# *BMI problems in control theory*


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# Static output feedback

Given a linear system with $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}$

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

we want to stabilize it by static output feedback

$$u = Ky\,.$$

Find $K \in \mathbb{R}^{m\times p}$ s.t. the eigenvalues of the closed-loop system $A + BKC$ belong to the left half-plane (continuous-time stability).

Lyapunov stability theory: $A + BKC$ has all its eigenvalues in the open left half-plane if and only if there exists a matrix $P$ such that

$$(A + BKC)^T P + P(A + BKC) \prec 0, \quad P = P^T \succ 0$$

Bilinear matrix inequality.

# SOF $H_2$ problem

Let $h_i(t) = Ce^{At}B_i$ be the $i$-th column of the impulse response of the system $G$.

**SOF $H_2$ problem:** Find a SOF gain $K$ such that $A + BKC$ is Hurwitz and $\|G\|_2^2 = \sum \|h_i\|_2^2$ is minimal.

The dual problem (in $K, Q$):

$$\begin{aligned} & \min \text{Trace}(CQC^T) \quad (= \|G\|_2^2) \\ \text{s.t.} \quad & (A + BKC)Q + Q(A + BKC)^T + BB^T \preceq 0 \\ & Q \succ 0 \end{aligned}$$

# SOF $H_2$ problem

The dual problem (in $K, Q$):

$$\begin{aligned} & \min \text{Trace}(CQC^T) \quad (= \|G\|_2^2) \\ \text{s.t.} \quad & (A + BKC)Q + Q(A + BKC)^T + BB^T \preceq 0 \\ & Q \succ 0 \end{aligned}$$

equivalent to

$$\begin{aligned} & \min \text{Trace}(X) \\ \text{s.t.} \quad & (A + BKC)Q + Q(A + BKC)^T + BB^T \preceq 0 \\ & Q \succ 0 \\ & \begin{bmatrix} X & C \\ C^T & Q \end{bmatrix} \succeq 0, \end{aligned}$$

in variables $K \in \mathbb{R}^{m \times p}, Q \in \mathbb{R}^{n \times n}, X \in \mathbb{R}^{p \times p}$.

# SOF $H_\infty$ problem

Find a SOF matrix $F$ such that $A + BKC$ is Hurwitz and the $\|G\|_\infty$ is minimal.

BMI (in variables $K \in \mathbb{R}^{m \times p}, P \in \mathbb{R}^{n \times n}, \gamma \in \mathbb{R}$):

$$\min \gamma$$

$$\text{s.t.} \quad \begin{bmatrix} (A+BKC)^T P + P(A+BKC) & PB & C^T \\ B^T P & -\gamma\, I & D^T \\ C & D & -\gamma\, I \end{bmatrix} \prec 0$$

$$P \succ 0$$

$$\gamma > 0$$

# Solving SOF problems by PENBMI

| Ex. | CPU (sec) | $N$ | $M$ | $n$ | $p$ | $m$ | maximal real EV of $A(F)$ | $\mathcal{H}_2$-perf | prec |
|---|---|---|---|---|---|---|---|---|---|
| AC1 | 7.900e-01 | 27 | 17 | 5 | 3 | 3 | -2.6507e-01 | 1.0070e-03 | |
| AC6 | 3.262e+00 | 64 | 28 | 7 | 4 | 2 | -8.7223e-01 | 3.7982e+00 | |
| HE3 | 1.956e+01 | 115 | 34 | 8 | 6 | 4 | -1.4347e-01 | 8.1179e-01 | |
| HE7 | 5.135e+03 | 370 | 76 | 20 | 6 | 4 | -5.0000e-03 | 6.3718e+01 | A |
| REA3 | 2.629e+01 | 159 | 48 | 12 | 3 | 1 | -2.0658e-02 | 1.2087e+01 | |
| DIS1 | 6.456e+00 | 88 | 32 | 8 | 4 | 4 | -4.3379e-01 | 2.6601e+00 | a |
| TG1 | 1.071e+02 | 114 | 40 | 10 | 2 | 2 | -3.3982e-01 | 2.2312e+01 | a |
| AGS | 1.675e+02 | 160 | 48 | 12 | 2 | 2 | -2.0302e-01 | 6.9954e+00 | |
| IH | 3.760e+02 | 407 | 74 | 21 | 10 | 11 | -4.7990e-01 | 8.2603e-04 | |
| CSE1 | 5.096e+01 | 308 | 72 | 20 | 10 | 2 | -5.2977e-02 | 1.2083e-02 | |
| EB4 | 4.992e+02 | 214 | 62 | 20 | 1 | 1 | -1.7161e-05 | 5.0428e+02 | A |
| PSM | 2.571e+00 | 49 | 26 | 7 | 3 | 2 | -7.8437e-01 | 1.5043e+00 | |
| NN11 | 2.856e+02 | 157 | 51 | 16 | 5 | 3 | -3.4506e-01 | 1.1982e+02 | A |
| NN13 | 3.006e+00 | 31 | 21 | 6 | 2 | 2 | -2.3798e+00 | 2.6217e+01 | |
| NN16 | 4.657e+01 | 62 | 28 | 8 | 4 | 4 | -8.0625e-06 | 3.0732e-01 | A |

# SOF—How to avoid Lyapunov?

Recall SOF $H_2$ variables: $K \in \mathbb{R}^{m\times p}, Q \in \mathbb{R}^{n\times n}, X \in \mathbb{R}^{p\times p}$.

Typically: $n > p, m$, often $n \gg p, m$. Lyapunov variable dominates.
But it is just auxiliary. Can we get rid of it?
Yes, using polynomial formulation.

Let $k = \mathrm{vec}K, k \in \mathbb{R}^{mp}$. Define the characteristic polynomial of $A + BKC$:

$$q(s,k) = \det(sI - A - BKC) = \sum_{i=0}^{n} q_i(k)s^i$$

Here $q_i(k) = \sum_\alpha q_{i\alpha}k^\alpha$ and $\alpha \in \mathbb{N}^{mp}$ are all monomial powers.

Stability: all roots of $q(s,k)$ belong to the stability region $\mathcal{D}$.

# Hermite stability criterion

The roots of $q(s,k)$ belong to the stability region D iff

$$H(q) = \sum_{i=0}^{n}\sum_{j=0}^{n} q_i(k)q_j(k)H_{ij} \succ 0 \,.$$

Coefficients $H_{ij}$ depend only on the stability region $\mathcal{D}$.

E.g., for $n = 3$, $\mathcal{D}$ left half-plane:

$$H(q) = \begin{bmatrix} 2q_0q_1 & 0 & 2q_0q_3 \\ 0 & 2q_1q_2 - 2q_0q_3 & 0 \\ 2q_0q_3 & 0 & 2q_2q_3 \end{bmatrix}$$

In our case, $H(q) = H(k)$ depends polynomially on $k$:

$$H(k) = \sum_{\alpha} H_\alpha k^\alpha \succ 0$$

# Polynomial matrix inequality

**Lemma:** Problem SOF is solved iff $K$ solves the PMI

$$H(k) = \sum_{\alpha} H_\alpha k^\alpha \succ 0$$

where $H_\alpha = H_\alpha^T \in \mathbb{R}^{n \times n}$ and $\alpha \in \mathbb{N}^{mp}$ describes all monomial powers.

**Corollary:** If rank $B = 1$ (single input) or rank $C = 1$ (single output), then $\deg q_i(k) = 1$ and hence $\deg H(k) = 2$. PMI becomes BMI.

**Compare:**

Lyapunov variables: $K \in \mathbb{R}^{m \times p}, Q \in \mathbb{R}^{n \times n}$.
PMI variables: $k \in \mathbb{R}^{mp}$ only.

# Example

COMPLib, problem NN1:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 13 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 5 & -1 \\ -1 & -1 & 0 \end{bmatrix}, \; K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

The characteristic polynomial:

$$\begin{aligned} q(s,k) &= \det(sI - A - BKC) \\ &= k_2 + (-13 - 5k_1 + k_2)s + k_1 s^2 + s^3 \\ &= q_0(k) + q_1(k)s + q_2(k)s^2 + q_3(k)s^3 \end{aligned}$$

Hermite matrix (permuted):

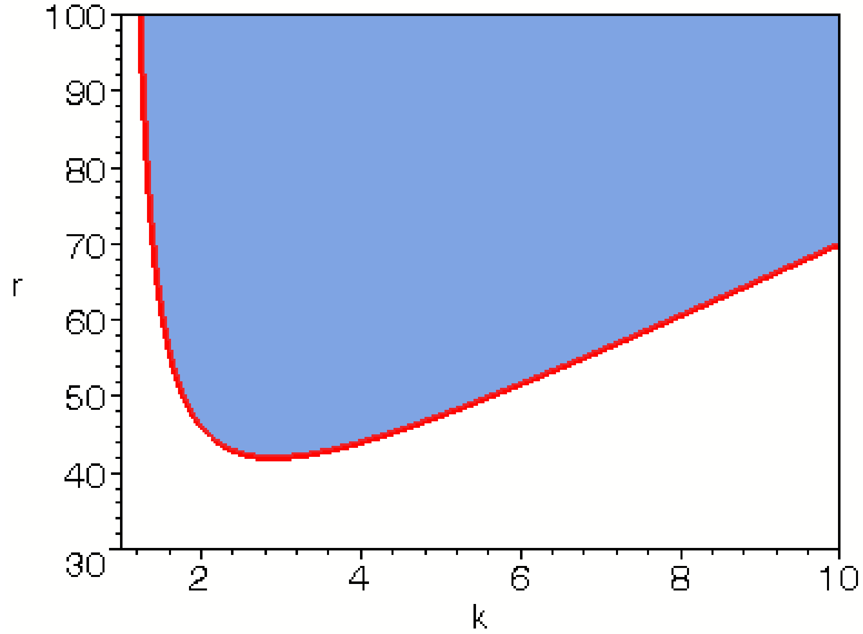
$$H(k) = \begin{bmatrix} q_0 q_1 & q_0 q_3 & 0 \\ q_0 q_3 & q_2 q_3 & 0 \\ 0 & 0 & q_1 q_2 - 2q_0 q_3 \end{bmatrix} = \begin{bmatrix} -13k_2 - 5k_1 k_2 + k_2^2 & k_2 & 0 \\ k_2 & k_1 & 0 \\ 0 & 0 & -13k_1 - k_2 - 5k_1^2 + k_1 k_2 \end{bmatrix}$$

# Example (cont.)

$$H(k) = \begin{bmatrix} -13k_2 - 5k_1k_2 + k_2^2 & k_2 & 0 \\ k_2 & k_1 & 0 \\ 0 & 0 & -13k_1 - k_2 - 5k_1^2 + k_1k_2 \end{bmatrix}$$

$H(k) \succ 0$ iff

$$k_1 > 0,\ -13 - 5k_1 + k_2 > 0,\ -13k_1 - k_2 - 5k_1^2 + k_1k_2 > 0.$$



# Simultaneous stabilization by BMI

$$P_i(s) = \frac{n_i(s)}{d_i(s)}, \quad i = 1, 2, \ldots, N$$

coprime polynomial fraction descriptions for $N$ linear plants.
We seek a controller

$$C(s) = \frac{x_n(s)}{x_d(s)}$$

of fixed order simultaneously stabilizing plants $P_i(s)$.

Given polynomials $n_i(s)$, $d_i(s)$ of degree $n_p$, find polynomials $x_n(s)$, $x_d(s)$ of given degree $n_x$ such that all the characteristic polynomials

$$p_i(s) = n_i(s)x_n(s) + d_i(s)x_d(s), \quad i = 1, 2, \ldots, N$$

of degree $n = n_p + n_x$ have their roots in some specified stability region $\mathcal{D}$.

# Hermite stability criterion

The roots of $p(s) = p_0 + p_1 s + p_2 s^2 + \ldots + p_n s^n$ belong to the region D iff

$$H(p) = \sum_{j=0}^{n} \sum_{k=0}^{n} p_j p_k H_{jk} \succ 0 \,.$$

Coefficients $H_{jk}$ depend only on the stability region $\mathcal{D}$.

The simultaneous stabilization problem of $N$ plants of order $n_p$ is solved with a controller of order $n_x$ iff the $2n_x + 1$ controller coefficients $x_k$ satisfy the $N$ BMIs of order $n = n_p + n_x$:

$$H(p_i) = \sum_{j=0}^{n} \sum_{k=0}^{n} p_j p_k H_{ijk} \succ 0 \,, \quad i = 1, 2, \ldots, N \,.$$

BMI feasibility problem

# BMI feasibility problem

$$\min_{x\in\mathbb{R}^n,\lambda\in\mathbb{R}} \lambda + w\|x\|_2^2$$

s.t.

$$-x_{\text{bound}} \leq x^k \leq x_{\text{bound}}, \qquad k = 1,\dots,n$$

$$A_0^i + \sum_{k=1}^n x_k A_k^i + \sum_{k=1}^n \sum_{\ell=1}^n x_k x_\ell K_{k\ell}^i \preccurlyeq \lambda I_{n\times n}, \qquad i = 1,\dots,N\,.$$

where $w$ is a “proper” weighting parameter (best results with $w = 0.0001$).

# SimStab problems

| problem | system order | contr. order | nb. of vertices | known feas. point |
|---|---|---|---|---|
| discrete | 3 | 2 | 1 | [200 100 50] |
| f4e | 3 | 0 | 4 | -0.8692 |
| helicopter | 3 | 2 | 4 | [1.865 2.061 1.992 4.335 10.50] |
| interval1_0 | 3 | 0 | 16 | 0.0380 |
| interval1_1 | 3 | 1 | 16 | [66.16 52.01 38.18] |
| interval2_0 | 2 | 0 | 8 | 226.5 |
| interval2_1 | 2 | 1 | 8 | [397.4 214.4 -135.8] |
| mass_spring | 1 | 4 | 2 | [0.2887 1.6761 -2.1434 3.0755 2.7278] |
| oblique_wing | 4 | 0 | 64 | 0.381 |
| parametric | 2 | 0 | 3 | 24.1489 |
| servo_motor | 2 | 1 | 4 | [1.300, 26.88, 5.439, 0] |
| toy1 | 3 | 1 | 1 | [0.5647 1.6138 1.5873] |
| toy2 | 3 | 1 | 1 | [-2.9633 -2.2693 1.2783] |

# SimStab initial guess

Initial points generated by the following heuristics.

For a system with three vertices, generate three initial points $\boldsymbol{x_1^0, x_2^0, x_3^0}$:

$\boldsymbol{x_i^0}$ – coefficient vector of a controller stabilizing vertex number $\boldsymbol{i}$

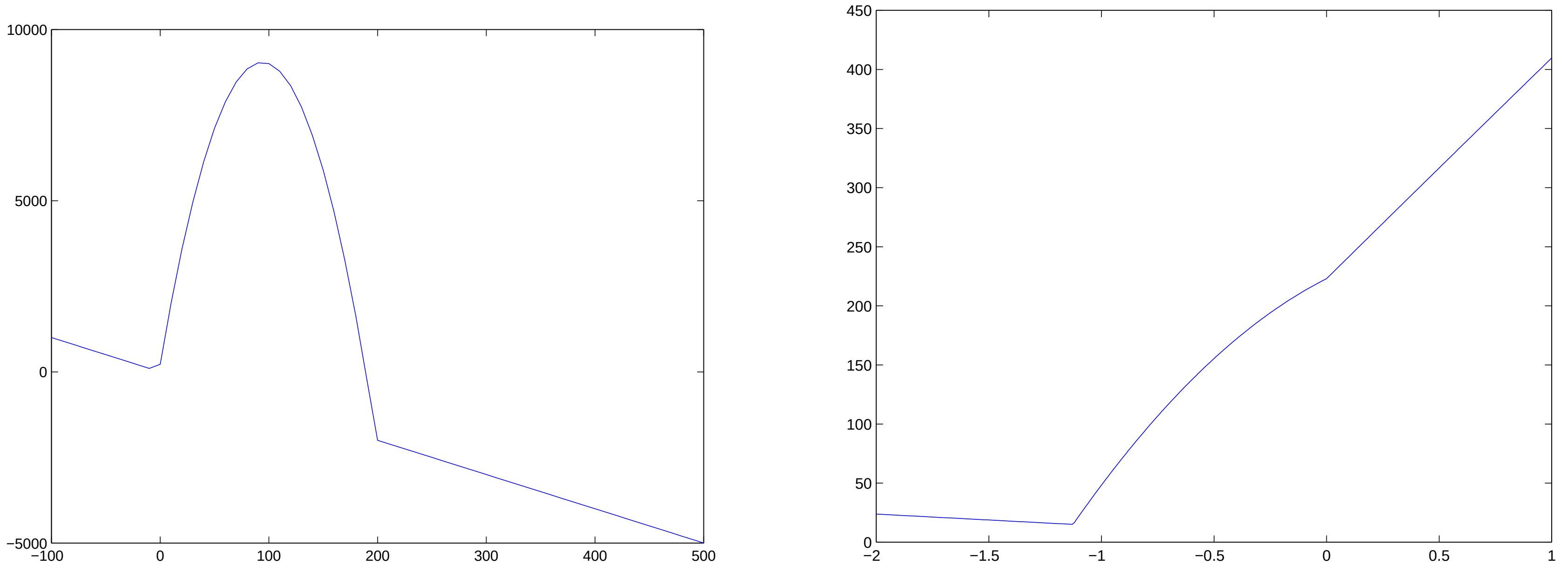
I.e., in $\mathbf{BMI}(\boldsymbol{i}),\ \boldsymbol{i = 1, 2, 3,}$ vector $\boldsymbol{x_i^0}$ is feasible for $\mathbf{BMI}(\boldsymbol{i})$, but not necessarily for $\mathbf{BMI}(\boldsymbol{j})$, $\boldsymbol{j \neq i}$.

Additionally, solve all problems with $\boldsymbol{x^0 = 0}$.

# PENBMI results

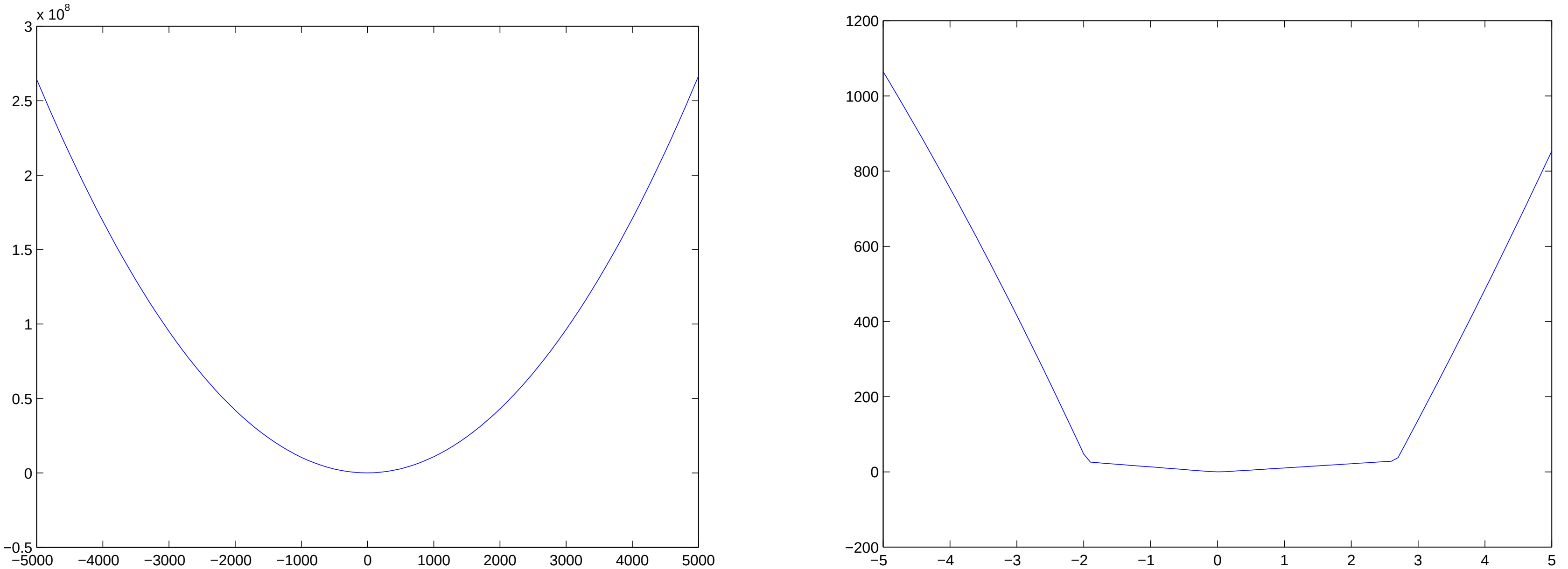
| problem | no. of init. pts | min. iter | max. iter | succ. cases | bound reached | unsucc. cases |
|---|---|---|---|---|---|---|
| discrete | 4 | 17 | 86 | 4 | 2 | 0 |
| f4e | 5 | 63 | 78 | 4 | 0 | 1 |
| helicopter | 5 | 85 | 107 | 3 | 0 | 2 |
| interval1_0 | 17 | 56 | 67 | 17 | 0 | 0 |
| interval1_1 | 17 | 66 | 104 | 16 | 0 | 1 |
| interval2_0 | 9 | 53 | 63 | 6 | 6 | 3 |
| interval2_1 | 9 | 63 | 131 | 4 | 4 | 5 |
| mass_spring | 2 | 348 | 397 | 2 | 0 | 0 |
| oblique_wing | 65 | 65 | 90 | 64 | 0 | 1 |
| parametric | 4 | 62 | 73 | 3 | 3 | 0 |
| patel11 | 4 | 3844 | 4311 | 0 | 0 | 4 |
| servo_motor | 5 | 84 | 179 | 5 | 5 | 0 |
| toy1 | 4 | 16 | 17 | 4 | 0 | 0 |
| toy2 | 4 | 66 | 243 | 4 | 3 | 0 |

# One-dimensional examples



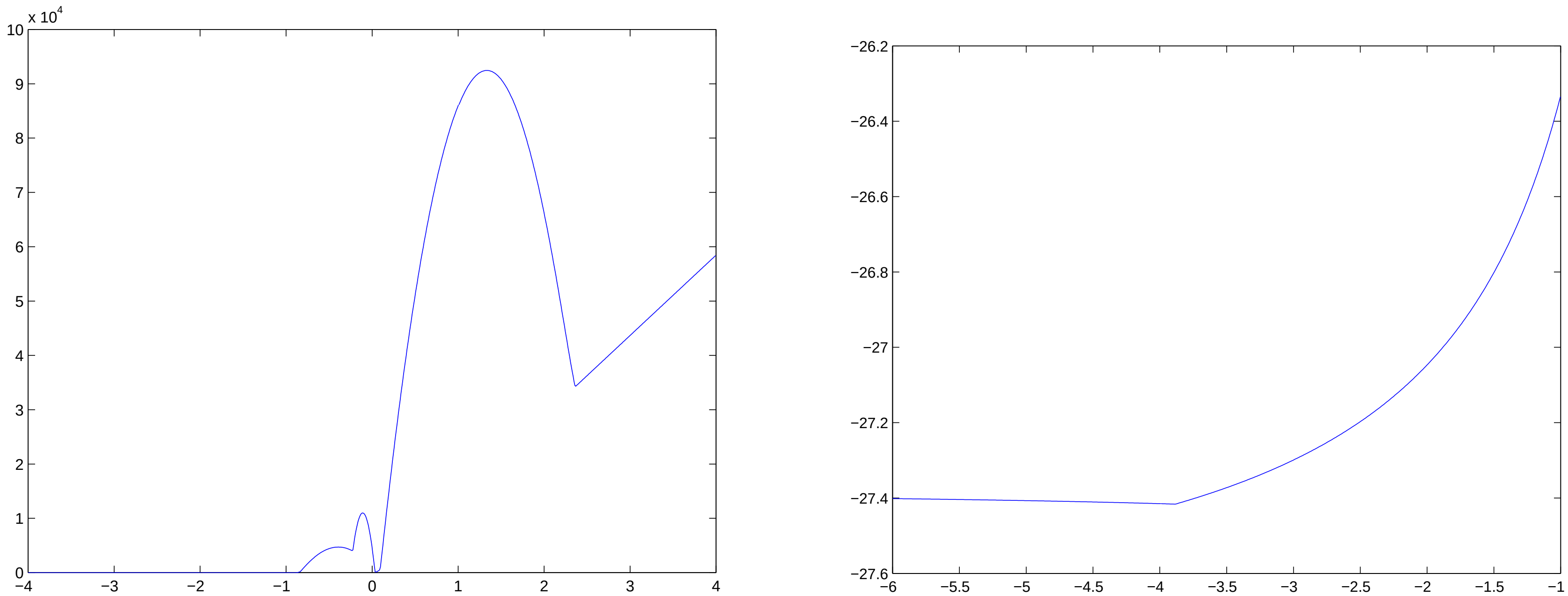

Problem `interval2_0`: global view (left) and detail of the local minimum (right). The $x$-axis is for the controller parameter, the $y$-axis for the maximum eigenvalue of the BMI.

# One-dimensional examples




Problem `interval1_0`: global view (left) and detail of the global minimum (right).

# One-dimensional examples




Problem `f4e`: global view (left) and detail of the global minimum (right).