{-1} U(z) + \ldots + b_0 z^{-m} U(z)$$

Z transfer function

General case :

$$a_n y[k] + a_{n-1} y[k-1] + \ldots + a_0 y[k-n] \;=\; b_m u[k] + b_{m-1} u[k-1] + \ldots + b_0 u[k-m]$$

$$\Big\downarrow \; \mathcal{Z}$$

$$a_n Y(z) + a_{n-1} z^{-1} Y(z) + \ldots + a_0 z^{-n} Y(z) = b_m U(z) + b_{m-1} z^{-1} U(z) + \ldots + b_0 z^{-m} U(z)$$

$$\Big\downarrow$$

Z transfer function :

$$\boxed{F(z) = \frac{Y(z)}{U(z)} = \frac{b_m + b_{m-1} z^{-1} + \ldots + b_0 z^{-m}}{a_n + a_{n-1} z^{-1} + \cdots + a_0 z^{-n}}}$$

Example

Let's consider the discrete-time system :

$$y[k] + 2y[k-1] - 4y[k-2] = 5u[k] + u[k-1]$$

Example

Let's consider the discrete-time system :

$$y[k] + 2y[k-1] - 4y[k-2] = 5u[k] + u[k-1]$$

Applying the Z transform :

$$Y(z) + 2z^{-1}Y(z) - 4z^{-2}Y(z) = 5U(z) + z^{-1}U(z)$$

Example

Let's consider the discrete-time system :

$$y[k] + 2y[k-1] - 4y[k-2] = 5u[k] + u[k-1]$$

Applying the Z transform :

$$Y(z) + 2z^{-1}Y(z) - 4z^{-2}Y(z) = 5U(z) + z^{-1}U(z)$$

Then, the transfer function of the system is :

$$F(z) = \frac{Y(z)}{U(z)} = \frac{5 + z^{-1}}{1 + 2z^{-1} - 4z^{-2}}$$

or

$$F(z) = \frac{Y(z)}{U(z)} = \frac{5z^2 + z}{z^2 + 2z - 4}$$

Parallel analog systems - digital systems

Discrete-time

Recurrence equation

$$\left\{y[k], y[k-1], y[k-2], \ldots, u[k], u[k-1], \ldots\right\}$$

Z transfer function

(Z trans.) $\left\{Y(z), U(z)\right\}$

Continuous-time

Differential equation

$$\left\{y(t), \dot{y}(t), \ddot{y}(t), \ldots, u(t), \dot{u}(t), \ldots\right\}$$

Transfer function

(Laplace trans.) $\left\{Y(p), U(p)\right\}$

Sommaire

Introduction
- Review
- General points
- Analog to digital converter
- Digital to analog converter

Modelling
- Discrete-time systems
- Sampled systems

Feeback system analysis
- Stability
- Tracking error

Controller design
- Principle
- Discretization methods
- Implementation

Sampled systems

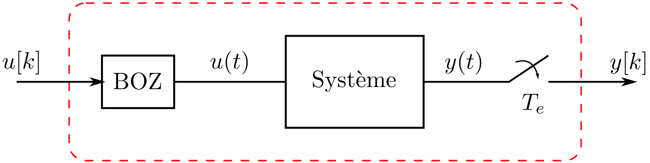
Control of an analog system with a digital device :

what is the point of view of the digital device ?

Sampled systems

Control of an analog system with a digital device :

what is the point of view of the digital device ?

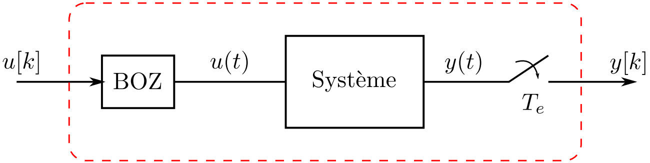


- digital to analog converter : $u[k] \longrightarrow u(t)$
- analog to digital converter : $y(t) \longrightarrow y[k]$
- the digital device only handles discrete signal : $u[k]$ and $y[k]$

Sampled systems

Control of an analog system with a digital device :

what is the point of view of the digital device ?



- digital to analog converter : $u[k] \longrightarrow u(t)$
- analog to digital converter : $y(t) \longrightarrow y[k]$
- the digital device only handles discrete signal : $u[k]$ and $y[k]$

★ Z transfer function of the sampled analog system ?

Given the transfer function of the analog system

$$Y(p) = G(p)U(p)$$

what is the corresponding Z transfer function from $u[k]$ to $y[k]$?

$$Y(z) \;=\; G(z)\;U(z)$$

Given the transfer function of the analog system

$$Y(p) = G(p)U(p)$$

what is the corresponding Z transfer function from $u[k]$ to $y[k]$?

$$Y(z) \; = \; G(z) \; U(z)$$

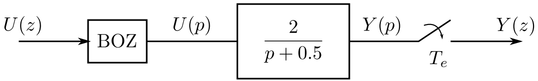
Z transfer function of a sampled system :

$$G(p) \qquad \xrightarrow{\text{sampling}} \qquad G(z) \; = \; \frac{z-1}{z} \mathcal{Z}\left\{ \frac{G(p)}{p} \right\}$$

The term $\mathcal{Z}\left\{\frac{G(p)}{p}\right\}$ is obtained from the table below.

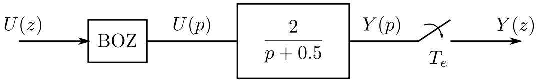
$F(p)$	$\mathcal{Z}\left\{F(p)\right\}$
$\frac{1}{p}$	$\frac{z}{z-1}$
$\frac{1}{p^2}$	$\frac{T_e z}{(z-1)^2}$
$\frac{1}{p+a}$	$\frac{z}{z-e^{-aT_e}}$
$\frac{1}{(p+a)^2}$	$\frac{T_e z\, e^{-aT_e}}{(z-e^{-aT_e})^2}$
$\frac{a}{p(p+a)}$	$\frac{z(1-e^{-aT_e})}{(z-1)(z-e^{-aT_e})}$
$\frac{\omega}{(p+a)^2+\omega^2}$	$\frac{ze^{-aT_e}\sin(\omega T_e)}{z^2-2ze^{-aT_e}\cos(\omega T_e)+e^{-2aT_e}}$
$\frac{p+a}{(p+a)^2+\omega^2}$	$\frac{z^2-ze^{-aT_e}\cos(\omega T_e)}{z^2-2ze^{-aT_e}\cos(\omega T_e)+e^{-2aT_e}}$

Example 1



Sampled transfer function ?

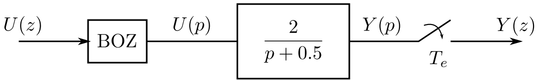
Example 1



Sampled transfer function ?

$$G(z) = \frac{z-1}{z}\mathcal{Z}\left\{\frac{G(p)}{p}\right\}$$

Example 1



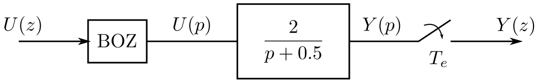
Sampled transfer function ?

$$G(z) = \frac{z-1}{z}\mathcal{Z}\left\{\frac{G(p)}{p}\right\}$$

Partial fraction decomposition :

$$\frac{G(p)}{p} = \frac{2}{p(p+0.5)} = \frac{4}{p} - \frac{4}{p+0.5}$$

Example 1



Sampled transfer function ?

$$G(z) = \frac{z-1}{z}\mathcal{Z}\left\{\frac{G(p)}{p}\right\}$$

Partial fraction decomposition :

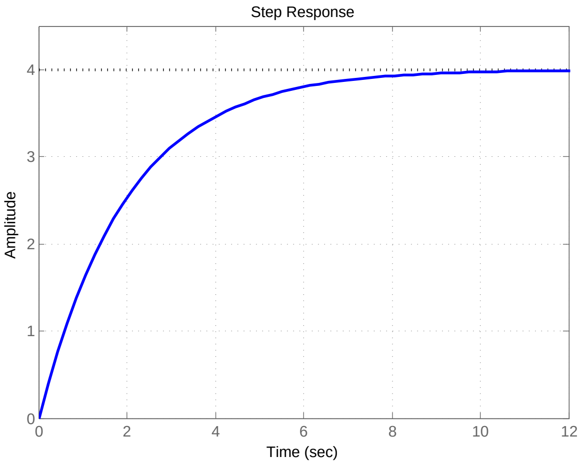
$$\frac{G(p)}{p} = \frac{2}{p(p+0.5)} = \frac{4}{p} - \frac{4}{p+0.5}$$

then with the table to have the corresponding ZTF :

$$G(z) = \frac{z-1}{z}\left[\frac{4z}{z-1} - \frac{4z}{z-e^{-0.5\,T_e}}\right] = 4\frac{1-e^{-0.5\,Te}}{z-e^{-0.5\,Te}}$$

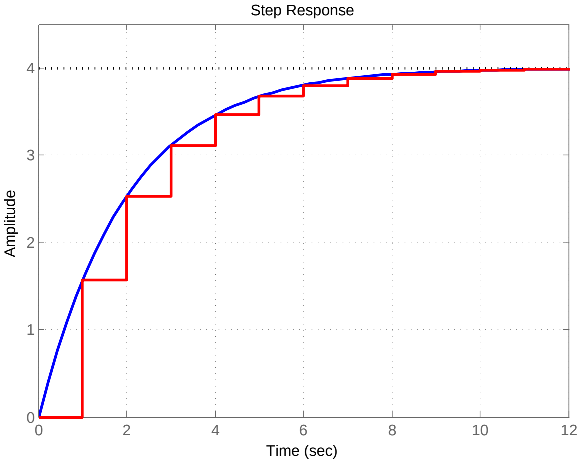
Simulation of the step responses

$$G(p) = \frac{2}{p + 0.5} \quad \text{and} \quad G(z) = 4\frac{1 - e^{-0.5\ Te}}{z - e^{-0.5\ Te}} \quad \text{with } T_e = 1s$$

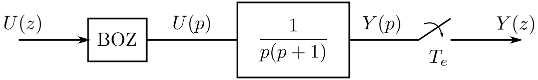


Simulation of the step responses

$$G(p) = \frac{2}{p + 0.5} \qquad \text{and} \qquad G(z) = 4\frac{1 - e^{-0.5\ Te}}{z - e^{-0.5\ Te}} \qquad \text{with } T_e = 1s$$

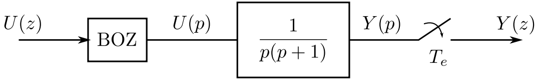


Exemple 2



Sampled transfer function ?

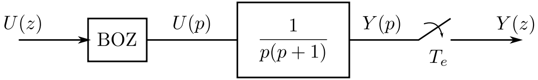
Exemple 2



Sampled transfer function ?

$$G(z) = \frac{z-1}{z}\mathcal{Z}\left\{\frac{G(p)}{p}\right\}$$

Exemple 2



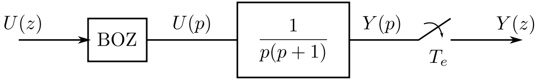
Sampled transfer function ?

$$G(z) = \frac{z-1}{z}\mathcal{Z}\left\{\frac{G(p)}{p}\right\}$$

Partial fraction decomposition :

$$\frac{G(p)}{p} = \frac{-1}{p} + \frac{1}{p^2} + \frac{1}{p+1}$$

Exemple 2



Sampled transfer function ?

$$G(z) = \frac{z-1}{z}\mathcal{Z}\left\{\frac{G(p)}{p}\right\}$$

Partial fraction decomposition :

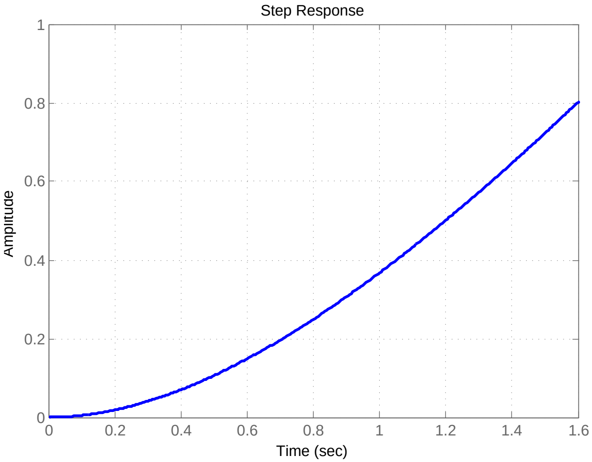
$$\frac{G(p)}{p} = \frac{-1}{p} + \frac{1}{p^2} + \frac{1}{p+1}$$

then with the table to have the corresponding ZTF :

$$G(z) = \frac{z-1}{z}\left[\frac{-z}{z-1} + \frac{T_e z}{(z-1)^2} + \frac{z}{z-e^{-T_e}}\right] = \frac{K(z-b)}{(z-1)(z-a)} \quad \begin{cases} K = e^{-T_e} - 1 + T_e \\ a = e^{-T_e} \\ b = 1 - \frac{T_e(1-e^{-T_e})}{K} \end{cases}$$

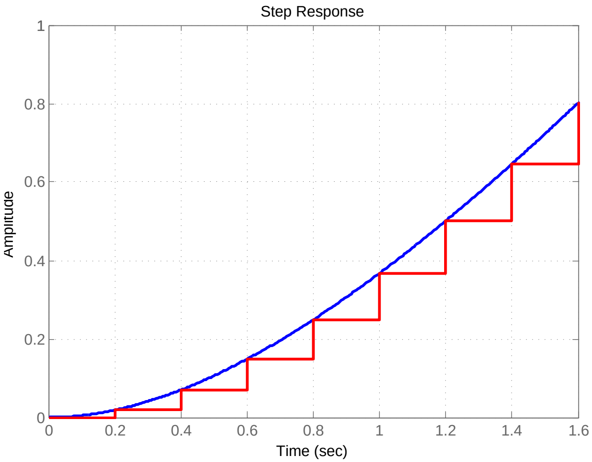
Simulation of the step responses

$$G(p) = \frac{1}{p(p+1)} \qquad \text{and} \qquad G(z) = K\frac{z-b}{(z-1)(z-a)} \qquad \text{with } T_e = 200ms$$



Simulation of the step responses

$$G(p) = \frac{1}{p(p+1)} \quad \text{and} \quad G(z) = K\frac{z-b}{(z-1)(z-a)} \quad \text{with } T_e = 200ms$$



Sommaire

Introduction
- Review
- General points
- Analog to digital converter
- Digital to analog converter

Modelling
- Discrete-time systems
- Sampled systems

Feeback system analysis
- Stability
- Tracking error

Controller design
- Principle
- Discretization methods
- Implementation

Feeback system analysis

Sommaire

Introduction
- Review
- General points
- Analog to digital converter
- Digital to analog converter

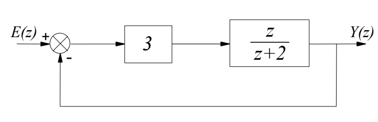
Modelling
- Discrete-time systems
- Sampled systems

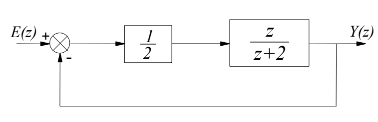
Feeback system analysis
- Stability
- Tracking error

Controller design
- Principle
- Discretization methods
- Implementation

Stability

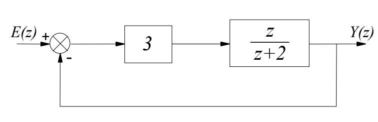
Let's consider two feedback systems with two proportional laws





Stability

Let's consider two feedback systems with two proportional laws

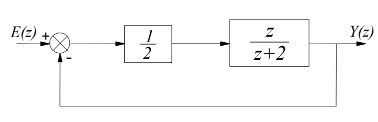


Closed-loop :

$$F(z) = \frac{3z}{4z + 2}$$

Impulse response :

$$y[k] = \frac{3}{4}\left(-\frac{1}{2}\right)^k$$

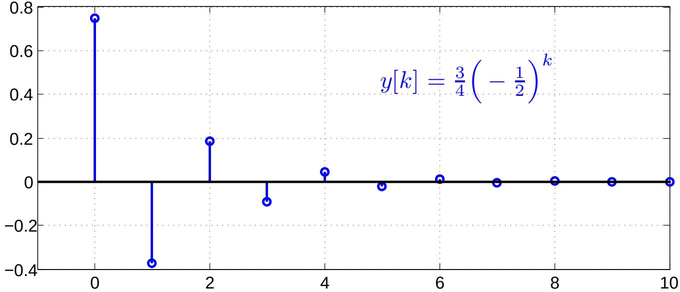


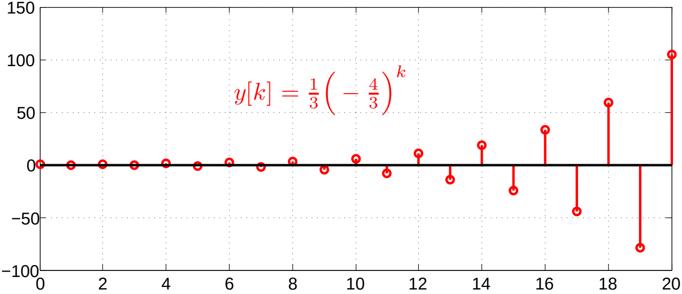
Closed-loop :

$$F(z) = \frac{\frac{1}{2}z}{\frac{3}{2}z + 2}$$

Impulse response :

$$y[k] = \frac{1}{3}\left(-\frac{4}{3}\right)^k$$


$y[k] = \frac{3}{4}\left(-\frac{1}{2}\right)^k$
0.8
0.6
0.4
0.2
0
−0.2
−0.4
0
2
4
6
8
10


$y[k] = \frac{1}{3}\left(-\frac{4}{3}\right)^k$
150
100
50
0
−50
−100
0
2
4
6
8
10
12
14
16
18
20

★ This is a **stability** issue.

Definition

A system is said to be stable if for any bounded input the output is also bounded

$$\text{if } \sup_{k\in\mathbb{N}} |u_k| < \infty, \quad \text{then} \quad \sup_{k\in\mathbb{N}} |y_k| < \infty$$

Theorem

A discrete-time linear system $F(z)$ is stable iff the module of all poles are strictly lower than 1.

$$\text{Charac. eq. } F(z): \quad a_n z^n + \ldots + a_1 z + a_0 = 0 \quad \Rightarrow \quad \text{poles } p_1,\ p_2, \ldots, p_n$$

Stability condition : $|p_i| < 1, \ \forall i = 1, \ldots n.$

Two methods for the stability analysis of discrete-time linear system $F(z)$:

- Direct calculation of poles of $F(z)$.
- Jury criterion.

Two methods for the stability analysis of discrete-time linear system $F(z)$:

- Direct calculation of poles of $F(z)$.
- Jury criterion.

Jury criterion

Set of inequality based on the $F(z)$ denominator coefficients :

$$a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$$

- avoid the calculation of poles
- inequalities can be tested w.r.t some parameters
- only the cases of orders n=2, 3 and 4 are considered

Jury criterion

$F(z)$ is stable iff the coefficients of its denominator satisfy :

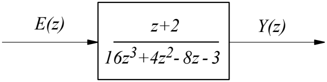
$$\text{for } n = 2 \ : \quad \begin{cases} a_0 + a_1 + a_2 & > & 0 \\ a_0 - a_1 + a_2 & > & 0 \\ a_2 - a_0 & > & 0 \end{cases}$$

$$\text{for } n = 3 \ : \quad \begin{cases} a_0 + a_1 + a_2 + a_3 & > & 0 \\ -a_0 + a_1 - a_2 + a_3 & > & 0 \\ a_3 - |a_0| & > & 0 \\ a_0 a_2 - a_1 a_3 - a_0^2 + a_3^2 & > & 0 \end{cases}$$

$$\text{for } n = 4 \ : \quad \begin{cases} a_0 + a_1 + a_2 + a_3 + a_4 & > & 0 \\ a_0 - a_1 + a_2 - a_3 + a_4 & > & 0 \\ a_4^2 - a_0^2 - |a_0 a_3 - a_1 a_4| & > & 0 \\ (a_0 - a_4)^2 (a_0 - a_2 + a_4) + (a_1 - a_3)(a_0 a_3 - a_1 a_4) & > & 0 \end{cases}$$

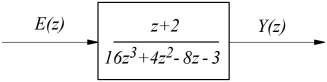
Example 1

Let's consider system



Example 1

Let's consider system

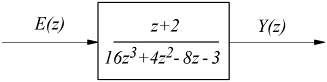


Applying the Jury criterion :

$$\left\{\begin{array}{rcccc} -3-8+4+16 & = & 9 & > & 0 \\ 3-8-4+16 & = & 7 & > & 0 \\ 16-3 & = & 13 & > & 0 \\ -3*4+8*16-9+256 & = & 363 & > & 0 \end{array}\right. \quad \Rightarrow \text{ system is stable}$$

Example 1

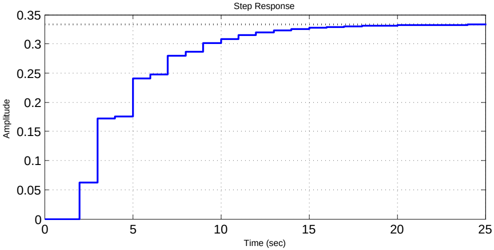
Let's consider system



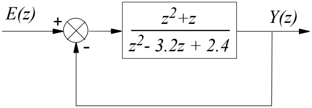
Applying the Jury criterion :

$$\left\{\begin{array}{rcccc} -3-8+4+16 & = & 9 & > & 0 \\ 3-8-4+16 & = & 7 & > & 0 \\ 16-3 & = & 13 & > & 0 \\ -3*4+8*16-9+256 & = & 363 & > & 0 \end{array}\right. \quad \Rightarrow \text{ system is stable}$$

Step response :

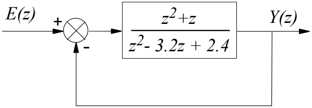


Example 2



$$F(z) = \frac{z^2 + z}{2z^2 - 2.2z + 2.4}$$

Example 2

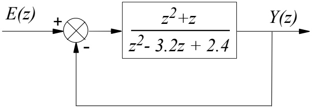


$$F(z) = \frac{z^2+z}{2z^2 - 2.2z + 2.4}$$

Applying the Jury criterion :

$$\begin{cases} 2.4 - 2.2 + 2 = 2.2 > 0 \\ 2.4 + 2.2 + 2 = 6.6 > 0 \\ 2 - 2.4 = -0.4 \ngtr 0 \end{cases} \Rightarrow \text{system is unstable}$$

Example 2

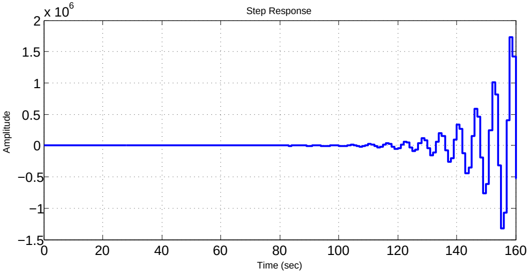


$$F(z) = \frac{z^2+z}{2z^2 - 2.2z + 2.4}$$

Applying the Jury criterion :

$$\left\{ \begin{array}{rcrcl} 2.4 - 2.2 + 2 & = & 2.2 & > & 0 \\ 2.4 + 2.2 + 2 & = & 6.6 & > & 0 \\ 2 - 2.4 & = & -0.4 & \not> & 0 \end{array} \right. \quad \Rightarrow \text{ system is unstable}$$

Step response :



Example 3

Let's consider the following system, K is a parameter

$$F(z) = \frac{2z^2 + z + 1}{z^3 + (K - 0.75)z - 0.25}$$

Example 3

Let's consider the following system, K is a parameter

$$F(z) = \frac{2z^2 + z + 1}{z^3 + (K - 0.75)z - 0.25}$$

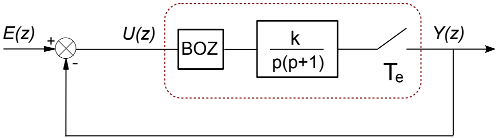
Applying the Jury criterion :

$$\begin{cases} a_0 + a_1 + a_2 + a_3 & = & K & > & 0 \\ -a_0 + a_1 - a_2 + a_3 & = & K + 0.5 & > & 0 \\ a_3 - |a_0| & = & 0.75 & > & 0 \\ a_0 a_2 - a_1 a_3 - a_0^2 + a_3^2 & = & 1.687 - K & > & 0 \end{cases}$$

The system is then stable for $0 < K < 1.687$.

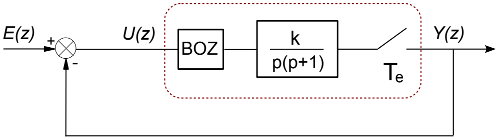
Exemple 4 : feedback of a sampled system

(k a tuning control parameter)



Exemple 4 : feedback of a sampled system

(k a tuning control parameter)



Z transfer function of the sampled system :

$$G(z) = \frac{z-1}{z}\mathcal{Z}\left\{\frac{k}{p^2(p+1)}\right\} = \frac{(z-a)kb}{(z-1)(z-e^{-T_e})}$$

with $a = \frac{e^{-T_e}(T_e+1)-1}{e^{-T_e}+T_e-1}$ and $b = e^{-T_e} - 1 + T_e$.

Transfer function of the closed loop system :

$$F(z) = \frac{(z-a)kb}{z^2 + (kb - 1 - e^{-T_e})z + e^{-T_e} - kba}$$

Applying the Jury criterion :

$$\begin{cases} e^{-T_e} - kba + kb - 1 - e^{-T_e} + 1 & > & 0 \\ e^{-T_e} - kba - kb + 1 + e^{-T_e} + 1 & > & 0 \\ 1 - e^{-T_e} + kba & > & 0 \end{cases}$$

2 cases :

$$T_e = 1\,s$$

$$\begin{cases} k & > & 0 \\ 26.4 & > & k \\ 2.4 & > & k \end{cases}$$

$$T_e = 10\,s$$

$$\begin{cases} k & > & 0 \\ 0.25 & > & k \\ 1 & > & k \end{cases}$$

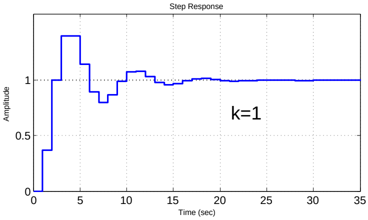
Applying the Jury criterion :

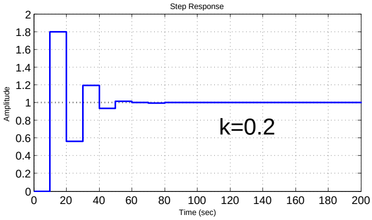
$$\begin{cases} e^{-T_e} - kba + kb - 1 - e^{-T_e} + 1 & > & 0 \\ e^{-T_e} - kba - kb + 1 + e^{-T_e} + 1 & > & 0 \\ 1 - e^{-T_e} + kba & > & 0 \end{cases}$$

2 cases :

$$T_e = 1\,s \qquad \begin{cases} k & > & 0 \\ 26.4 & > & k \\ 2.4 & > & k \end{cases}$$

$$T_e = 10\,s \qquad \begin{cases} k & > & 0 \\ 0.25 & > & k \\ 1 & > & k \end{cases}$$





Sommaire

Introduction
- Review
- General points
- Analog to digital converter
- Digital to analog converter

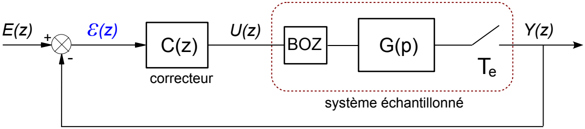
Modelling
- Discrete-time systems
- Sampled systems

Feeback system analysis
- Stability
- Tracking error

Controller design
- Principle
- Discretization methods
- Implementation

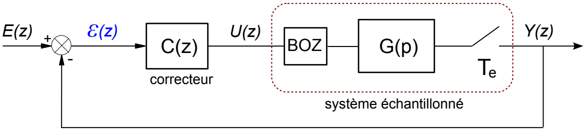
Tracking error

Does the system output converge to the reference ?



Tracking error

Does the system output converge to the reference ?

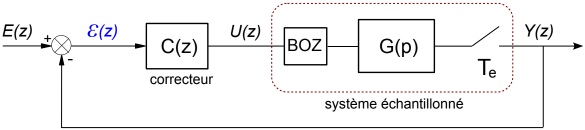


Precision performance

calculation of the error $\varepsilon[k] = y[k] - e[k]$ at the steady state.

Tracking error

Does the system output converge to the reference ?



Precision performance

calculation of the error $\varepsilon[k] = y[k] - e[k]$ at the steady state.

which value will the error converge to ?

$$\lim_{k \to \infty} \varepsilon[k] = ?$$

Tracking error

Error expression ?

$$\begin{aligned} \varepsilon(z) &= E(z) - Y(z) \\ &= E(z) - G(z)U(z) \\ &= E(z) - G(z)C(z)\varepsilon(z) \end{aligned}$$

$$\varepsilon(z)\Big(1 + G(z)C(z)\Big) = E(z)$$

Tracking error

Error expression ?

$$\begin{aligned}\varepsilon(z) &= E(z) - Y(z)\\ &= E(z) - G(z)U(z)\\ &= E(z) - G(z)C(z)\varepsilon(z)\end{aligned}$$

$$\varepsilon(z)\Big(1 + G(z)C(z)\Big) = E(z)$$

Thus : $$\varepsilon(z) = \frac{1}{1 + G(z)C(z)}E(z)$$

Tracking error

Error expression ?

$$\begin{aligned}\varepsilon(z) &= E(z) - Y(z)\\ &= E(z) - G(z)U(z)\\ &= E(z) - G(z)C(z)\varepsilon(z)\end{aligned}$$

$$\varepsilon(z)\Big(1 + G(z)C(z)\Big) = E(z)$$

Thus : $$\varepsilon(z) = \frac{1}{1 + G(z)C(z)}E(z)$$

And using the final value theorem

$$\boxed{\lim_{k\to\infty} \varepsilon[k] = \lim_{z\to 1}(1 - z^{-1})\,\varepsilon(z)}$$

★ Basically, tracking error is studied for specific given input ($\to e[k]$).

The **static error** is defined as the tracking error for a step input

$$e[k] = e_0, \quad \forall k \geq 0 \qquad \longrightarrow \qquad E(z) = e_0 \frac{z}{z-1}$$

The **static error** is defined as the tracking error for a step input

$$e[k] = e_0, \quad \forall k \geq 0 \qquad \longrightarrow \qquad E(z) = e_0 \frac{z}{z-1}$$

Express $\varepsilon(z)$, and apply the final value theorem :

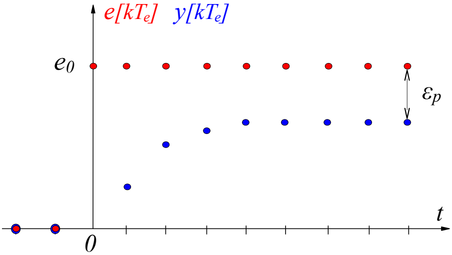
$$\varepsilon_s = \lim_{k\to\infty} \varepsilon[k] = \lim_{z\to 1} (1 - z^{-1}) \underbrace{\frac{1}{1 + G(z)C(z)} \frac{z}{z-1} e_0}_{\varepsilon(z)}$$

The **static error** is defined as the tracking error for a step input

$$e[k] = e_0, \quad \forall k \geq 0 \qquad \longrightarrow \qquad E(z) = e_0 \frac{z}{z-1}$$

Express $\varepsilon(z)$, and apply the final value theorem :

$$\varepsilon_s = \lim_{k\to\infty} \varepsilon[k] = \lim_{z\to 1} (1 - z^{-1}) \underbrace{\frac{1}{1 + G(z)C(z)} \frac{z}{z-1} e_0}_{\varepsilon(z)}$$



The **speed error** is defined as the tracking error for a ramp input

$$e[k] = e_0\, kT_e, \quad \forall k \geq 0 \qquad \longrightarrow \qquad E(z) = e_0\, T_e \frac{z}{(z-1)^2}$$

The **speed error** is defined as the tracking error for a ramp input

$$e[k] = e_0\, kT_e, \quad \forall k \geq 0 \qquad \longrightarrow \qquad E(z) = e_0\, T_e \frac{z}{(z-1)^2}$$

Express $\varepsilon(z)$, and apply the final value theorem :

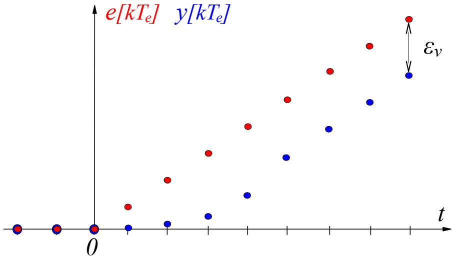
$$\varepsilon_t \quad = \quad \lim_{k\to\infty} \varepsilon[k] \;=\; \lim_{z\to 1}(1 - z^{-1})\, \underbrace{\frac{1}{1 + G(z)C(z)}\, \frac{z}{(z-1)^2}\, T_e e_0}_{\varepsilon(z)}$$

The **speed error** is defined as the tracking error for a ramp input

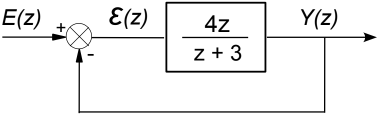
$$e[k] = e_0\, kT_e, \quad \forall k \geq 0 \qquad \longrightarrow \qquad E(z) = e_0\, T_e \frac{z}{(z-1)^2}$$

Express $\varepsilon(z)$, and apply the final value theorem :

$$\varepsilon_t = \lim_{k\to\infty} \varepsilon[k] = \lim_{z\to 1}(1 - z^{-1}) \underbrace{\frac{1}{1 + G(z)C(z)} \frac{z}{(z-1)^2} T_e e_0}_{\varepsilon(z)}$$

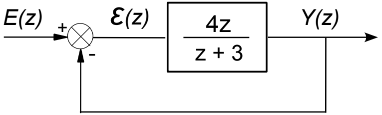


Example 1



Error expression :
$$\varepsilon(z) = \frac{1}{1 + \frac{4z}{z+3}}\, E(z) = \frac{z+3}{5z+3}\, E(z)$$

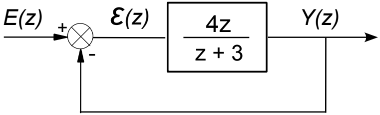
Example 1



Error expression : $$\varepsilon(z) = \frac{1}{1+\frac{4z}{z+3}}\,E(z) = \frac{z+3}{5z+3}\,E(z)$$

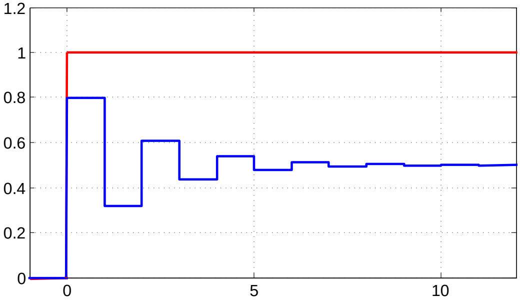
Static error : $$\varepsilon_s \;=\; \lim_{z\to 1}(1-z^{-1})\,\varepsilon(z) \;=\; 0.5\,e_0$$

Example 1



Error expression : $$\varepsilon(z) = \frac{1}{1 + \frac{4z}{z+3}} \; E(z) = \frac{z+3}{5z+3} \; E(z)$$

Static error : $$\varepsilon_s = \lim_{z \to 1} (1 - z^{-1}) \, \varepsilon(z) = 0.5 \, e_0$$

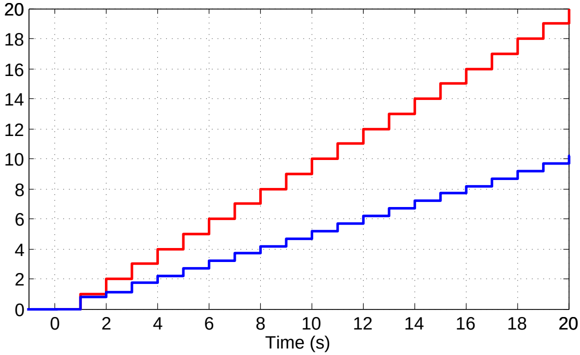


speed error :

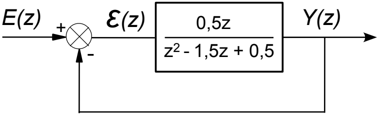
$$\begin{aligned}\varepsilon_t &= \lim_{z\to 1}(1-z^{-1})\,\varepsilon(z)\\ &= \lim_{z\to 1}\frac{z-1}{z}\,\frac{z+3}{5z+3}\,\underbrace{\frac{z}{(z-1)^2}\,e_0}_{E(z)}\\ &= +\infty\end{aligned}$$

speed error :

$$\begin{aligned}\varepsilon_t &= \lim_{z\to 1}(1-z^{-1})\,\varepsilon(z)\\ &= \lim_{z\to 1}\frac{z-1}{z}\,\frac{z+3}{5z+3}\,\underbrace{\frac{z}{(z-1)^2}\,e_0}_{E(z)}\\ &= +\infty\end{aligned}$$



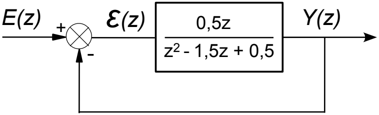
Example 2



Error expression :

$$\varepsilon(z) = \frac{1}{1 + \frac{0.5z}{z^2 - 1.5z + 0.5}} E(z) = \frac{z^2 - 1.5z + 0.5}{z^2 - z + 0.5} E(z)$$

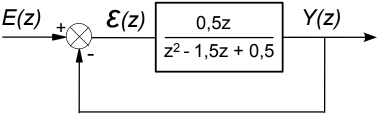
Example 2



Error expression : $\varepsilon(z) = \dfrac{1}{1 + \frac{0.5z}{z^2 - 1.5z + 0.5}} E(z) = \dfrac{z^2 - 1.5z + 0.5}{z^2 - z + 0.5} E(z)$

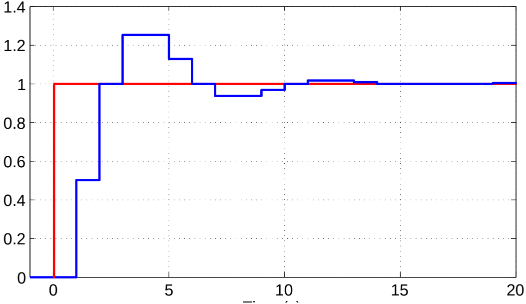
Static error : $\varepsilon_s = \lim_{z \to 1} (1 - z^{-1})\, \varepsilon(z) = 0$

Example 2



Error expression : $\varepsilon(z) = \dfrac{1}{1+\frac{0.5z}{z^2-1.5z+0.5}}\,E(z) = \dfrac{z^2-1.5z+0.5}{z^2-z+0.5}\,E(z)$

Static error : $\varepsilon_s \;=\; \lim_{z\to 1}(1-z^{-1})\,\varepsilon(z) \;=\; 0$

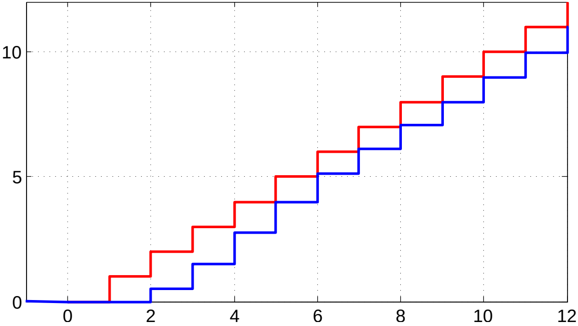


speed error :

$$\begin{aligned}\varepsilon_t &= \lim_{z\to 1}(1-z^{-1})\,\varepsilon(z)\\ &= \lim_{z\to 1}\frac{z-1}{z}\,\frac{z^2-1.5z+0.5}{z^2-z+0.5}\,\underbrace{\frac{z}{(z-1)^2}\,e_0}_{E(z)}\\ &= \lim_{z\to 1}\frac{(z-1)(z-0.5)}{z^2-z+0.5}\,\frac{e_0}{z-1} \;=\; e_0\end{aligned}$$

speed error :

$$\begin{aligned}\varepsilon_t &= \lim_{z\to 1}(1-z^{-1})\,\varepsilon(z)\\ &= \lim_{z\to 1}\frac{z-1}{z}\,\frac{z^2-1.5z+0.5}{z^2-z+0.5}\,\underbrace{\frac{z}{(z-1)^2}\,e_0}_{E(z)}\\ &= \lim_{z\to 1}\frac{(z-1)(z-0.5)}{z^2-z+0.5}\,\frac{e_0}{z-1} \;=\; e_0\end{aligned}$$



Sommaire

Introduction
- Review
- General points
- Analog to digital converter
- Digital to analog converter

Modelling
- Discrete-time systems
- Sampled systems

Feeback system analysis
- Stability
- Tracking error

Controller design
- Principle
- Discretization methods
- Implementation

Controller design

Sommaire

Introduction
- Review
- General points
- Analog to digital converter
- Digital to analog converter

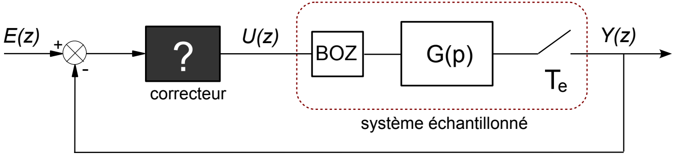
Modelling
- Discrete-time systems
- Sampled systems

Feeback system analysis
- Stability
- Tracking error

Controller design
- Principle
- Discretization methods
- Implementation

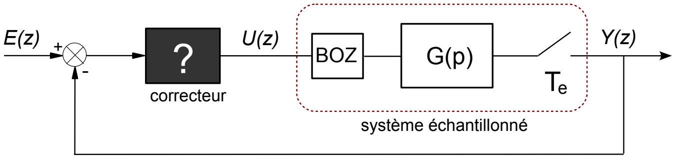
Principle

★ **Goal** : design a controller to drive the output of the system with some desired performances



Principle

★ **Goal** : design a controller to drive the output of the system with some desired performances



Two approaches :

- *Direct* design of a discrete-time controller.
- *Discretization* of a continuous-time controller.

Discretization of a continuous-time controller

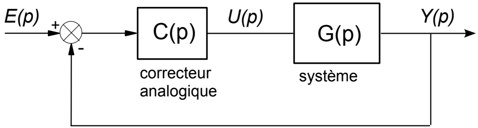
Key idea : approximate a continuous-time controller into a discrete-time one

$\Rightarrow$ the continuous-time controller must be designed beforehand with the continuous-time system

Discretization of a continuous-time controller

Key idea : approximate a continuous-time controller into a discrete-time one

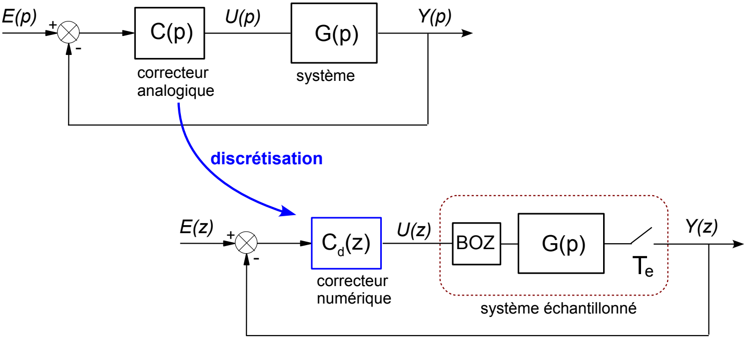
$\Rightarrow$ the continuous-time controller must be designed beforehand with the continuous-time system



Discretization of a continuous-time controller

Key idea : approximate a continuous-time controller into a discrete-time one

⇒ the continuous-time controller must be designed beforehand with the continuous-time system



Sommaire

Introduction
- Review
- General points
- Analog to digital converter
- Digital to analog converter

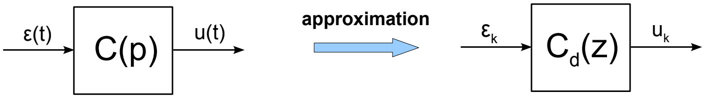
Modelling
- Discrete-time systems
- Sampled systems

Feeback system analysis
- Stability
- Tracking error

Controller design
- Principle
- Discretization methods
- Implementation

Discretization methods

How can we get $C_d(z)$ an approximation of $C(p)$?



- Several methods exist.
- Only two are presented here.
- Those methods are based on an approximation the Laplace variable p.

Euler method

The method is based on the numerical approximation of the derivation :

$$\frac{dx(t)}{dt} \simeq \frac{x_k - x_{k-1}}{T_e}$$

Euler method

The method is based on the numerical approximation of the derivation :

$$\frac{dx(t)}{dt} \simeq \frac{x_k - x_{k-1}}{T_e}$$

- In Laplace domain, the derivation is the multiplication by p.
- Let's apply the Z transform to the right hand side : $\dfrac{1 - z^{-1}}{T_e} X(z)$.

By analogy, a change of variable is proposed :

$$p \longrightarrow \frac{1 - z^{-1}}{T_e} = \frac{z - 1}{zT_e}$$

Bilinear (or Tustin's) method

The method is based on the numerical approximation of the integration :

$$y(t) = \int x(t)dt \quad \xrightarrow{\text{numerical approx.}} \quad y_k = y_{k-1} + \frac{x_{k-1} + x_k}{2} T_e$$

Bilinear (or Tustin's) method

The method is based on the numerical approximation of the integration :

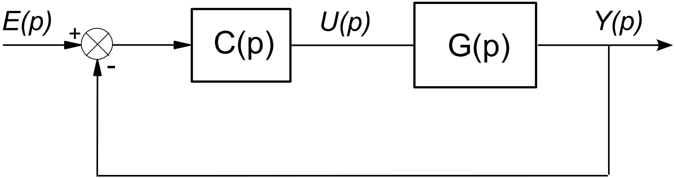
$$y(t) = \int x(t)dt \quad \xrightarrow{\text{numerical approx.}} \quad y_k = y_{k-1} + \frac{x_{k-1} + x_k}{2} T_e$$

- In the Laplace domain, integration is the division by p : $Y(p) = \frac{1}{p} X(p)$.
- Let's apply the Z transform to the right hand side : $Y(z) = \frac{T_e}{2} \frac{z+1}{z-1} X(z)$.

By analogy, a change of variable is proposed :

$$p \longrightarrow \frac{2}{T_e} \frac{z-1}{z+1}$$

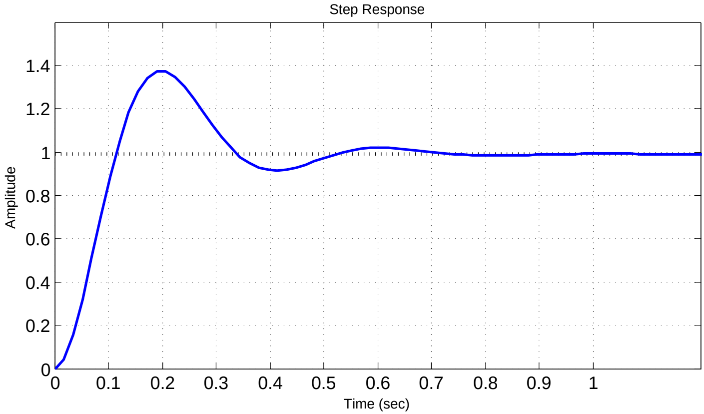
Example



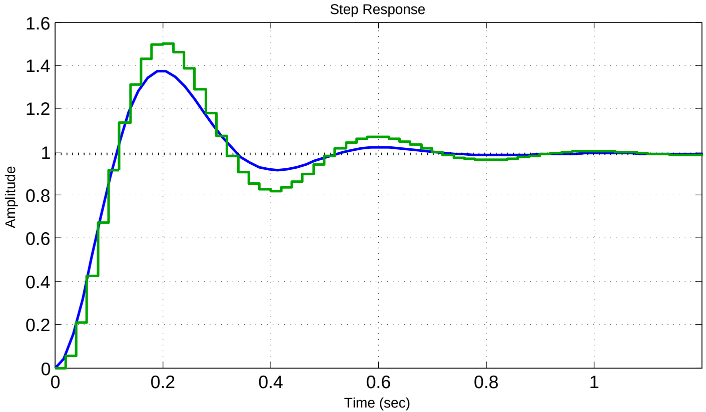
- System to be controlled : $G(p) = \dfrac{100}{(p+1)^2}$
- An analog controller has be designed : $C(p) = \dfrac{0.19p+1}{0.06p+1}$
- Let's discretize the controller with the bilinear method (with $T_e = 0.02s$) :

$$C_d(z) = \frac{2.8z - 2.6}{z - 0.7}$$

Example : output response of the closed-loop system



Example : output response of the closed-loop system



The digital control based on a numerical approx. of an analog controller allows to have similar performances.

Sommaire

Introduction
- Review
- General points
- Analog to digital converter
- Digital to analog converter

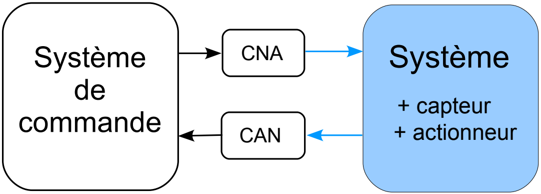
Modelling
- Discrete-time systems
- Sampled systems

Feeback system analysis
- Stability
- Tracking error

Controller design
- Principle
- Discretization methods
- Implementation

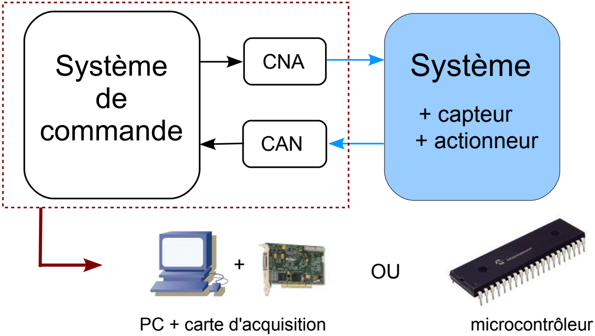
Implementation

General structure of a digital control system



Implementation

General structure of a digital control system



- Converters ADC and DAC are at the interface digital system / physical system.
- It is implemented on programmable electronic boards or PC.

The controller synthesis phase allows us to obtain the controller transfer function : $C_d(z)$.

$\Rightarrow$ How do we implement such a transfer function ?

The controller synthesis phase allows us to obtain the controller transfer function : $C_d(z)$.

$\Rightarrow$ How do we implement such a transfer function ?

$\Rightarrow$ we need to go back to time-domain formulation.

Let's express the input/output relationship of the controller :

The controller synthesis phase allows us to obtain the controller transfer function : $C_d(z)$.

$\Rightarrow$ How do we implement such a transfer function ?

$\Rightarrow$ we need to go back to time-domain formulation.

Let's express the input/output relationship of the controller :

$\Rightarrow$ **recurrence equation**

The controller synthesis phase allows us to obtain the controller transfer function : $C_d(z)$.

$\Rightarrow$ How do we implement such a transfer function ?

$\Rightarrow$ we need to go back to time-domain formulation.

Let's express the input/output relationship of the controller :

$\Rightarrow$ **recurrence equation**

Example :

$$C_d(z) = \frac{U(z)}{\varepsilon(z)} = \frac{3z+2}{z-0.5} \qquad \Longleftrightarrow \qquad u[k] = 0.5u[k-1] + 3\varepsilon[k] + 2\varepsilon[k-1]$$

Programming of the control law in C language. (example)

```c
float commande(float reference, float measure)
  {
    float output;
    float epsilon;
    static float output_previous = 0.0;
    static float epsilon_previous = 0.0;

    // calculate the difference between the
        reference and the measure
    epsilon = reference - measure;

    // recurrence equation of the controller
    output = 0.5*output_previous + 3*epsilon +
        2*epsilon_previous;

    // update previous values (instant k-1)
    output_previous = output;
    epsilon_previous = epsilon;

    return output;
  }
```

This function must be executed at each sampling period T_e.