Design of Stable Mechanical Structures

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Structural design problems

Mathematical program with equilibrium constraints:

$$\min_{\rho, u} F(\rho, u)$$

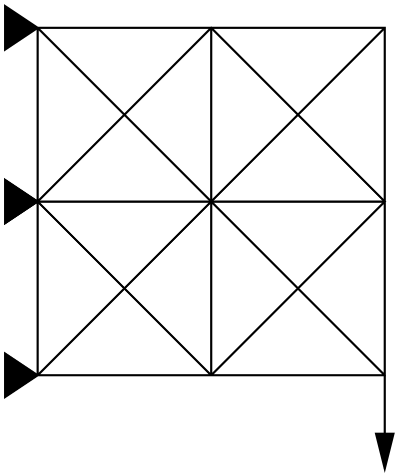
s.t.

$$\rho \in U_{\mathrm{ad}}$$

$$u \text{ solves } \mathcal{E}(\rho, u)$$

$F(\rho, u)$	...	cost functional (weight, stiffness, peak stress. . .)
ρ	...	design variable (thickness, material properties, shape. . .)
u	...	state variable (displacements, stresses)
U_{ad}	...	admissible designs

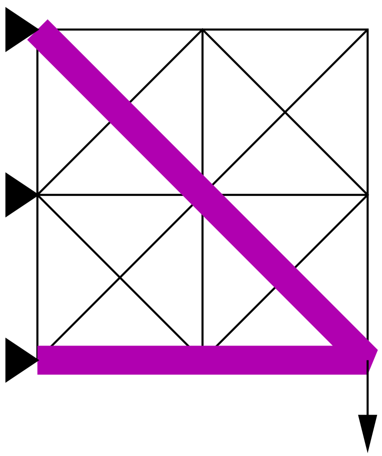
Truss design problem



$$K(\rho) = \sum_{i=1}^{m} \rho_i K_i, \qquad K_i = \frac{E_i}{l_i^2}\gamma_i\gamma_i^T$$

$\rho_i \ldots$ bar volumes
$u_i \ldots$ displacements

Truss design problem



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$\rho_i \ldots$ bar volumes
$u_i \ldots$ displacements

$$\min_{\rho \in \mathbb{R}^m, u \in \mathbb{R}^n} \sum_{i=1}^{m} \rho_i$$

$$\text{s.t.} \quad K(\rho)u = f, \quad f^T u \le C, \quad \rho_i \ge 0, \quad i = 1, 2, \ldots, m$$

Structural design problems

WEIGHT versus STIFFNESS:

- W weight $\quad \sum \rho_i$
- C stiffness (compliance) $\quad f^T u$

Equilibrium constraint: $\quad u$ solves $\mathcal{E}(\rho, u) \longrightarrow \sum(\rho_i K_i) u = f$

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$$\min C$$

s.t.

$$W \leq \widehat{W}$$

equilibrium

Structural design problems

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$\min C$ s.t. $W \leq \widehat{W}$, equilibrium	$\min W$ s.t. $C \leq \widehat{C}$, equilibrium

S. Timoshenko:

Experience showed that structures like bridges or aircrafts may fail in some cases not on account of high stresses but owing to insufficient elastic stability.

Self-vibrations of a mechanical system

Free vibrations (motions in absence of external loads) of a mechanical system

$$M\ddot{x}(t) = -Kx(t) \qquad (\star)$$

$K \succeq 0 \ldots$ stiffness matrix, $\quad M \succ 0 \ldots$ mass matrix
$M = \sum \rho_i \beta_i \beta_i^T, \quad \beta_i = \sqrt{\mu}\gamma_i$

The solutions to $(\star)$ are of the form

$$x(t) = \sum_{j=1}^{n} [a_j \cos(\omega_j t) + b_j \sin(\omega_j)t] e_j$$

where a_j, b_j are free parameters, e_j are the eigenvectors of

$$(\lambda_j M - K)e_j = 0$$

and $\omega_j = \sqrt{\lambda_j}$.

Structural design with free vibration control

Three quantities to control:

- W weight $\sum \rho_i$
- C stiffness (compliance) $f^T u$
- λ min. eigenfrequency $K(\rho)u = \lambda M(\rho)u$

Three quantities to control:

- $\boldsymbol{W}$ weight $\sum \boldsymbol{\rho_i}$
- $\boldsymbol{C}$ stiffness (compliance) $\boldsymbol{f^T u}$
- $\boldsymbol{\lambda}$ min. eigenfrequency $\boldsymbol{K(\rho)u = \lambda M(\rho)u}$

$$\min \boldsymbol{C}$$

s.t.

$$\boldsymbol{W} \leq \widehat{\boldsymbol{W}}$$

$$\boldsymbol{\lambda} \geq \widehat{\boldsymbol{\lambda}}$$

equilibrium

Three quantities to control:

- $\boldsymbol{W}$ weight $\quad \sum \boldsymbol{\rho}_i$
- $\boldsymbol{C}$ stiffness (compliance) $\quad \boldsymbol{f}^T \boldsymbol{u}$
- $\boldsymbol{\lambda}$ min. eigenfrequency $\quad \boldsymbol{K}(\boldsymbol{\rho})\boldsymbol{u} = \boldsymbol{\lambda} \boldsymbol{M}(\boldsymbol{\rho})\boldsymbol{u}$

$\min \boldsymbol{C}$ s.t. $\boldsymbol{W} \le \widehat{\boldsymbol{W}}$ $\boldsymbol{\lambda} \ge \widehat{\boldsymbol{\lambda}}$ equilibrium	$\min \boldsymbol{W}$ s.t. $\boldsymbol{C} \le \widehat{\boldsymbol{C}}$ $\boldsymbol{\lambda} \ge \widehat{\boldsymbol{\lambda}}$ equilibrium

Three quantities to control:

- W weight $\quad \sum \rho_i$
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$\min C$ s.t. $W \le \widehat{W}$ $\lambda \ge \hat{\lambda}$ equilibrium	$\min W$ s.t. $C \le \hat{C}$ $\lambda \ge \hat{\lambda}$ equilibrium	$\max \lambda$ s.t. $W \le \widehat{W}$ $C \le \hat{C}$ equilibrium

Structural design with stability control

Three quantities to control:

- W weight $\quad \sum \rho_i$
- C stiffness (compliance) $\quad f^T u$
- λ critical buckling force $\quad K(\rho)u = \lambda G(\rho, u)u$

$\min C$ s.t. $W \leq \widehat{W}$ $\lambda \geq 1$ equilibrium	$\min W$ s.t. $C \leq \widehat{C}$ $\lambda \geq 1$ equilibrium	$\max \lambda$ s.t. $W \leq \widehat{W}$ $C \leq \widehat{C}$ equilibrium

Structural design with stability control

Lowest (positive) eigenvalue of

$$K(\rho)u = \lambda G(\rho, u)u$$

(critical force) should be bigger than 1 (than $\widehat{\lambda}$ for vibration constraint)

$$\min_{\rho, u} W(\rho)$$

s.t.

$$K(\rho)u = f$$

$$f^T u \leq \widehat{C}$$

$$\rho_i \geq 0, \quad i = 1, \ldots, m$$

$$\lambda \geq 1$$

Two standard tricks:

$$\boldsymbol{K}(\boldsymbol{\rho}) \succ \mathbf{0}, \qquad \boldsymbol{u} = \boldsymbol{K}(\boldsymbol{\rho})^{-1}\boldsymbol{f}$$

$$\boldsymbol{f}^T\boldsymbol{K}(\boldsymbol{\rho})^{-1}\boldsymbol{f} \leq \widehat{C} \quad \Longleftrightarrow \quad \begin{pmatrix} \widehat{C} & \boldsymbol{f}^T \\ \boldsymbol{f} & \boldsymbol{K}(\boldsymbol{\rho}) \end{pmatrix} \succeq \mathbf{0}$$

Structural design with stability control

Two standard tricks:

$$K(\rho) \succ 0, \qquad u = K(\rho)^{-1} f$$

$$f^T K(\rho)^{-1} f \le \widehat{C} \quad \Longleftrightarrow \quad \begin{pmatrix} \widehat{C} & f^T \\ f & K(\rho) \end{pmatrix} \succeq 0$$

$$\left.\begin{aligned} K(\rho)u = \lambda G(\rho, u)u \\ \lambda \ge 1 \end{aligned}\right\} \quad \Longleftrightarrow \quad K(\rho) - G(\rho, u) \succeq 0$$

$$\Longleftrightarrow \quad K(\rho) - \widetilde{G}(\rho) \succeq 0$$

$$\widetilde{G}(\rho) = G(\rho, K(\rho)^{-1} f)$$

Structural design with stability control

Formulated as SDP problem:

$$\min_{\rho} W(\rho)$$

subject to

$$K(\rho) - \widetilde{G}(\rho) \succeq 0$$

$$\begin{pmatrix} c & f^T \\ f & K(\rho) \end{pmatrix} \succeq 0$$

$$\rho_i \geq 0, \quad i = 1, \ldots, m$$

where

$$K(\rho) = \sum \rho_i K_i, \qquad \widetilde{G}(\rho) = \sum \widetilde{G}_i$$

Structural design with free vibration control

Formulated as SDP problem:

$$\min_{\rho} W(\rho)$$

subject to

$$K(\rho) - \hat{\lambda} M(\rho) \succeq 0$$

$$\begin{pmatrix} c & f^T \\ f & K(\rho) \end{pmatrix} \succeq 0$$

$$\rho_i \geq 0, \quad i = 1, \ldots, m$$

where

$$K(\rho) = \sum \rho_i K_i, \qquad M(\rho) = \sum \rho_i Mi$$

Solving vibration problem as GEVP

Another option (vibration problems):
Solve the maximum eigenvalue problem formulated as GEVP:

$\boldsymbol{\lambda}$ min. eigenfrequency of $\boldsymbol{K}(\boldsymbol{\rho})\boldsymbol{u} = \boldsymbol{\lambda}\boldsymbol{M}(\boldsymbol{\rho})\boldsymbol{u}$

$$\max \boldsymbol{\lambda}$$

s.t.

$$\boldsymbol{W} \leq \widehat{\boldsymbol{W}}$$

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equilibrium

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$$\max \boldsymbol{\lambda}$$

s.t.

$$\boldsymbol{W} \leq \widehat{\boldsymbol{W}}$$

$$\boldsymbol{C} \leq \widehat{\boldsymbol{C}}$$

equilibrium

$$\max_{\boldsymbol{\rho},\boldsymbol{\lambda}} \boldsymbol{\lambda}$$

s.t.

$$\boldsymbol{K}(\boldsymbol{\rho}) - \boldsymbol{\lambda}\boldsymbol{M}(\boldsymbol{\rho}) \succeq \mathbf{0}$$

$$\sum \boldsymbol{\rho}_i \leq \widehat{\boldsymbol{W}}$$

$$\boldsymbol{\rho}_i \geq \mathbf{0}, \quad \boldsymbol{i} = \mathbf{1}, \ldots, \boldsymbol{m}$$

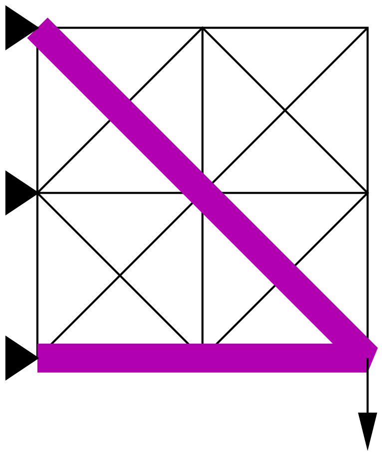
$$\begin{pmatrix} \widehat{\boldsymbol{C}} & \boldsymbol{f}^T \\ \boldsymbol{f} & \boldsymbol{K}(\boldsymbol{\rho}) \end{pmatrix} \succeq \mathbf{0}$$

(quasiconvex) SDP problem with BMI constraints — solve by PENBMI

Truss design problem

$$\min_{\rho \in \mathbb{R}^m, u \in \mathbb{R}^n} f^T u$$

$$\text{s.t.} \quad K(\rho)u = f, \quad \sum_{i=1}^{m} \rho_i = V, \quad \rho_i \geq 0, \quad i = 1, 2, \ldots, m$$



$$K(\rho) = \sum_{i=1}^{m} \rho_i K_i, \qquad K_i = \frac{E_i}{l_i^2} \gamma_i \gamma_i^T$$

$\rho_i \ldots$ bar volumes

$u_i \ldots$ displacements

Truss design with stability constraint

$$\min_{\rho} \sum \rho_i$$ (minimize weight)

subject to

$$K(\rho) + \widetilde{G}(\rho) \succeq 0$$

$$\begin{pmatrix} c & f^T \\ f & K(\rho) \end{pmatrix} \succeq 0$$

$$\rho_i \geq 0, \quad i = 1, \ldots, m$$

with

$$K(\rho) = \sum \rho_i K_i \qquad K_i = \frac{E_i}{\ell_i^2} \gamma_i \gamma_i^T$$

$$\widetilde{G}(\rho) = \sum \widetilde{G}_i(\rho) \qquad \widetilde{G}_i(\rho) = \frac{E \rho_i}{\ell_i^3} (\gamma_i^T K(\rho)^{-1} f)(\delta_i \delta_i^T).$$

Truss design with free vibration constraint

$$\min_{\rho} \sum \rho_i$$ (minimize weight)

subject to

$$K(\rho) + \widehat{\lambda} M(\rho) \succeq 0$$

$$\begin{pmatrix} c & f^T \\ f & K(\rho) \end{pmatrix} \succeq 0$$

$$\rho_i \geq 0, \quad i = 1, \ldots, m$$

with

$$K(\rho) = \sum \rho_i K_i \qquad K_i = \frac{E_i}{\ell_i^2} \gamma_i \gamma_i^T$$

$$M(\rho) = \sum \rho_i M_i \qquad M_i = \text{diag}(\delta_i \delta_i^T)$$

Free Material Optimization

Aim:

Given an amount of material, boundary conditions and external load f, find the material (distribution) so that the body is as stiff as possible under f.

The design variables are the material properties at each point of the structure.

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$$\inf_{\substack{\rho \geq 0 \\ \int \rho \, dx \leq 1}} \sup_{u \in U} -\frac{1}{2} \int_\Omega \rho \langle e(u), e(u) \rangle \, dx + \int_{\Gamma_2} f \cdot u \, dx$$

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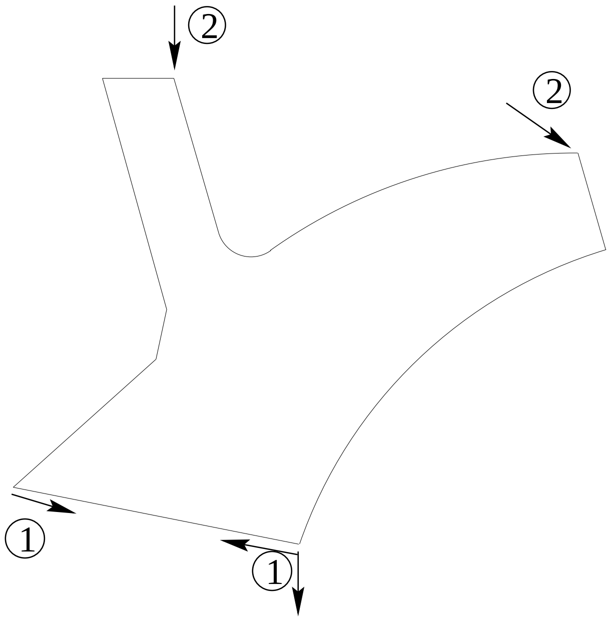
$$\inf_{\alpha \in \mathbb{R}, u \in U} \left\{ \alpha - f^T u \,|\, \alpha \ge \frac{m}{2} u^T A_i \, u \quad \text{for} \quad i = 1, \dots, m \right\}$$

FMO, example



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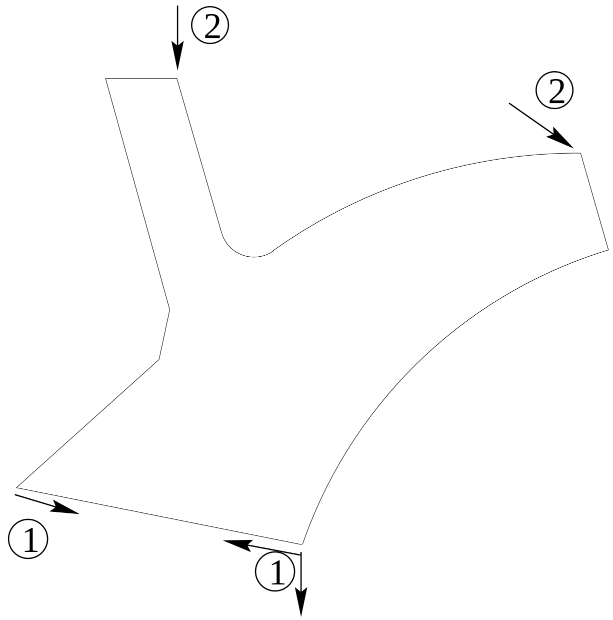
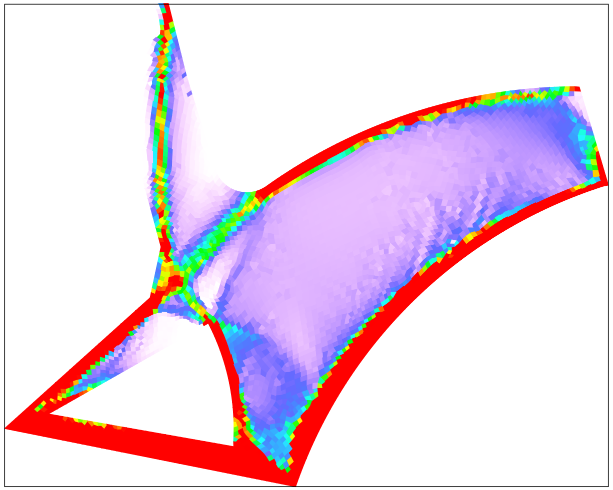
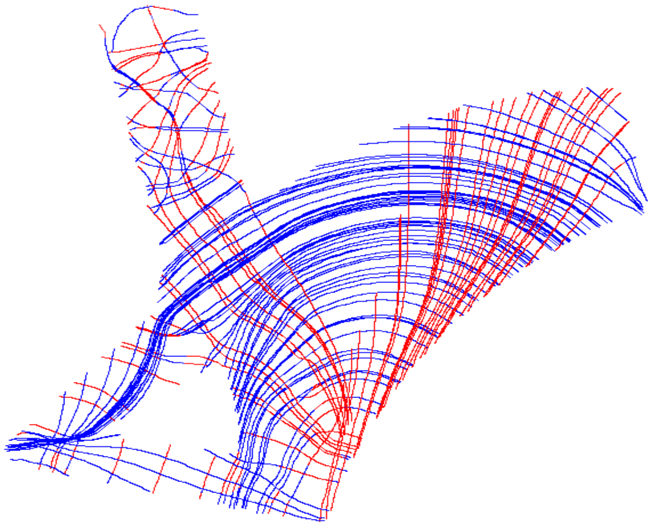
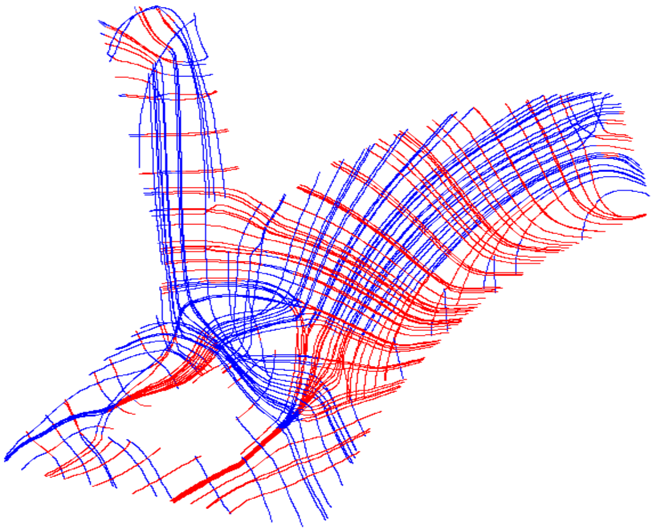
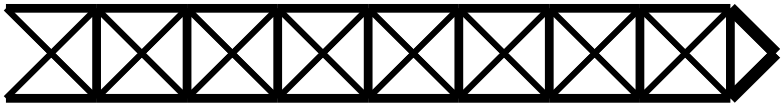








Truss design w. stability, vibration constraint



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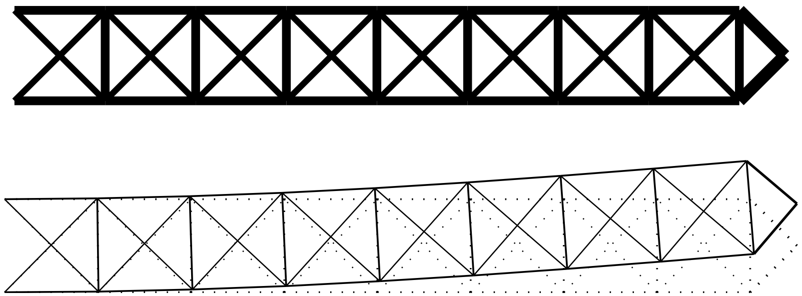

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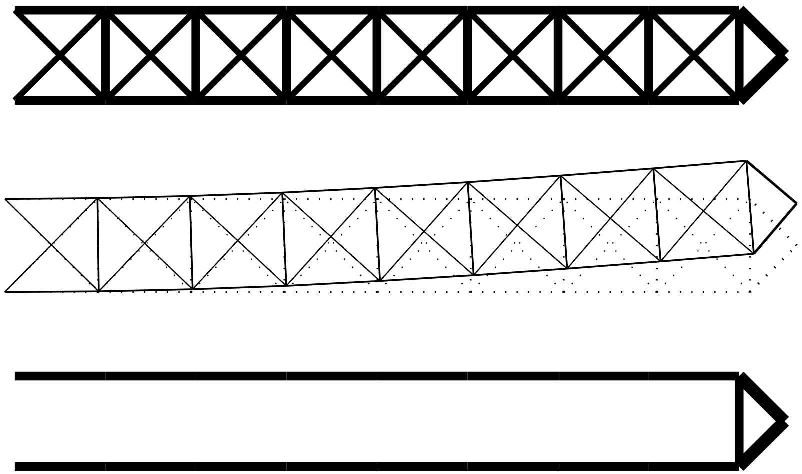

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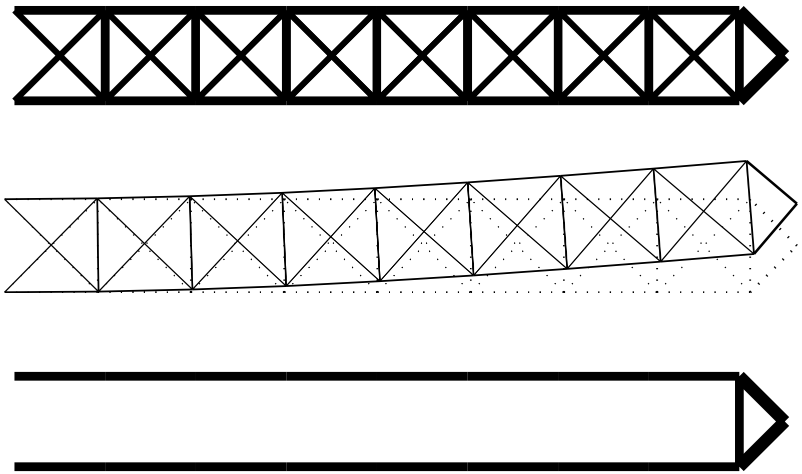

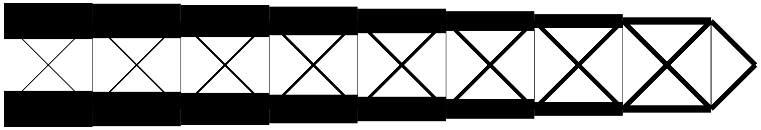

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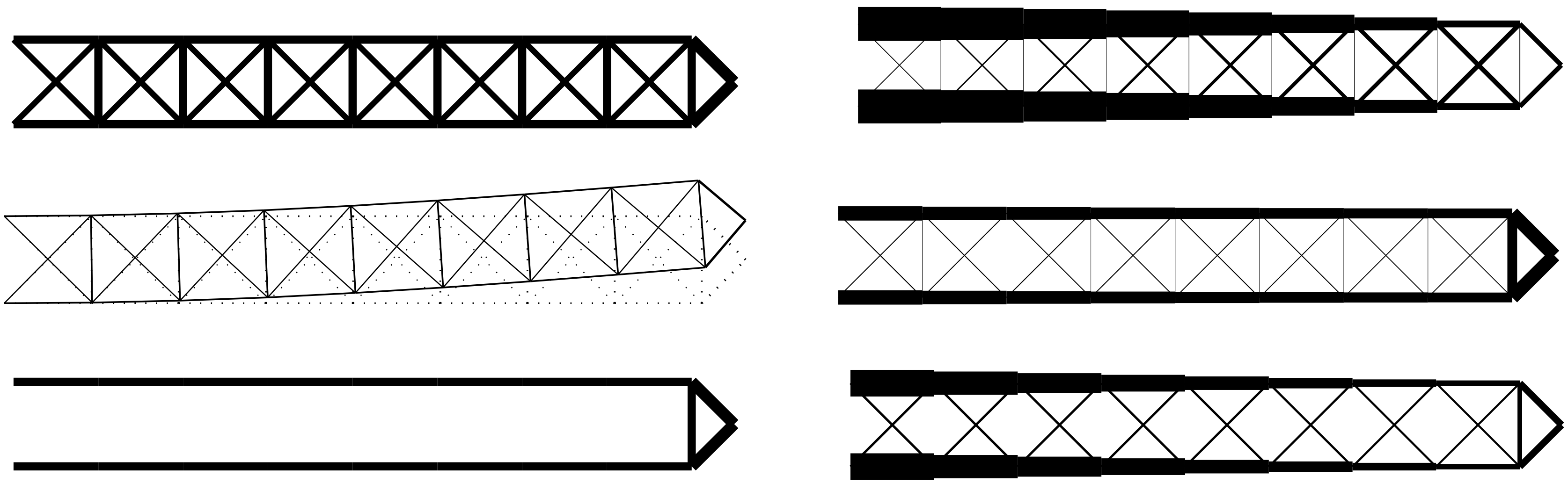
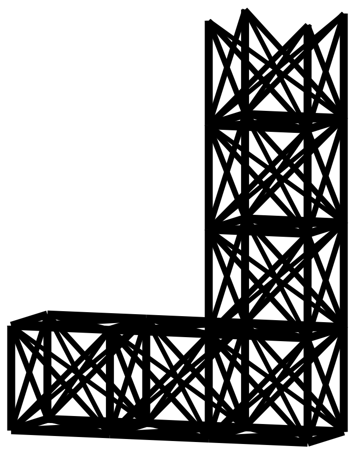
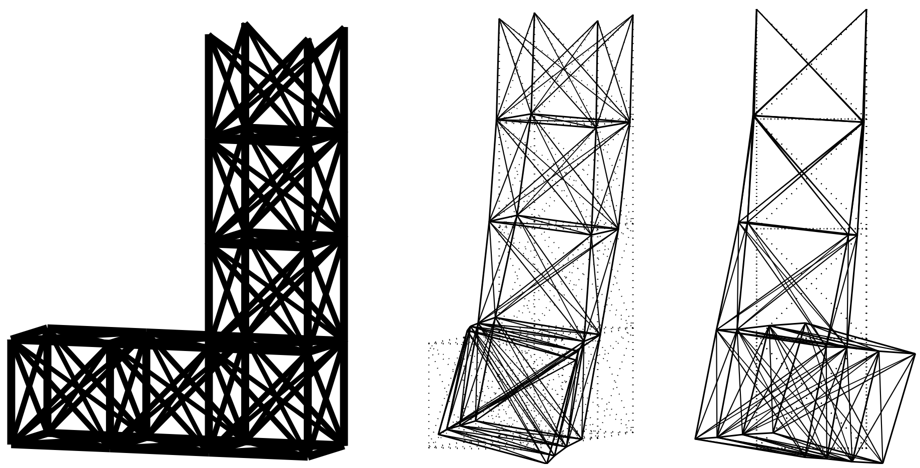


Truss design w. stability constraint



Truss design w. stability constraint



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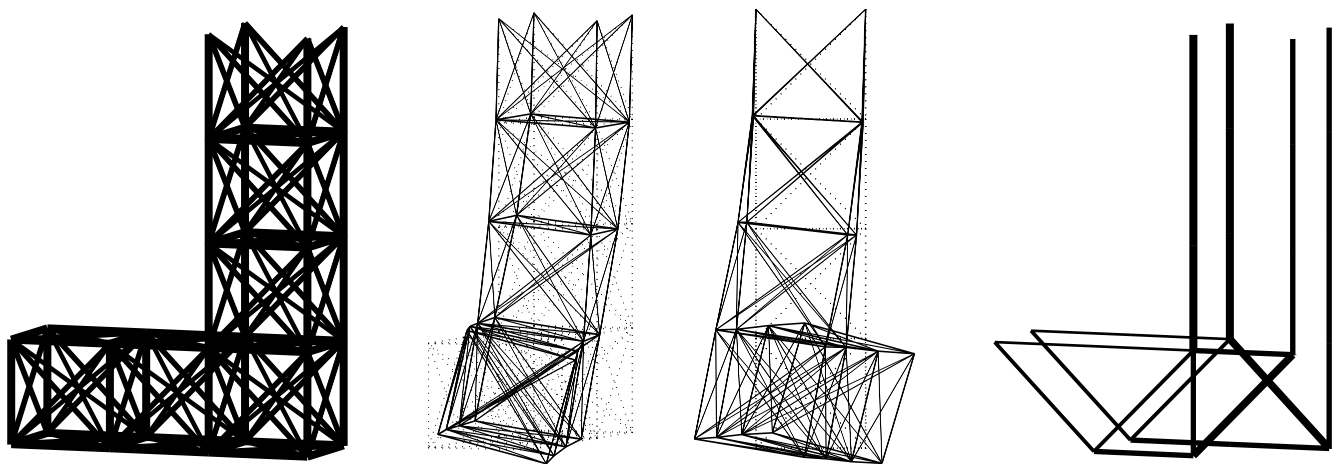

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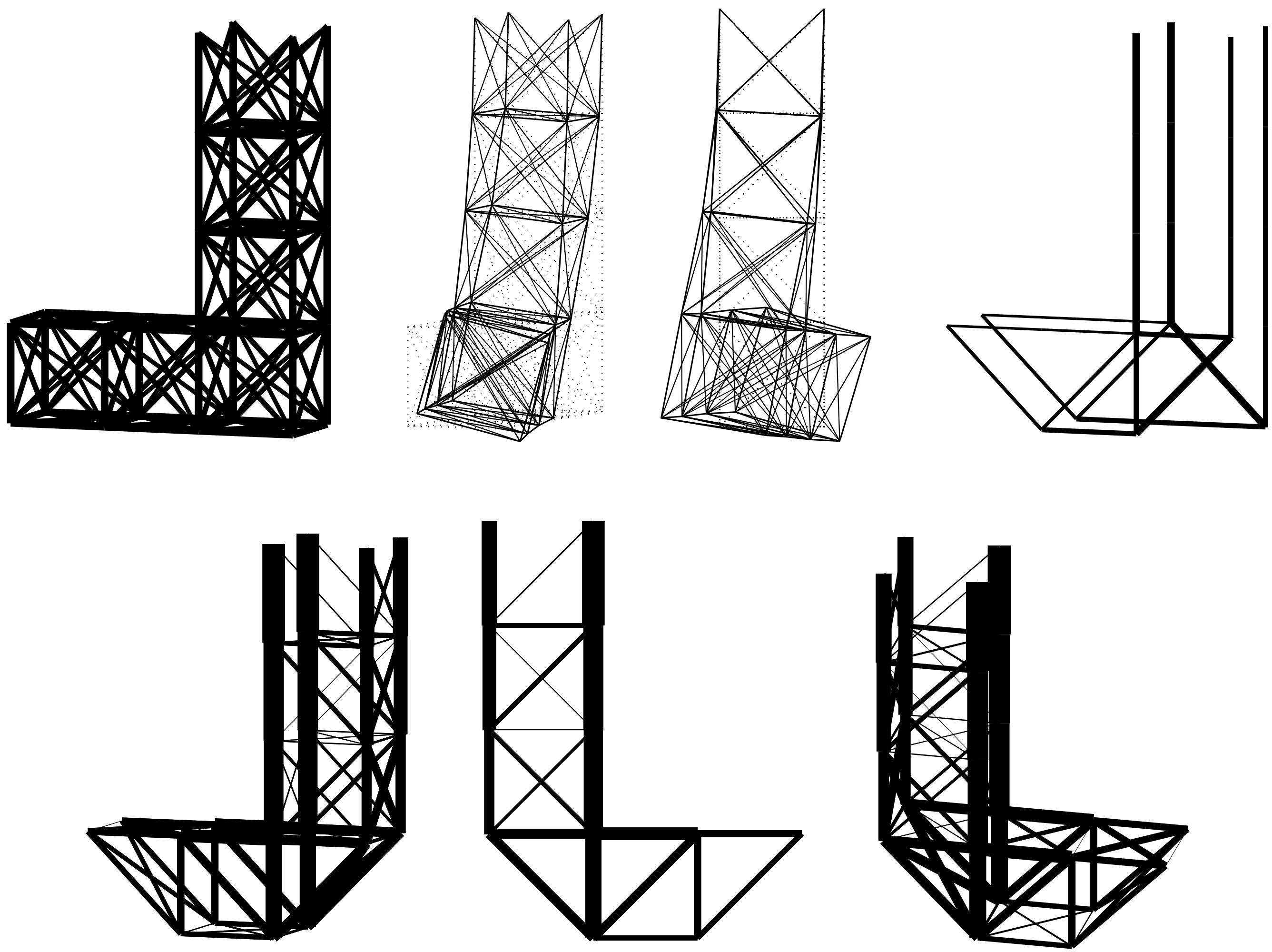
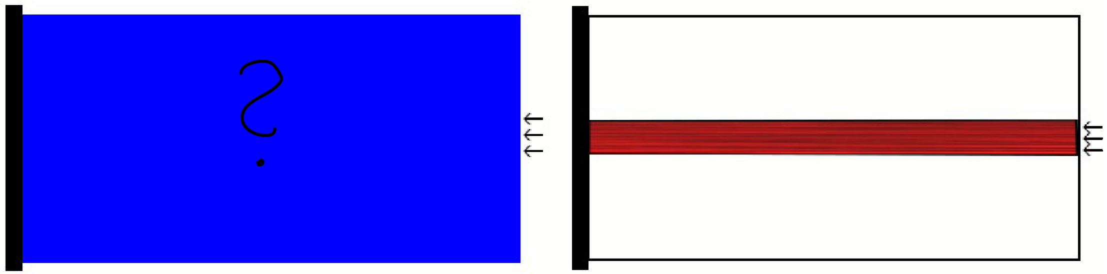


Examples, FMO w. vibration constraint



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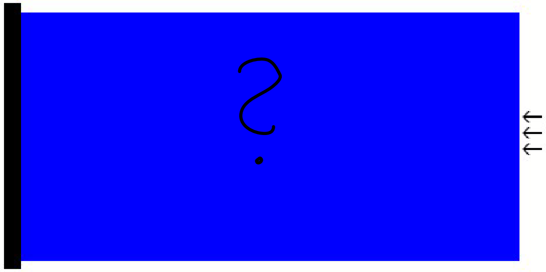

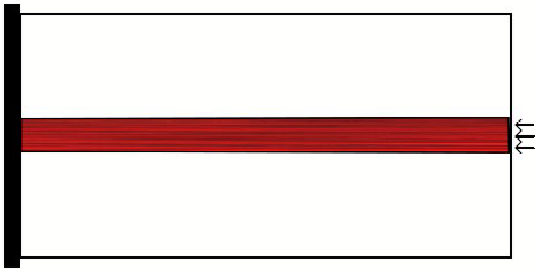

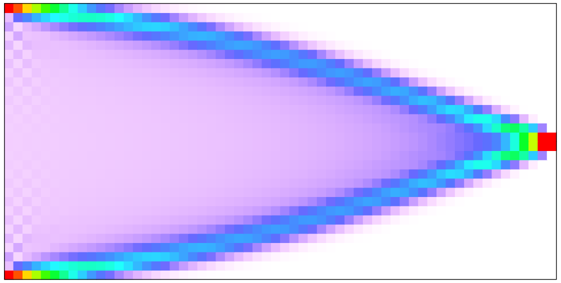

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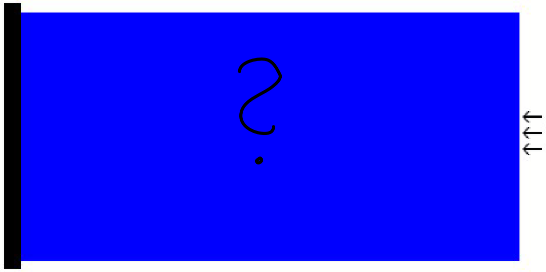

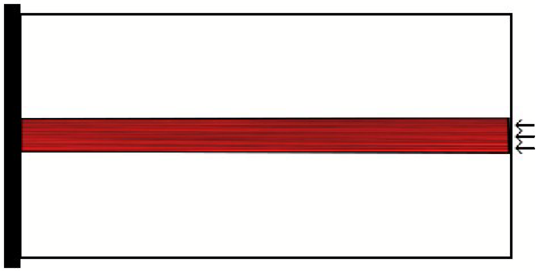

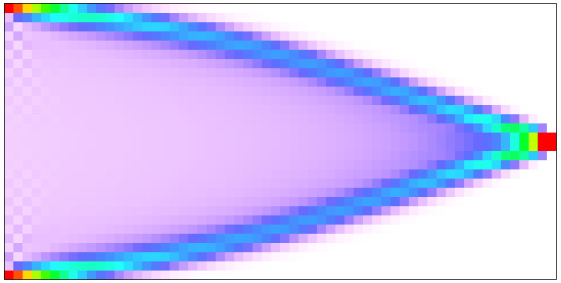

FMO with vibration constraint: BMI formulation