Part I
Logical agents

Outline

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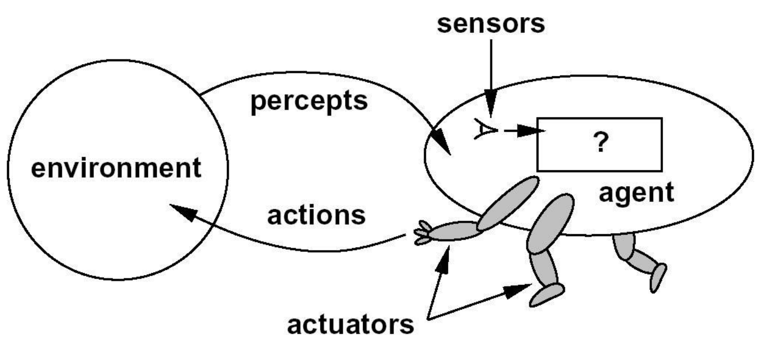
1 Knowledge-based agents

What is an agent?



Agents interact with environments through sensors and actuators

Knowledge-based agent

Definition 1. A *Knowledge base* is a set of sentences in a *formal* language.

Definition 2. *Knowledge-based agent*: Agents can be viewed at the knowledge level i.e., *what they know*, regardless of how implemented

Generic knowledge-based agent:

1. *PERCEIVE* an input

2. *TELL* the knowledge base what it perceives (same language)
3. *ASK* the knowledge base for an action to return (reasoning in the KB for a query, inference)
4. *TELL* the knowledge base about this action that has been executed (update of the KB)

A good way to represent and interact with a KB is *Logic*.

2 Logic in general models and entailment

Logic? But what is it?

Logics are *formal languages* for representing information such that conclusions can be drawn. To define a logic, we need:

1. *syntax*: how a *sentence* of the logic looks like?
2. *semantic*: what is the meaning of the sentence?
   - Given a *world*, is the sentence *true* or *false*?

Example 3. The language of arithmetic
Syntax: $x + 2 \geq y$ is a sentence; $x2 + y \geq y$ is not a sentence
Semantic: $x + 2 \geq y$ is `true` in a world where $x = 7, y = 1$ $x + 2 \geq y$ is `false` in a world where $x = 0, y = 6$

Entailment

Entailment means that one sentence (α) *follows from* other sentences (KB) and is denoted:

$$KB \models \alpha$$

We say that the Knowledge Base KB *entails* α *if and only if* α is `true` in all worlds where KB is `true`. Entailment is a relationship between sentences (i.e. syntax) that is based on semantics.

Example 4. Knowledge Base = { “The car is blue” “The bicycle is green or yellow” }
KB entails sentences α like:

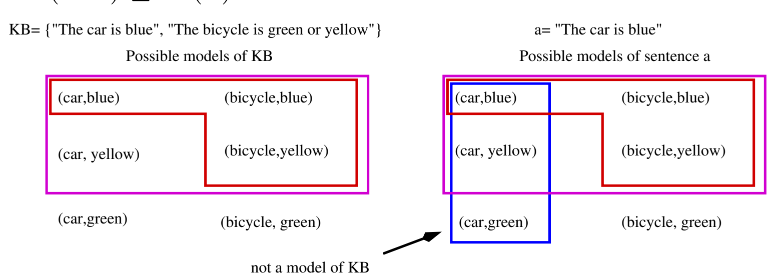
- “The car is blue”
- `true`
- “The car is blue or the bicycle is yellow”

The sentence “The car is blue and the bicycle is yellow” is not entailed by KB.

World in Logic = Model

Definition 5. We say m is a *model* of a sentence α if α is true in the world m. We denote by $M(\alpha)$ the set of models

KB entails α if and only if $M(KB) \subseteq M(\alpha)$.



Example 6.

Inference

Definition 7. *Inference*: A sentence β can be inferred from another sentence α by some inference algorithm i. This is denoted:

$$\alpha \vdash_i \beta$$

Definition 8. *Soundness*: An inference algorithm is *sound* if it produces entailed sentences

Definition 9. *Completeness*: An inference algorithm is *complete* if it can derive all the sentences which it entails.

Well-known logics

1. Propositional logic
2. First-order logic
3. Default logic
4. Circumscription
5. Temporal logic
6. Modal logic
7. ..

Every logic has its Pros and Cons (expressivity, soundness and completeness of inference algorithm)

3 Propositional Logic: a very simple logic

Propositional Logic: a very simple logic

We consider a set of proposition symbols $\{p_1, p_2, \cdots\}$.

Definition 10. *Syntax*: What is a sentence in the propositional logic?

1. any proposition symbol p_i is a sentence (*atomic sentence*)
2. if S is a sentence then $\neg S$ is a sentence
3. if S_1 and S_2 are sentences then $S_1 \wedge S_2$ is a sentence
4. if S_1 and S_2 are sentences then $S_1 \vee S_2$ is a sentence
5. if S_1 and S_2 are sentences then $S_1 \Rightarrow S_2$ is a sentence
6. if S_1 and S_2 are sentences then $S_1 \Leftrightarrow S_2$ is a sentence

Example 11. $p_1, p_1 \wedge p_2, p_1 \vee (\neg p_2 \wedge p_3), (p_3 \Rightarrow p_4) \wedge (p_4 \Rightarrow p_3)$, ...

Propositional Logic: a very simple logic

Definition 12. *Semantics*: What is the meaning of a sentence? A model m is a mapping between the proposition symbols $\{p_1, p_2, \cdots\}$ and $\{true, false\}$. Given m, we have:

1. $\neg S$ is $true$ iff S is $false$ ("not" S)
2. $S_1 \wedge S_2$ is $true$ iff S_1 is $true$ and S_2 is $true$ (S_1 "and" S_2)
3. $S_1 \vee S_2$ is $true$ iff S_1 is $true$ or S_2 is $true$ (S_1 "or" S_2)
4. $S_1 \Rightarrow S_2$ is $true$ iff S_1 is $false$ or S_2 is $true$ (S_1 "implies" S_2)
    - i.e. $S_1 \Rightarrow S_2$ is $false$ iff S_1 is $true$ or S_2 is $false$
5. $S_1 \Leftrightarrow S_2$ is $true$ iff $S_1 \Rightarrow S_2$ is $true$ and $S_2 \Rightarrow S_1$ is $true$ (S_1 "is equivalent to" S_2)

Propositional Logic: a very simple logic

Example 13. Symbols = $\{abcd\}$

Given the model $m = \{a = true, b = false, c = true, d = false\}$, then

- $\neg a = false$,
- $a \wedge \neg b = true$,
- $a \vee (b \wedge \neg c) = true$,
- $d \Rightarrow c = true$,
- $d \Rightarrow \neg c = true$,
- $\neg d \Rightarrow \neg c = false$,
- $\neg(a \vee b) \Leftrightarrow d = true$

4 Equivalence, validity, satisfiability

Logical equivalence

Definition 14. Two sentences α, β are *logically equivalent* IF AND ONLY IF they are true in the same models. α entails β and vice-versa.

$$
\begin{array}{rcl}
(\alpha \wedge \beta) & \equiv & (\beta \wedge \alpha) \quad \text{commutativity of } \wedge \\
(\alpha \vee \beta) & \equiv & (\beta \vee \alpha) \quad \text{commutativity of } \vee \\
((\alpha \wedge \beta) \wedge \gamma) & \equiv & (\alpha \wedge (\beta \wedge \gamma)) \quad \text{associativity of } \wedge \\
((\alpha \vee \beta) \vee \gamma) & \equiv & (\alpha \vee (\beta \vee \gamma)) \quad \text{associativity of } \vee \\
\neg(\neg\alpha) & \equiv & \alpha \quad \text{double-negation elimination} \\
(\alpha \Rightarrow \beta) & \equiv & (\neg\beta \Rightarrow \neg\alpha) \quad \text{contraposition} \\
(\alpha \Rightarrow \beta) & \equiv & (\neg\alpha \vee \beta) \quad \text{implication elimination} \\
(\alpha \Leftrightarrow \beta) & \equiv & ((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)) \quad \text{biconditional elimination} \\
\neg(\alpha \wedge \beta) & \equiv & (\neg\alpha \vee \neg\beta) \quad \text{De Morgan} \\
\neg(\alpha \vee \beta) & \equiv & (\neg\alpha \wedge \neg\beta) \quad \text{De Morgan} \\
(\alpha \wedge (\beta \vee \gamma)) & \equiv & ((\alpha \wedge \beta) \vee (\alpha \wedge \gamma)) \quad \text{distributivity of } \wedge \text{ over } \vee \\
(\alpha \vee (\beta \wedge \gamma)) & \equiv & ((\alpha \vee \beta) \wedge (\alpha \vee \gamma)) \quad \text{distributivity of } \vee \text{ over } \wedge
\end{array}
$$

Validity and satisfiability

Definition 15. A sentence is *valid* if it is true in *ALL models*:
$a \vee \neg a$, $a \Rightarrow a$, $(a \wedge (a \Rightarrow b)) \Rightarrow b$

KB entails α ($KB \models \alpha$) iff the sentence $KB \Rightarrow \alpha$ is valid. Validity is then connected to inference.

Definition 16.
1. A sentence is *satisfiable* if it is true in *SOME models*. A valid sentence is satisfiable, but a satisfiable sentence may be not valid.
2. A sentence is *unsatisfiable* if it is true in *NO models*:
   $a \wedge \neg a$, $(a \wedge b) \Leftrightarrow (\neg a \wedge c)$

KB entails α ($KB \models \alpha$) iff the sentence $KB \wedge \neg\alpha$ is unsatisfiable.

5 Inference rules and theorem proving in propositional logic

Inference, Theorem proving

Given KB, can I prove the sentence α? Is α satisfiable in KB? $KB \models \alpha$? Is $KB \wedge \neg\alpha$ unsatisfiable?

Model (Truth table) enumeration, we check that $M(KB) \subseteq M(\alpha)$.
Bad news: exponential in the number of proposition symbols involved in KB, α.
Some improved methods: Davis-Putnam-Logemann-Loveland (complete), min-conflicts-like (incomplete) hill-climbing (incomplete).

- Sound generation of new sentences from old.
- Proof = a sequence of *inference rules* that finally generate α

Methods: Forward-chaining, Backward chaining, Resolution

Inference rules: examples

Example 17. Modus Ponens:

$$\frac{a, a \Rightarrow b}{b}$$

And-elimination:

$$\frac{a \wedge b}{a}$$

Factoring:

$$\frac{a \vee a}{a}$$

Logical equivalences:

$$\frac{\neg a \vee \neg b}{\neg(a \wedge b)}$$

$$\frac{a \Leftrightarrow b}{a \Rightarrow b \wedge b \Rightarrow a}$$

Forward and backward chaining methods

Proof methods that are simple and efficient. They need a restriction on the form of KB.

$$KB = \text{conjunction of Horn clauses}$$

Definition 18. A *Horn clause* is

- a propositional symbol a, or
- something like $p_1 \cdots p_n \Rightarrow c$ (p_i is a *premisse* and c the *conclusion*)

Inference problem

Rule 1 $P \Rightarrow Q$

Rule 2 $L \wedge M \Rightarrow P$

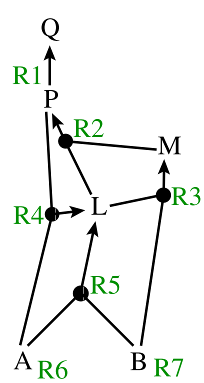
Rule 3 $B \wedge L \Rightarrow M$

Rule 4 $A \wedge P \Rightarrow L$

Rule 5 $A \wedge B \Rightarrow L$

Rule 6 A

Rule 7 B



Is the proposition Q true or not?

Forward chaining method

Forward chaining:

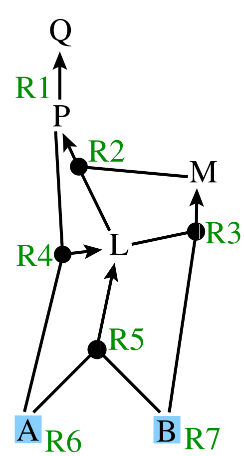
1. Fire any rule whose premises are satisfied in the Knowledge Base
2. Add its conclusion to the Knowledge Base
   - Management of a *Working Memory* (an Agenda)
3. Repeat 1 and 2 until the proposition Q is true or no more conclusion can be derived

$$\frac{a_1, \cdot, a_n \quad a_1 \wedge \cdots \wedge a_n \Rightarrow b}{b}$$

Forward chaining: example

A and B are known in the knowledge base (Rules 6 and 7).

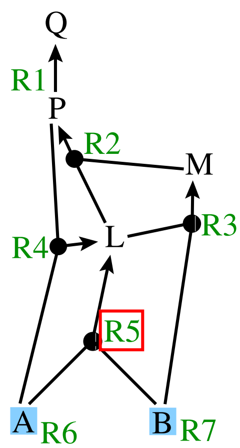
$$Agenda = \{A, B\}$$



Forward chaining: example

Rule 5 can be fired since A and B are known.

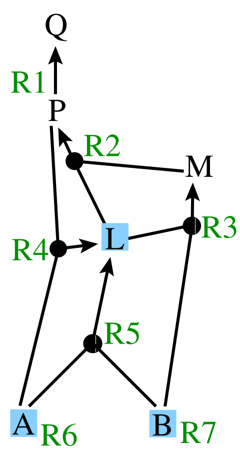
$$A \wedge B \Rightarrow L$$



Forward chaining: example

The agenda is updated with the conclusion (head) of Rule 5:

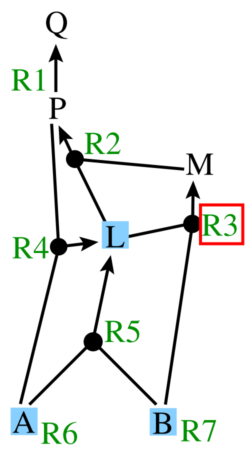
$$Agenda = \{A, B, L\}$$



Forward chaining: example

Rule 3 can be fired since B and L are known.

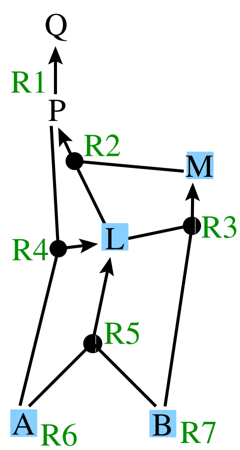
$$B \wedge L \Rightarrow M$$



Forward chaining: example

The agenda is updated with the conclusion (head) of Rule 3:

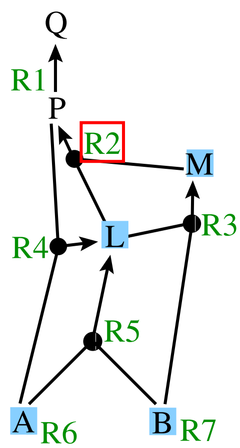
$$Agenda = \{A, B, L, M\}$$



Forward chaining: example

Rule 2 can be fired since M and L are known.

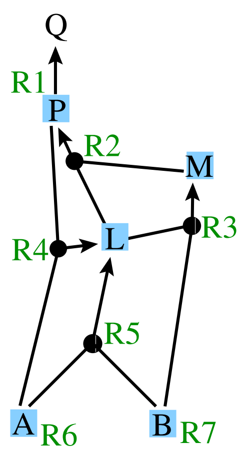
$$M \wedge L \Rightarrow P$$



Forward chaining: example

The agenda is updated with the conclusion (head) of Rule 2:

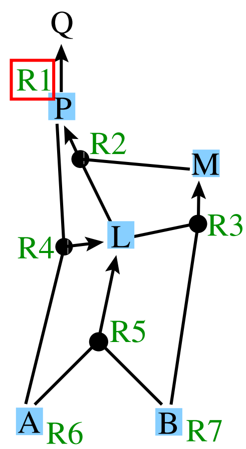
$$Agenda = \{A, B, L, M, P\}$$



Forward chaining: example

Rule 1 can be fired since P is known.

$$P \Rightarrow Q$$

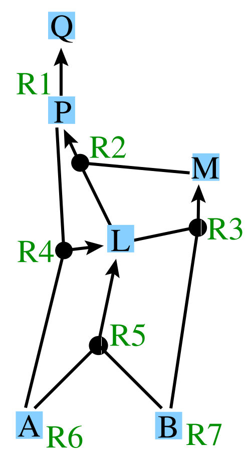


Forward chaining: example

The agenda is updated with the conclusion (head) of Rule 1:

$$Agenda = \{A, B, L, M, P, Q\}$$

Q has been derived. STOP



Properties of the FC algorithm

Theorem 19. *The forward chaining algorithm is complete.*

- Every atomic proposition that is provable in the knowledge base can be derived thanks to the algorithm.
- The result is due to the fact that:
  1. The knowledge base is a conjunction of Horn clauses.
  2. In the worst case of FC, every clause is fired.

Theorem 20. *The forward chaining algorithm runs in linear time.*

- Every rule is fired at most once.

Backward chaining method

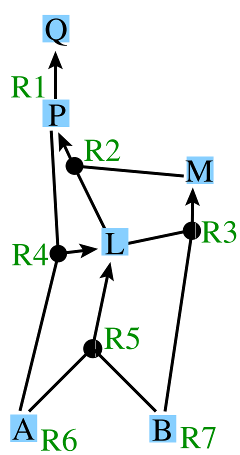
At a given step, the choice between the firable rules is *random*. FC may apply rules that are *useless* for the proof of Q! *Data-driven algorithm*

Backward chaining: *Goal-driven algorithm*

- To prove Q every premise of one rule which concludes Q has to be at least proved.
- Basically, BackwardChaining(Q) =
  1. If Q is known in KB then Q is proved
  2. Otherwise, select a rule $P_1 \wedge \cdots \wedge P_n \Rightarrow Q$ in KB
  3. Recusirvely apply BackwardChaining(P_1), ... , BackwardChaining(P_n) to prove the premises.
  4. Repeat 2-3 until Q is proved or no more rules can be selected.

Backward chaining: example

To prove Q, we need to prove P. To prove P we need to prove L and M etc. So it goes until A and B are reached. A and B are known in KB so Q is proved.



Backward chaining method (2)

Theorem 21. *The backward chaining algorithm is complete.*

Theorem 22. *The backward chaining algorithm runs in linear time.*

- Every rule is fired at most once
- In practice, it is much less than linear in size of KB: it is linear in the size of the set of rules that are involved in the proof of the premises of Q.

Propositional logic inference: resolution algorithm

FC and BC are efficient algorithms but they make the asssumption that KB is a set of Horn clauses.

Given a KB, is there an algorithm which is sound and complete?

YES, this is called a *resolution algorithm* based on the resolution inference rule.

Resolution inference rule

$$\frac{\ell_1 \vee \cdots \vee \boxed{\ell_i} \vee \cdots \vee \ell_k, \qquad \ell'_1 \vee \cdots \vee \boxed{\ell'_j} \vee \cdots \vee \ell'_n}{\ell_1 \vee \cdots \vee \ell_{i-1} \vee \ell_{i+1} \vee \cdots \vee \ell_k \vee \ell'_1 \vee \cdots \vee \ell'_{j-1} \vee \ell'_{j+1} \vee \cdots \vee \ell'_n}$$

with $\boxed{\ell_i = a}$ and $\boxed{\ell_j = \neg a}$ or $\boxed{\ell_i = \neg a}$ and $\boxed{\ell_j = a}$
(a is an atomic sentence)

Example 23.

$$\frac{\boxed{e}, \boxed{\neg e}}{\emptyset}$$

$$\frac{v \vee \boxed{\neg w} \vee \neg x, \boxed{w} \vee \neg x \vee y \vee z}{v \vee \neg x \vee y \vee z}$$

Resolution inference rule (2): CNF

The rule is applied on sentences like $(\ell_1 \vee \cdots\cdots \vee \ell_k) \wedge \cdots \wedge (\ell_m \vee \cdots\cdots \vee \ell_n)$ where ℓ_i are positive/negative literals.

This form is called *Conjunctive Normal Form* (CNF for short).

Every sentence in proposition logic is logically equivalent to a CNF sentence.

Example 24. $(a \vee b) \Leftrightarrow (c \wedge d)$ is logically equivalent to the CNF $(\neg a \vee c) \wedge (\neg a \vee d) \wedge (\neg b \vee c) \wedge (\neg b \vee d) \wedge (\neg c \vee \neg d \vee a \vee b)$.

Conversion to a CNF

1. Eliminate $\Leftrightarrow$, replacing $\alpha \Leftrightarrow \beta$ with $(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$.
2. Eliminate $\Rightarrow$, replacing $\alpha \Rightarrow \beta$ with $\neg\alpha \vee \beta$.
3. Move $\neg$ inwards using de Morgan's rules $\{\neg(a \wedge b) \equiv (\neg a \vee \neg b), \neg(a \vee b) \equiv (\neg a \wedge \neg b)\}$ and double negation rule $(\neg\neg a \equiv a)$
4. Apply distributivity law ($\vee$ over $\wedge$) and flatten $(a \wedge b) \vee c \equiv (a \vee c) \wedge (b \vee c)$

Resolution algorithm

Definition 25. *Proof by contradiction:* given KB, to prove α, we prove that $KB \wedge \neg\alpha$ is not satisfiable.

Example 26. Symbols:

- Und: “The students have understood this lecture”
- Gt: “I am a good teacher”
- $Party$: “The students went to a party last night”

Knowledge base:

$$KB = (\neg Und \Leftrightarrow (\neg Gt \vee Party)) \wedge Und$$

Query to prove: *I am a good teacher*

$$\alpha = Gt$$

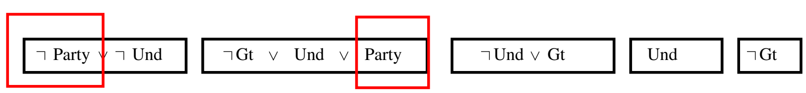
Resolution algorithm

Example 27. Conversion to CNF: $KB \wedge \neg\alpha \rightarrow (KB \wedge \neg\alpha)_{CNF}$

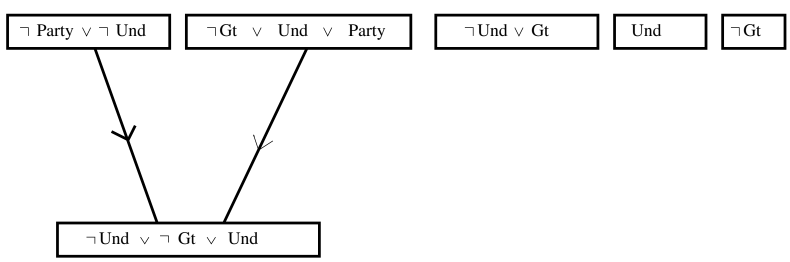
$$\begin{aligned}(KB \wedge \neg\alpha)_{CNF} = &(\neg Party \vee \neg Und) \wedge \\ &(\neg Gt \vee Und \vee Party) \wedge \\ &(\neg Und \vee Gt) \wedge \\ &Und \wedge \neg Gt\end{aligned}$$

Example 28. 

Resolution algorithm

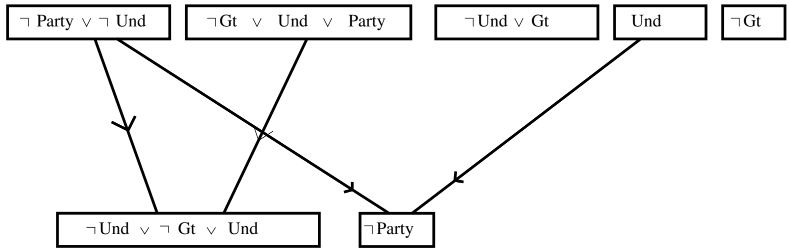
Example 29. 

Resolution algorithm



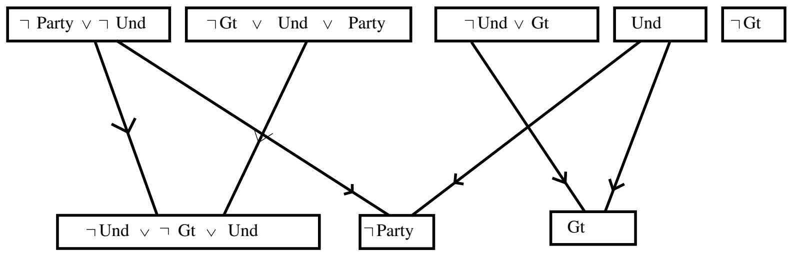
Example 30.

Resolution algorithm



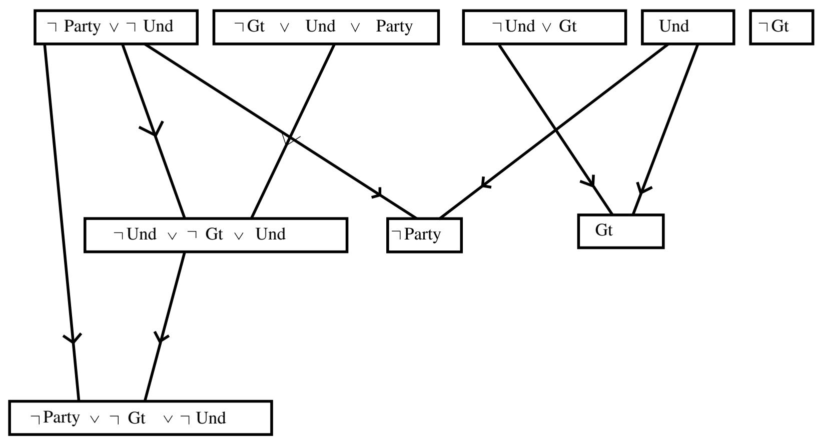
Example 31.

Resolution algorithm



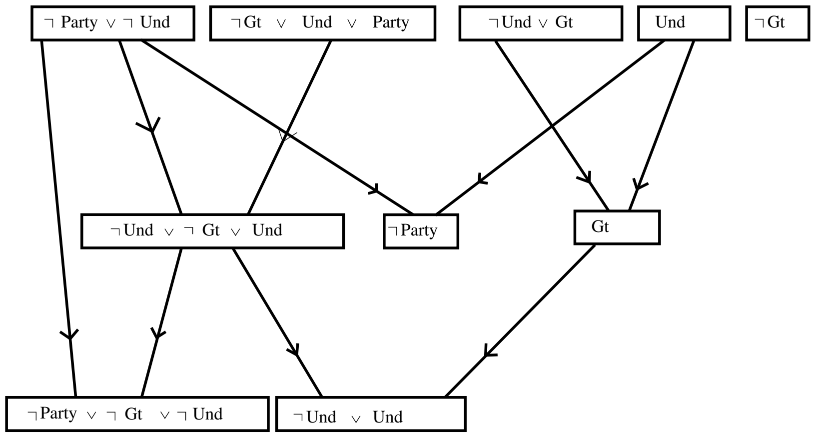
Example 32.

Resolution algorithm



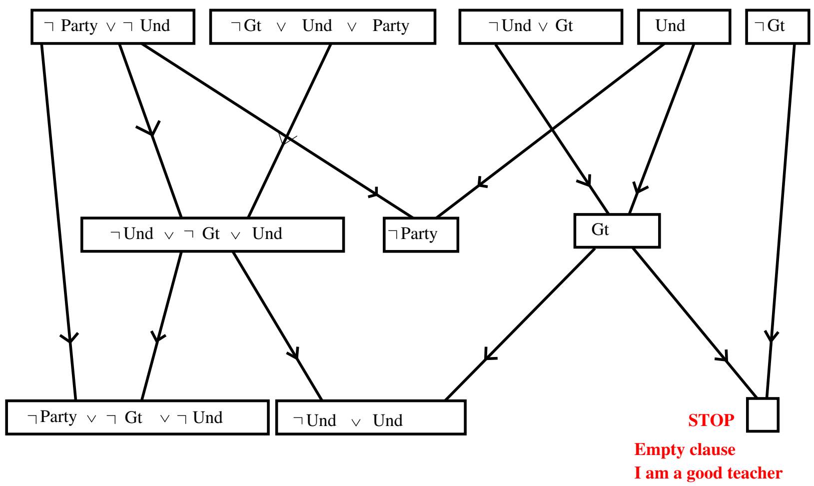
Example 33.

Resolution algorithm



Example 34.

Resolution algorithm



Example 35.

Summary

Logical agents apply *inference* to a *knowledge base* to derive new information and make decisions
Basic concepts of logic:

- *syntax*: formal structure of *sentences*
- *semantics*: *truth* of sentences wrt *models*
- *entailment*: necessary truth of one sentence given another
- *inference*: deriving sentences from other sentences
- *soundess*: derivations produce only entailed sentences
- *completeness*: derivations can produce all entailed sentences

Forward, backward chaining are linear-time, complete for Horn clauses Resolution is complete for propositional logic

Propositional logic lacks expressive power