KRR2: First-order logic

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Outline

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1. First-order logic, why?
2. Syntax and semantics

Pros and cons of propositional logic

Pros

- Propositional logic is declarative: pieces of syntax correspond to facts
- Propositional logic allows partial/disjunctive/negated information
  (unlike most data structures and databases)
- Propositional logic is compositional: meaning of $a \wedge b$ is derived from meaning of a and of b
- Meaning in propositional logic is context-independent
  (unlike natural language, where meaning depends on context)

Cons

Propositional logic has very limited expressive power (unlike natural language)

Expressiveness of propositional logic

Example

I want to declare:

Every friend of my sisters has a blue car

In the natural language, I need one simple sentence. What about in propositional logic?

I need symbols!! Lots of them!! Because I have a big family and my sisters are very friendly.

- Sister_1_Friend_1_Has_Blue_Car
- Sister_1_Friend_2_Has_Blue_Car
- ...
- Sister_4_Friend_23_Has_Blue_Car

First-order logic

Whereas propositional logic assumes world contains *facts*, first-order logic (like natural language) assumes the world contains

- Objects: people, houses, numbers, theories, colors, cricket games, centuries . . . and me, and cars!!
- Relations: red, round, bogus, prime, multistoried . . ., brother of, bigger than, inside, part of, has color, occurred after, owns, comes between, . . . and blue!!
- Functions: father of, third inning of, one more than, end of . . . and friend of, sister of!!

Outline

Syntax of First-Order Logic

Basic elements

1. Constants: $KingJohn, 2, ANU, Yannick \cdots$
2. Predicate: $Sister, > \cdots$
3. Functions: $Sqrt, FriendOf \cdots$
4. Variables: $x, y, a, b \cdots$
5. Connectives: $\vee, \wedge, \neg, \Rightarrow, \Leftrightarrow$
6. Equality: $=$
7. Quantifiers: $\forall \; \exists$

Syntax of First-Order Logic

Term

A term represents an object in FOL. Its syntax is:

- a constant, or
- a variable, or
- a function of terms $\mathit{function}(\mathit{term}_1, \cdots, \mathit{term}_n)$

Atomic sentence

An atomic sentence represents an elementary relation between terms. Its syntax is:

- a predicate $\mathit{predicate}(\mathit{term}_1, \cdots, \mathit{term}_n)$
- an equality of terms $\mathit{term}_1 = \mathit{term}_2$

Example

$\mathit{Brother}(\mathit{KingJohn}, \mathit{RichardTheLionheart})$
$> (\mathit{Length}(\mathit{LeftLegOf}(\mathit{Richard})), \mathit{Length}(\mathit{LeftLegOf}(\mathit{KingJohn})))$
$\mathit{carOf}(\mathit{friendOf}(\mathit{oneSisterOf}(\mathit{Yannick}))) = \mathit{colorOf}(\mathit{Ocean})$

Syntax of First-Order Logic

Complex sentences

Complex sentences are made from atomic sentences using connectives

$$\neg S, \quad S_1 \wedge S_2, \quad S_1 \vee S_2, \quad S_1 \Rightarrow S_2, \quad S_1 \Leftrightarrow S_2$$

Example

$Sibling(KingJohn, Richard) \Rightarrow Sibling(Richard, KingJohn)$
$>(1,2) \vee \leq(1,2)$
$>(1,2) \wedge \neg>(1,2)$
$Sister(Marie, Yannick) \Rightarrow CarColor(FriendOf(Marie), blue)$

Truth in first-order logic

Semantics

Sentences are true with respect to a model and an interpretation.

Model

Model contains `objects` (domain elements) and `relations` among them.

Interpretation

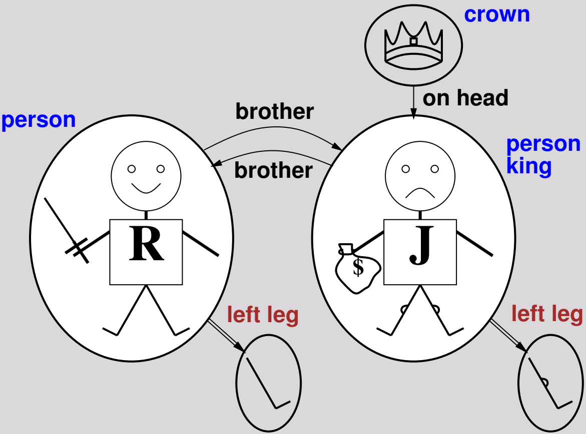
Interpretation specifies referents for

- *constant symbols* $\rightarrow$ `objects`
- *predicate symbols* $\rightarrow$ `relations`
- *function symbols* $\rightarrow$ `functional relations`

An atomic sentence $\mathit{predicate}(\mathit{term}_1, \ldots, \mathit{term}_n)$ is true iff the `objects` referred to by $\mathit{term}_1, \ldots, \mathit{term}_n$ are in the `relation` referred to by *predicate*.

Models for FOL: Example

Example



Models for FOL: Example

Example

Consider the interpretation in which

- *Richard* $\rightarrow$ `Richard the lionheart`
- *John* $\rightarrow$ `the evil King John`
- *Brother* $\rightarrow$ `the brotherhood relation`

Under this interpretation, $Brother(Richard, John)$ is true just in case `Richard the Lionheart` and `the evil King John` are in the brotherhood relation in the model

Models for FOL: Example

Example

Consider this new interpretation on the model about Yannick and his sisters in which

- *Richard* $\rightarrow$ `Yannick`
- *John* $\rightarrow$ `Marie`
- *Brother* $\rightarrow$ `the brother-sister relation`

Under this interpretation, $Brother(Richard, John)$ is true just in case `Yannick` and `Marie` are in the brother-sister relation in the model.
This interpretation is ok but the symbols are not very well-chosen!

Universal quantification

Universal quantification

Symbol: $\forall$

Syntax: $\forall \quad < variables > \quad < sentence >$

Semantics: $\forall \quad x \quad P$ is true in a model m iff P is true with x being each possible object in the model

Example

Quantified sentence: $\forall x \quad Sister(x, Yannick) \Rightarrow ColorCar(FriendOf(x), blue)$

Roughly speaking, it is equivalent to:

$Sister(Marie, Yannick) \Rightarrow ColorCar(FriendOf(Marie), Blue)$

$\wedge Sister(Claire, Yannick) \Rightarrow ColorCar(FriendOf(Claire), Blue)$

$\wedge Sister(KingJohn, Yannick) \Rightarrow ColorCar(FriendOf(KingJohn), Blue)$

$\wedge Sister(Blue, Yannick) \Rightarrow ColorCar(FriendOf(Blue), Blue)$

$\wedge Sister(Yannick, Yannick) \Rightarrow ColorCar(FriendOf(Yannick), Blue) \wedge \ldots$

A common mistake to avoid

Main connective with $\forall$

Typically, $\Rightarrow$ is the main connective with $\forall$
Common mistake: using $\land$ as the main connective with $\forall$

Example

$$\forall x \quad Sister(x, Yannick) \land ColorCar(FriendOf(x), Blue)$$

means “Everyone is the sister of Yannick and every friend of everyone has a blue car”

Existential quantification

Existential quantification

Symbol: $\exists$

Syntax: $\exists < variables > \quad < sentence >$

Semantics: $\exists x \quad P$ is true in the model m iff P is true with x beging some object in the model.

Example

Quantified sentence: $\exists x \quad Sister(x, Yannick) \land ColorCar(FriendOf(x), blue)$

Roughly speaking, it is equivalent to:

$Sister(Marie, Yannick) \land ColorCar(FriendOf(Marie), Blue)$

$\lor Sister(Claire, Yannick) \land ColorCar(FriendOf(Claire), Blue)$

$\lor Sister(KingJohn, Yannick) \land ColorCar(FriendOf(KingJohn), Blue)$

$\lor Sister(Blue, Yannick) \land ColorCar(FriendOf(Blue), Blue)$

$\lor Sister(Yannick, Yannick) \land ColorCar(FriendOf(Yannick), Blue) \lor \ldots$

Another common mistake to avoid

Main connective with $\exists$

Typically, $\wedge$ is the main connective with $\exists$
Common mistake: using $\Rightarrow$ as the main connective with $\exists$

Example

$$\exists x \quad Sister(x, Yannick) \Rightarrow ColorCar(FriendOf(x), Blue)$$

This sentence is true if you find someone who is not my sister!
It does not matter if this person has a blue car or not.

Properties of quantifiers

Properties

- $\forall x \forall y$ is the same as $\forall y \forall x$
- $\exists x \exists y$ is the same as $\exists y \exists x$
- $\exists x \forall y$ is not the same as $\forall y \exists x$
  - $\exists x \forall y \quad Loves(x, y)$
  - "There is a person who loves everyone in the world"
  - $\forall y \exists x \quad Loves(x, y)$
  - "Everyone in the world is loved by at least one person"

Quantifier duality

Each quantifier can be expressed using the other

- $\forall x \quad Likes(x, IceCream)$ is the same as $\neg \exists x \quad \neg Likes(x, IceCream)$
- $\exists x \quad Likes(x, Broccoli)$ is the same as $\neg \forall x \quad \neg Likes(x, Broccoli)$

Fun with sentences

Example

- Brothers are siblings

Fun with sentences

Example

- Brothers are siblings

$$\forall x, y \; Brother(x, y) \Rightarrow Sibling(x, y)$$

Fun with sentences

Example

- Brothers are siblings

$$\forall x, y \; Brother(x, y) \Rightarrow Sibling(x, y)$$

- "Sibling" is symmetric

Fun with sentences

Example

- Brothers are siblings

$$\forall x, y \; Brother(x, y) \Rightarrow Sibling(x, y)$$

- "Sibling" is symmetric

$$\forall x, y \; Sibling(x, y) \Leftrightarrow Sibling(y, x)$$

Fun with sentences

Example

- Brothers are siblings

$$\forall x, y \ Brother(x, y) \Rightarrow Sibling(x, y)$$

- "Sibling" is symmetric

$$\forall x, y \ Sibling(x, y) \Leftrightarrow Sibling(y, x)$$

- "One's mother is one's female parent"

Fun with sentences

Example

- Brothers are siblings

$$\forall x, y\ Brother(x, y) \Rightarrow Sibling(x, y)$$

- "Sibling" is symmetric

$$\forall x, y\ Sibling(x, y) \Leftrightarrow Sibling(y, x)$$

- "One's mother is one's female parent"

$$\forall x, y\ Mother(x, y) \Leftrightarrow (Female(x) \wedge Parent(x, y))$$

Fun with sentences

Example

- Brothers are siblings

$$\forall x, y \; Brother(x, y) \Rightarrow Sibling(x, y)$$

- "Sibling" is symmetric

$$\forall x, y \; Sibling(x, y) \Leftrightarrow Sibling(y, x)$$

- "One's mother is one's female parent"

$$\forall x, y \; Mother(x, y) \Leftrightarrow (Female(x) \wedge Parent(x, y))$$

- "A first cousin is a child of a parent's sibling"

Fun with sentences

Example

- Brothers are siblings

$$\forall x, y \; Brother(x, y) \Rightarrow Sibling(x, y)$$

- "Sibling" is symmetric

$$\forall x, y \; Sibling(x, y) \Leftrightarrow Sibling(y, x)$$

- "One's mother is one's female parent"

$$\forall x, y \; Mother(x, y) \Leftrightarrow (Female(x) \wedge Parent(x, y))$$

- "A first cousin is a child of a parent's sibling"

$$\forall x, y \; FirstCousin(x, y) \Leftrightarrow \exists p, ps \; Parent(p, x) \wedge Sibling(ps, p) \wedge Parent(ps, y)$$

Equality

Equality

$term_1 = term_2$ is true under a given interpretation if and only if $term_1$ and $term_2$ refer to the same object.

Example

- $1 = 2$ is satisfiable (if the symbols 1 and 2 refer to the same object in the interpretation)
- $2 = 2$ is valid

Example

Definition of *Sibling* thanks to *Parent*:

$$\forall x, y \; Sibling(x, y) \Leftrightarrow (\neg(x = y) \wedge \exists m, f \; \neg(m = f) \wedge$$

$$Parent(m, x) \wedge Parent(f, x) \wedge Parent(m, y) \wedge Parent(f, y))$$

Summary

Summary

- Knowledge representation language:
  - declarative, compositional, expressive, context-independent, unambiguous
- Model: set of objects, functions and their relation
- Knowledge-base in first-order logic
  - careful process
    1. analyzing the domain (objects, functions, relations),
    2. choosing a vocabulary (interpretation)
    3. encoding the axioms (what is known in *KB*) to support the desired inferences