COURSE ON LMI OPTIMIZATION
WITH APPLICATIONS IN CONTROL
PART II.3

LMIs IN SYSTEMS CONTROL
ROBUSTNESS ANALYSIS
POLYNOMIAL METHODS

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Vertical and diagonal Planes (1913-14)
František Kupka (1871-1957)

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Polynomial methods

Based on the algebra of polynomials and polynomial matrices, typically involve

- linear Diophantine equations
- quadratic spectral factorization

Pioneered in central Europe during the 70s mainly by Vladimír Kučera from the former Czechoslovak Academy of Sciences

Network funded by the European commission

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Polynomial matrices also occur in Jan Willems' behavorial approach to systems theory

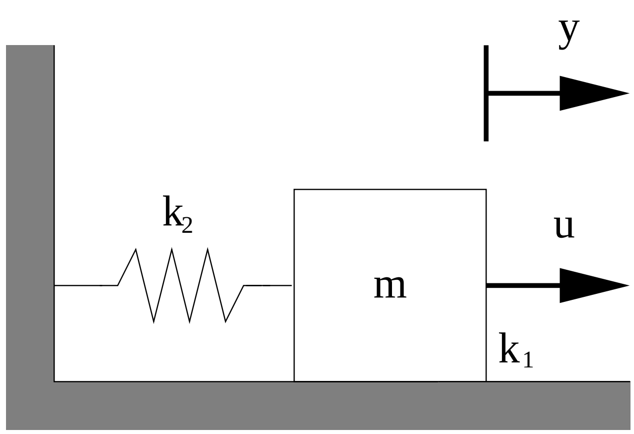
Alternative to state-space methods developed during the 60s most notably by Rudolf Kalman in the USA, rather based on

- linear Lyapunov equations
- quadratic Riccati equations

Ratio of polynomials

A scalar transfer function can be viewed as the ratio of two polynomials

Example
Consider the mechanical system



- y displacement
- u external force
- k_1 viscous friction coeff
- k_2 spring constant
- m mass

Neglecting static and Coloumb frictions, we obtain the linear transfer function

$$G(s) = \frac{y(s)}{u(s)} = \frac{1}{ms^2 + k_1 s + k_2}$$

Ratio of polynomial matrices

Similarly, a MIMO transfer function can be viewed as the ratio of polynomial matrices

$$G(s) = N_R(s)D_R^{-1}(s) = D_L^{-1}(s)N_L(s)$$

the so-called matrix fraction description (MFD)

Lightly damped structures such as oil derricks, regional power models, earthquakes models, mechanical multi-body systems, damped gyroscopic systems are most naturally represented by second order polynomial MFDs

$$(D_0 + D_1 s + D_2 s^2)y(s) = N_0 u(s)$$

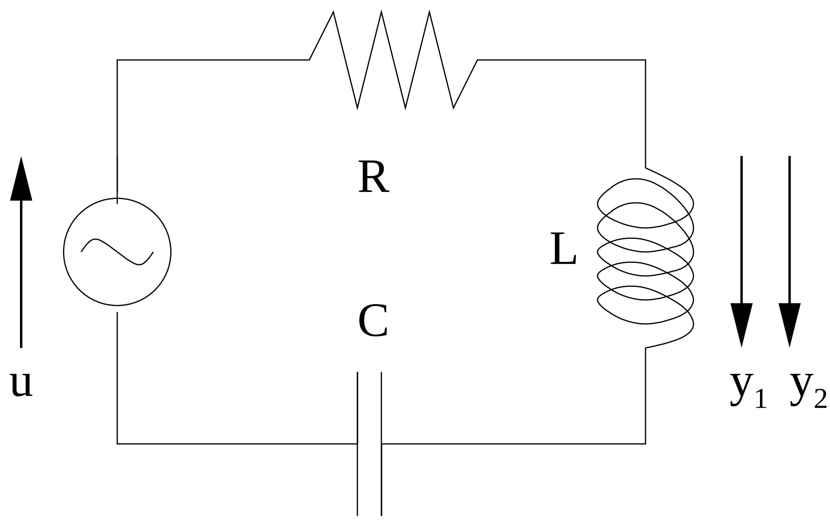
Example
The (simplified) oscillations of a wing in an air stream is captured by properties of the quadratic polynomial matrix [Lancaster 1966]

$$\begin{aligned} D(s) \;=\; & \left[\begin{array}{ccc} 121 & 18.9 & 15.9 \\ 0 & 2.7 & 0.145 \\ 11.9 & 3.64 & 15.5 \end{array}\right] + \left[\begin{array}{ccc} 7.66 & 2.45 & 2.1 \\ 0.23 & 1.04 & 0.223 \\ 0.6 & 0.756 & 0.658 \end{array}\right] s + \\ & \left[\begin{array}{ccc} 17.6 & 1.28 & 2.89 \\ 1.28 & 0.824 & 0.413 \\ 2.89 & 0.413 & 0.725 \end{array}\right] s^2 \end{aligned}$$

First-order polynomial MFD

Example

RCL network



- y_1 voltage through inductor
- y_2 current through inductor
- u voltage

Applying Kirchoff's laws and Laplace transform we get

$$\begin{bmatrix} 1 & -Ls \\ Cs & 1+RCs \end{bmatrix} \begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} 0 \\ Cs \end{bmatrix} u(s)$$

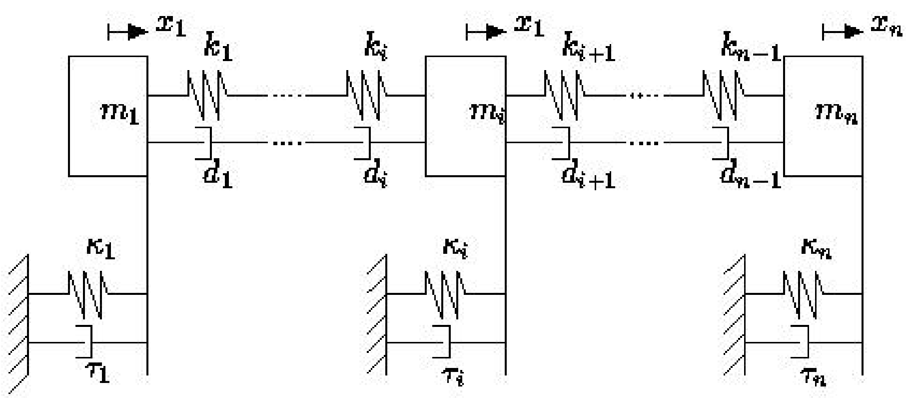
and thus the first-order left system MFD

$$G(s) = \begin{bmatrix} 1 & -Ls \\ Cs & 1+RCs \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ Cs \end{bmatrix}.$$

Second-order polynomial MFD

Example

mass-spring system



Vibration of system governed by 2nd-order differential equation $M\ddot{x} + C\dot{x} + Kx = 0$ where e.g. $n = 250$, $m_i = 1, \kappa_i = 5, \tau_i = 10$ except $\kappa_1 = \kappa_n = 10$ and $\tau_1 = \tau_n = 20$

Quadratic matrix polynomial

$$D(s) = Ms^2 + Cs + K$$

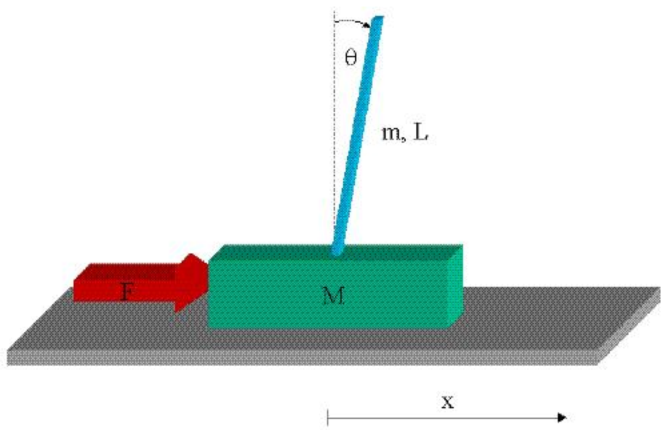
with

$$M = I$$
$$C = \text{tridiag}(-10, 30, -10)$$
$$K = \text{tridiag}(-5, 15, -5).$$

Another second-order polynomial MFD

Example

Inverted pendulum on a cart



Linearization around the upper vertical position yields the left polynomial MFD

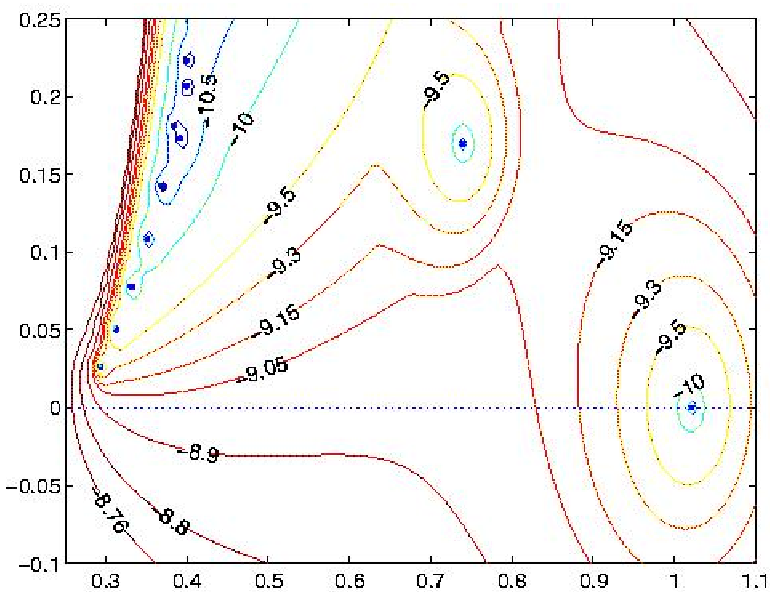
$$\begin{bmatrix} (M+m)s^2+bs & lms^2 \\ lms^2 & (J+l^2m)s^2+ks-lmg \end{bmatrix} \begin{bmatrix} x(s) \\ \phi(s) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} f(s)$$

With $J = mL^2/12$, $l = L/2$ and $g = 9.8$, $M = 2$, $m = 0.35$, $l = 0.7$, $b = 4$, $k = 1$, we obtain the denominator polynomial matrix

$$D(s) = \begin{bmatrix} 5s+3s^2 & 0.35s^2 \\ 0.35s^2 & -3.4+s+0.16s^2 \end{bmatrix}$$

More examples of polynomial MFDs

Higher degree polynomial matrices can also be found in aero-acoustics (3rd degree) or in the study of the spatial stability of the Orr-Sommerfeld equation for plane Poiseuille flow in fluid mechanics (4rd degree)



Pseudospectra of Orr-Sommerfeld equation

For more info see Nick Higham's homepage at www.ma.man.ac.uk/∼higham

Stability analysis for polynomials

Well established theory - LMIs are of no use here !

Given a continuous-time polynomial

$$p(s) = p0 + p1s + \cdots + p_{n-1}s^{n-1} + p_n s^n$$

with $p_n > 0$ we define its $n \times n$ Hurwitz matrix

$$H(p) = \begin{bmatrix} p_{n-1} & p_{n-3} & & 0 & 0 \\ p_n & p_{n-2} & & \vdots & \vdots \\ 0 & p_{n-1} & \ddots & 0 & 0 \\ 0 & p_n & & p_0 & 0 \\ \vdots & \vdots & & p_1 & 0 \\ 0 & 0 & & p_2 & p_0 \end{bmatrix}$$

Hurwitz stability criterion: Polynomial $p(s)$ is stable iff all principal minors of $H(p)$ are > 0

Adolf Hurwitz
(Hanover 1859 - Zürich 1919)

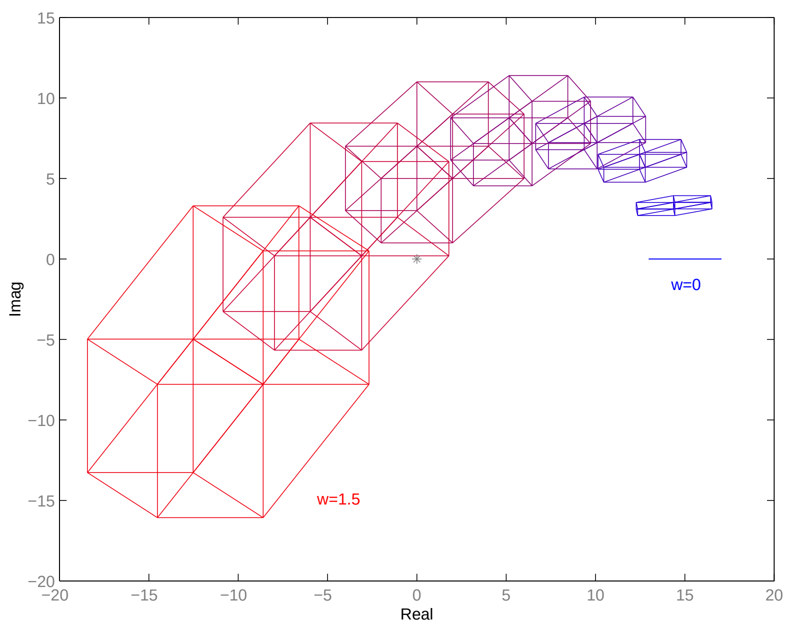
Robust stability analysis for polynomials

Analyzing stability robustness of polynomials is a little bit more interesting..

Here too computational complexity depends on the uncertainty model

In increasing order of complexity, we will distinguish between
- single parameter uncertainty $q \in [q_{\min}, q_{\max}]$
- interval uncertainty $q_i \in [q_{i\min}, q_{i\max}]$
- polytopic uncertainty $\lambda_1 q_1 + \cdots + \lambda_N q_N$
- multilinear uncertainty $q_0 + q_1 \cdot q_2 \cdot q_3$

LMIs will not show up very soon..
..just basic linear algebra



Graphic stability analysis of an interval plant controlled with a first-order compensator

Single parameter uncertainty and eigenvalue criterion

Consider the uncertain polynomial

$$p(s,q) = p_0(s) + q p_1(s)$$

where
- $p_0(s)$ nominally stable with positive coefs
- $p_1(s)$ such that $\deg p_1(s) < \deg p_0(s)$

The largest stability interval

$$q \in]q_{\max}, q_{\max}[$$

such that $p(s,q)$ is robustly stable is given by

$$\begin{aligned} q_{\max} &= 1/\lambda^{+}_{\max}(-H_0^{-1}H_1) \\ q_{\min} &= 1/\lambda^{-}_{\min}(-H_0^{-1}H_1) \end{aligned}$$

where $\lambda^{+}_{\max}$ is the max positive real eigenvalue
$\lambda^{-}_{\min}$ is the min negative real eigenvalue
H_i is the Hurwitz matrix of $p_i(s)$

Higher powers of a single parameter

Now consider the continuous-time polynomial

$$p(s,q) = p_0(s) + qp_1(s) + q^2 p_2(s) + \cdots + q^m p_m(s)$$

with $p_0(s)$ stable and $\deg p_0(s) > \deg p_i(s)$

Using the zeros (roots of determinant) of the polynomial Hurwitz matrix

$$H(p) = H(p_0) + qH(p_1) + q^2 H(p_2) + \cdots + q^m H(p_m)$$

we can show that

$$\begin{array}{rcl} q_{\min} & = & 1/\lambda^-_{\min}(M) \\ q_{\max} & = & 1/\lambda^+_{\max}(M) \end{array}$$

where

$$M = \begin{bmatrix} 0 & & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & & I & 0 \\ 0 & & 0 & I \\ -H_0^{-1}H_m & \cdots & -H_0^{-1}H_2 & -H_0^{-1}H_1 \end{bmatrix}$$

is a block companion matrix

MIMO systems

Uncertain multivariable systems are modeled by uncertain polynomial matrices

$$P(s,q) = P_0(s) + qP_1(s) + q^2P_2(s) + \cdots + q^mP_m(s)$$

where $p_0(s) = \det P_0(s)$ is a stable polynomial

We can apply the scalar procedure to the determinant polynomial

$$\det P(s,q) = p_0(s) + qp_1(s) + q^2p_2(s) + \cdots + q^rp_r(s)$$

Example

MIMO design on the plant with left MFD

$$A^{-1}(s,q)B(s,q) = \begin{bmatrix} s^2 & q \\ q^2+1 & s \end{bmatrix}^{-1} \begin{bmatrix} s+1 & 0 \\ q & 1 \end{bmatrix}$$
$$= \frac{\begin{bmatrix} s^2+s-q^2 & -q \\ qs^2-(q^2+1)s-(q^2+1) & s^2 \end{bmatrix}}{s^3-q^2-q}$$

with uncertain parameter $q \in [0,1]$

MIMO systems: example

Using some design method, we obtain a controller with right MFD

$$Y(s)X^{-1}(s) = \begin{bmatrix} 94-51s & -18+17s \\ -55 & 100 \end{bmatrix} \begin{bmatrix} 55+s & -17 \\ -1 & 18+s \end{bmatrix}$$

Closed-loop system with characteristic denominator polynomial matrix

$$\begin{aligned} D(s,q) &= A(s,q)X(s) + B(s,q)Y(s) \\ &= D_0(s) + qD_1(s) + q^2D_2(s) \end{aligned}$$

Nominal system poles: roots of $\det D_0(s)$

Applying the eigenvalue criterion on $\det D(s,q)$ yields the stability interval

$$q \in]-0.93,\ 1.17[\ \supset\ [0,\ 1]$$

so the closed-loop system is robustly stable

Independent uncertainty

So far we have studied polynomials affected by a single uncertain parameter

$$p(s, q) = (6 + q) + (4 + q)s + (2 + q)s^2$$

However in practice several parameters can be uncertain, such as in

$$p(s, q) = (6 + q_0) + (4 + q_1)s + (2 + q_2)s^2$$

Independent uncertainty structure: each component q_i enters into only one coefficient

Interval uncertainty: independent structure and uncertain parameter vector q belongs to a given box, i.e. $q_i \in [q_i^-, q_i^+]$

Example

Uncertain polynomial

$$(6 + q_0) + (4 + q_1)s + (2 + q_2)s^2, \quad |q_i| \leq 1$$

has interval uncertainty, also denoted as

$$[5, 7] + [3, 5]s + [1, 3]s^2$$

Some coefficients can be fixed, e.g.

$$6 + [3, 5]s + 2s^2$$

Kharitonov's polynomials

Associated with the interval polynomial

$$p(s,q) = \sum_{i=0}^{n} [q_i^-,\ q_i^+] s^i$$

are four Kharitonov's polynomials

$$\begin{aligned}
p^{--}(s) &= q_0^- + q_1^- s + q_2^+ s^2 + q_3^+ s^3 + q_4^- s^4 + q_5^- s^5 + \cdots \\
p^{-+}(s) &= q_0^- + q_1^+ s + q_2^+ s^2 + q_3^- s^3 + q_4^- s^4 + q_5^+ s^5 + \cdots \\
p^{+-}(s) &= q_0^+ + q_1^- s + q_2^- s^2 + q_3^+ s^3 + q_4^+ s^4 + q_5^- s^5 + \cdots \\
p^{++}(s) &= q_0^+ + q_1^+ s + q_2^- s^2 + q_3^- s^3 + q_4^+ s^4 + q_5^+ s^5 + \cdots
\end{aligned}$$

where we assume $q_n^- > 0$ and $q_n^+ > 0$

Example
Interval polynomial

$$p(s,q) = [1,2] + [3,4]s + [5,6]s^2 + [7,8]s^3$$

Kharitonov's polynomials

$$\begin{aligned}
p^{--}(s) &= 1 + 3s + 6s^2 + 8s^3 \\
p^{-+}(s) &= 1 + 4s + 6s^2 + 7s^3 \\
p^{+-}(s) &= 2 + 3s + 5s^2 + 8s^3 \\
p^{++}(s) &= 2 + 4s + 5s^2 + 7s^3
\end{aligned}$$

Kharitonov's theorem

In 1978 the Russian researcher Vladimír Kharitonov proved the following fundamental result

> A continuous-time interval polynomial
> is robustly stable iff its
> four Kharitonov polynomials are stable

Instead of checking stability of an infinite number of polynomials we just have to check stability of four polynomials, which can be done using the classical Hurwitz criterion

Peter and Paul fortress in St Petersburg

Affine uncertainty

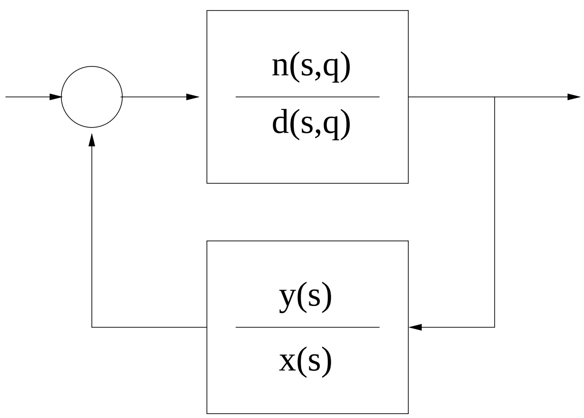
Sadly, Kharitonov's theorem is valid only
- for continuous-time polynomials
- for independent interval uncertainty

so that we have to use more general tools in practice

When coefficients of an uncertain polynomial $p(s, q)$ or a rational function $n(s, q)/d(s, q)$ depend affinely on parameter q, such as in

$$a^T q + b$$

we speak about affine uncertainty



The above feedback interconnection

$$\frac{n(s, q)x(s)}{d(s, q)x(s) + n(s, q)y(s)}$$

preserves the affine uncertainty structure of the plant

Polytopes of polynomials

A family of polynomials $p(s, q)$, $q \in Q$ is said to be a polytope of polynomials if

- $p(s, q)$ has an affine uncertainty structure
- Q is a polytope

There is a natural isomorphism between a polytope of polynomials and its set of coefficients

Example

$p(s, q) = (2q_1 - q_2 + 5) + (4q_1 + 3q_2 + 2)s + s^2$, $|q_i| \leq 1$

Uncertainty polytope has 4 generating vertices

$$\begin{array}{rclrcl} q^1 & = & [-1,\ -1] & q^2 & = & [-1,\ 1] \\ q^3 & = & [1,\ -1] & q^4 & = & [1,\ 1] \end{array}$$

Uncertain polynomial family has 4 generating vertices

$$\begin{array}{rclrcl} p(s, q^1) & = & 4 - 5s + s^2 & p(s, q^2) & = & 2 + s + s^2 \\ p(s, q^3) & = & 8 + 3s + s^2 & p(s, q^4) & = & 6 + 9s + s^2 \end{array}$$

Any polynomial in the family can be written as

$$p(s, q) = \sum_{i=1}^{4} \lambda_i p(s, q^i),\ \sum_{i=1}^{4} \lambda_i = 1,\ \lambda_i \geq 0$$

Interval polynomials

Interval polynomials are a special case of polytopic polynomials

$$p(s,q) = \sum_{i=0}^{n} [q_i^-,\ q_i^+] s^i$$

with at most 2^{n+1} generating vertices

$$p(s,q^k) = \sum_{i=0}^{n} q_i^k s^i, \quad q_i^k = \begin{cases} q_i^- \\ \text{or} \\ q_i^+ \end{cases} \quad 1 \leq k \leq 2^{n+1}$$

Example

The interval polynomial

$$p(s,q) = [5,6] + [3,4]s + 5s^2 + [7,8]s^3 + s^4$$

can be generated by the $2^3 = 8$ vertex polynomials

$$\begin{array}{rcl}
p(s,q^1) & = & 5 + 3s + 5s^2 + 7s^3 + s^4 \\
p(s,q^2) & = & 6 + 3s + 5s^2 + 7s^3 + s^4 \\
p(s,q^3) & = & 5 + 4s + 5s^2 + 7s^3 + s^4 \\
p(s,q^4) & = & 6 + 4s + 5s^2 + 7s^3 + s^4 \\
p(s,q^5) & = & 5 + 3s + 5s^2 + 8s^3 + s^4 \\
p(s,q^6) & = & 6 + 3s + 5s^2 + 8s^3 + s^4 \\
p(s,q^7) & = & 5 + 4s + 5s^2 + 8s^3 + s^4 \\
p(s,q^8) & = & 6 + 4s + 5s^2 + 8s^3 + s^4
\end{array}$$

The edge theorem

Let $p(s,q)$, $q \in Q$ be a polynomial with invariant degree over polytopic set Q

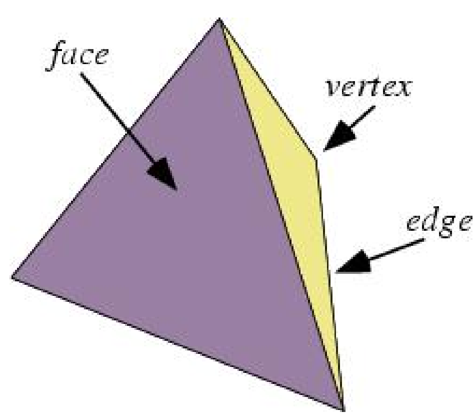
> Polynomial $p(s,q)$ is robustly stable over the whole uncertainty polytope Q iff $p(s,q)$ is stable along the edges of Q

In other words, it is enough to check robust stability of the single parameter polynomial

$$\lambda p(s, q^{i_1}) + (1-\lambda) p(s, q^{i_2}), \quad \lambda \in [0,\ 1]$$

for each pair of vertices q^{i_1} and q^{i_2} of Q

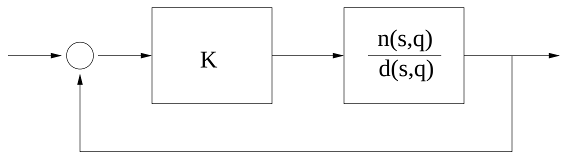
This can be done with the eigenvalue criterion



Interval feedback system

Example

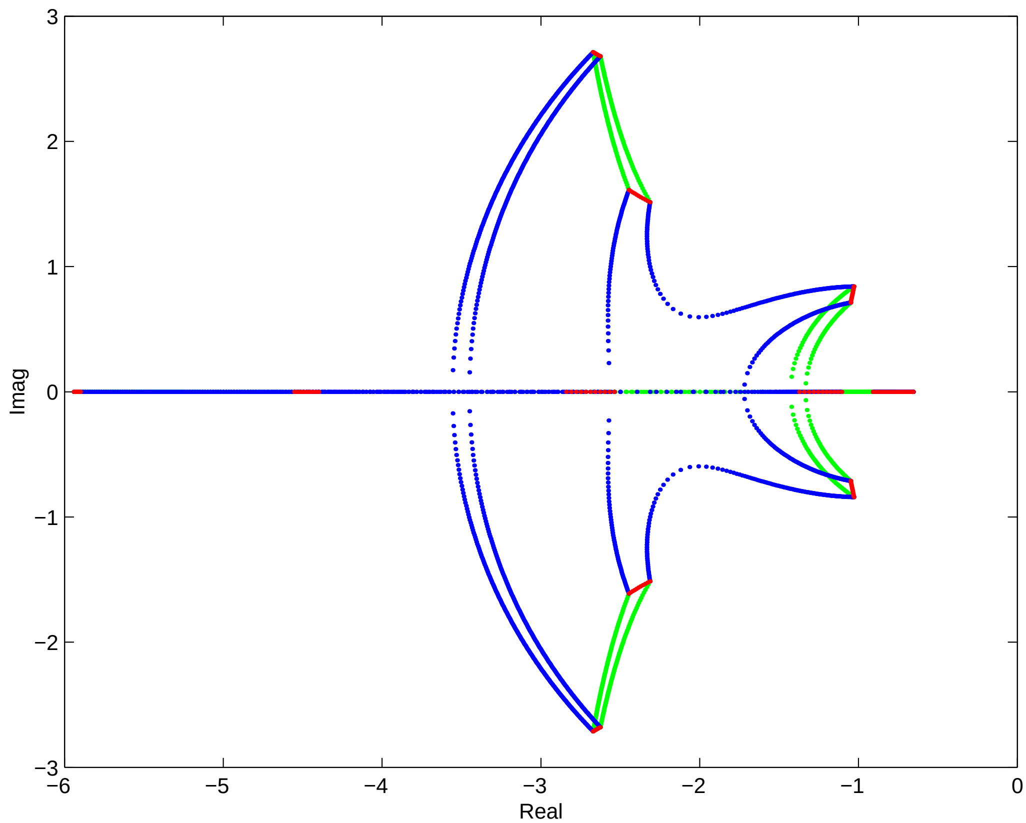
We consider the interval control system



with $n(s,q) = [6,\ 8]s^2 + [9.5,\ 10.5]$, $d(s,q) = s(s^2 + [14,\ 18])$ and characteristic polynomial

$$K[9.5,\ 10.5] + [14,\ 18]s + K[6,\ 8]s^2 + s^3$$

For $K = 1$ we draw the 12 edges of its root set



The closed-loop system is robustly stable

More about uncertainty structure

In typical applications, uncertainty structure is more complicated than interval or affine

Usually, uncertainty enters highly non-linearly in the closed-loop characteristic polynomial

We distinguish between

- multilinear uncertainty, when each uncertain parameter q_i is linear when other parameters q_j, $i \neq j$ are fixed
- polynomic uncertainty, when coefficients are multivariable polynomials in parameters q_i

We can define the following hierarchy on the uncertainty structures

$$\text{interval} \subset \text{affine} \subset \text{multilinear} \subset \text{polynomic}$$

Examples of uncertainty structures

Examples

The uncertain polynomial

$$(5q_1 - q_2 + 5) + (4q_1 + q_2 + q_3)s + s^2$$

has affine uncertainty structure

The uncertain polynomial

$$(5q_1 - q_2 + 5) + (4q_1q_3 - 6q_1q_3 + q_3)s + s^2$$

has multilinear uncertainty structure

The uncertain polynomial

$$(5q_1 - q_2 + 5) + (4q_1 - 6q_1 - q_3^2)s + s^2$$

has polynomic (here quadratic) uncertainty structure

The uncertain polynomial

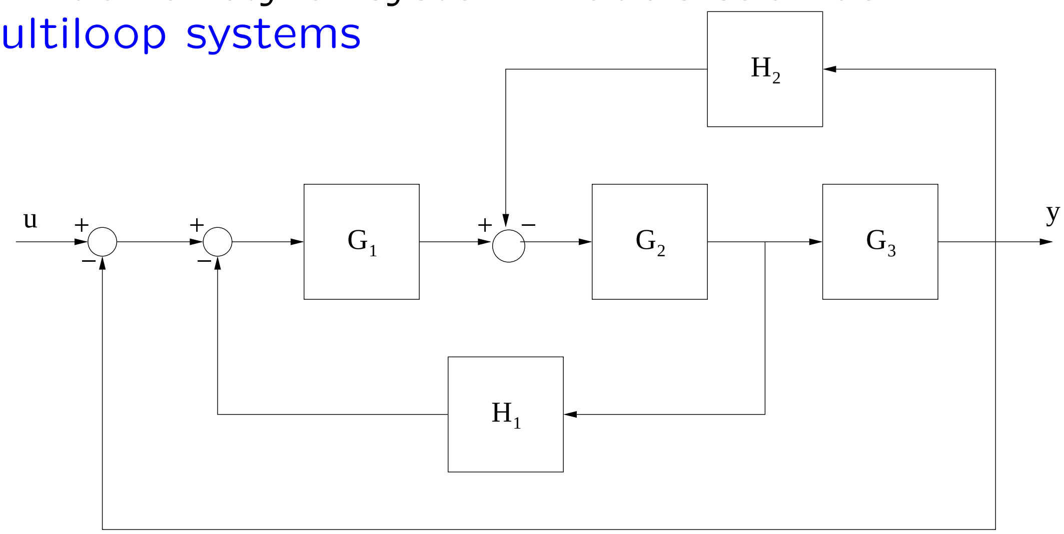
$$(5q_1 - q_2 + 5) + (4q_1 - 6q_1q_3^2 + q_3)s + s^2$$

has polynomic uncertainty structure

Multilinear uncertainty

We will focus on multilinear uncertainty because it arises in a wide variety of system models such as:

- multiloop systems



Closed-loop transfer function

$$\frac{y}{u} = \frac{G_1G_2G_3}{1 + G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3}$$

- state-space models with rank-one uncertainty

$$\dot{x} = A(q)x,\ A(q) = \sum_{i=1}^{n} q_i A_i,\ \text{rank } A_i = 1$$

and characteristic polynomial

$$p(s,q) = \det(sI - A(q))$$

- polynomial MFDs with MIMO interval uncertainty

$$G(s) = A^{-1}(s,q)B(s,q), \quad C(s) = Y(s)X^{-1}(s)$$

and closed-loop characteristic polynomial

$$p(s,q) = \det(A(s,q)X(s) + B(s,q)Y(s))$$

Robust stability analysis for multilinear and polynomic uncertainty

Unfortunately, there is no systematic computational tractable necessary and sufficient robust stability condition

On the one hand, sufficient condition through polynomial value sets, the zero exclusion condition and the mapping theorem

On the other hand, brute-force method: intensive parameter gridding, NP-hard in general

No easy trade-off between computational complexity and conservatism

Polynomial stability analysis: summary

Checking robust stability can be
- easy (polynomial-time algorithms) or more
- difficult (NP-hard problem)

depending namely on the uncertainty model

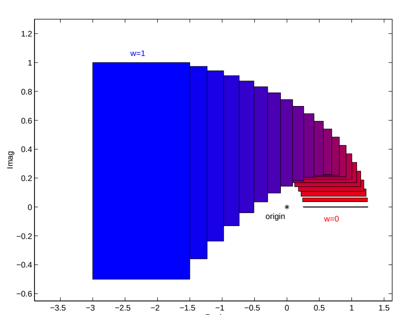
We focused on polytopic uncertainty:
- Interval scalar polynomials

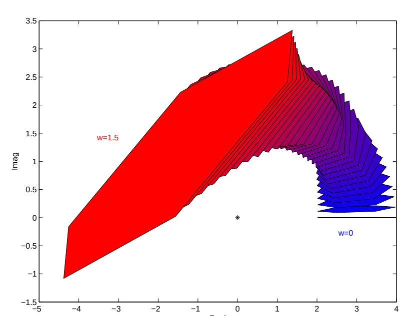
Kharitonov's theorem (ct only)
- Polytope of scalar polynomials

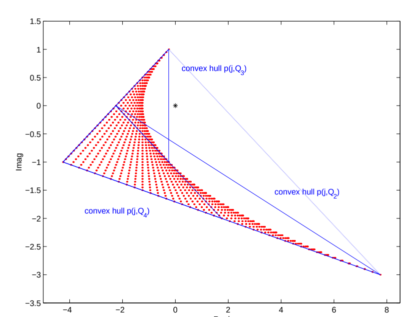
(affine polynomial families)
Edge theorem
- Interval matrix polynomials

(multiaffine polynomial families)
Mapping theorem
- Polytopes of matrix polynomials

(polynomic polynomial families)







Lessons from robust analysis: lack of extreme point results

Ensuring robust stability of the parametrized polynomial

$$p(s,q) = p_0(s) + qp_1(s)$$
$$q \in [q_{\text{min}},\ q_{\text{max}}]$$

amounts to ensuring robust stability of the whole segment of polynomials

$$\lambda p(s,q_{\text{min}}) + (1-\lambda)p(s,q_{\text{max}})$$
$$\lambda = \frac{q_{\text{max}}-q}{q_{\text{max}}-q_{\text{min}}} \in [0,\ 1]$$

A natural question arises: does stability of two vertices imply stability of the segment ?

Unfortunately, the answer is no

Example
First vertex: $0.57 + 6s + s^2 + 10s^3$ stable
Second vertex: $1.57 + 8s + 2s^2 + 10s^3$ stable
But middle of segment:
$1.07 + 7s + 1.50s^2 + 10s^3$ unstable

Lessons from robust analysis: lack of edge results

In the same way there is lack of vertex results for affine uncertainty, there is a lack of edge results for multilinear uncertainty

Example
Consider the uncertain polynomial

$$\begin{array}{rcl} p(s,q) & = & (4.032q_1q_2 + 3.773q_1 + 1.985q_2 + 1.853) \\ & & +(1.06q_1q_2 + 4.841q_1 + 1.561q_2 + 3.164)s \\ & & +(q_1q_2 + 2.06q_1 + 1.561q_2 + 2.871)s^2 \\ & & +(q_1 + q_2 + 2.56)s^3 + s^4 \end{array}$$

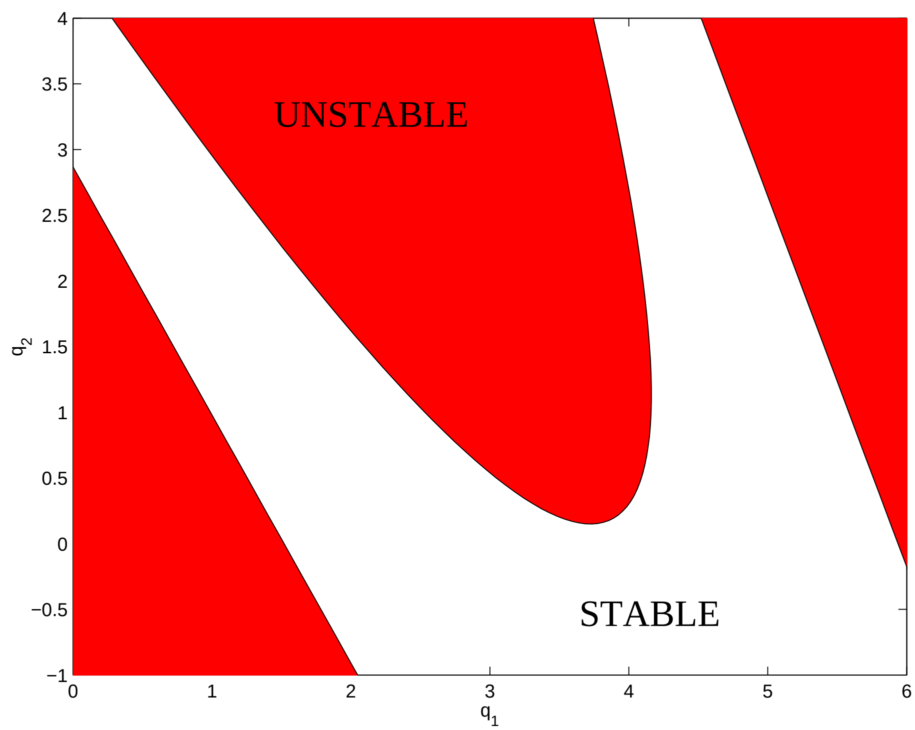
with multilinear uncertainty over the polytope $q_1 \in [0,\ 1]$, $q_2 \in [0,\ 3]$, corresponding to the state-space interval matrix

$$p(s,q) = \det(sI - \begin{bmatrix} [-1.5,\ -0.5] & -12.06 & -0.06 & 0 \\ -0.25 & -0.03 & 1 & 0.5 \\ 0.25 & -4 & -1.03 & 0 \\ 0 & 0.5 & 0 & [-4,\ 1] \end{bmatrix})$$

The four edges of the uncertainty bounding set are stable, however for $q_1 = 0.5$ and $q_2 = 1$ polynomial $p(s,q)$ is unstable..

Non-convexity of stability domain

Main problem: the stability domain in the space of polynomial coefficients p_i is non-convex in general



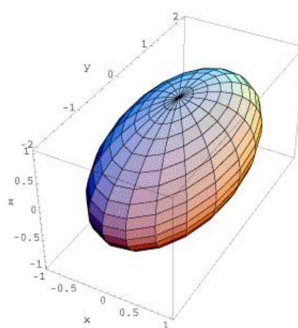
Discrete-time stability domain in (q_1, q_2) plane for polynomial $p(z,q) = (-0.825 + 0.225q_1 + 0.1q_2) + (0.895 + 0.025q_1 + 0.09q_2)z + (-2.475 + 0.675q_1 + 0.3q_2)z^2 + z^3$

How can we overcome the non-convexity of the stability conditions in the coefficient space ?

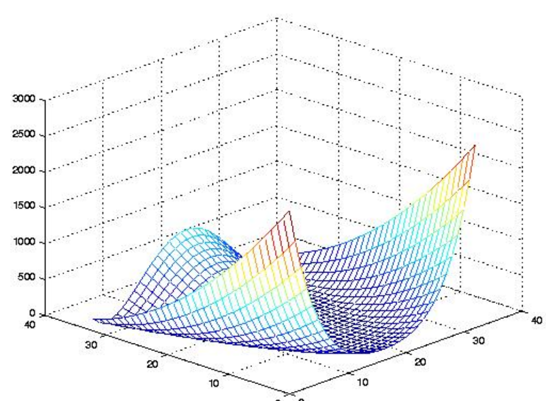
Handling non-convexity

Basically, we can pursue two approaches:

• we can approximate the non-convex stability domain with a convex domain (segment, polytope, sphere, ellipsoid, LMI)



• we can address the non-convexity with the help of non-convex optimization (global or local optimization)



Approximation of the stability domain

From the tools of robust stability analysis, we can build around a nominally stable polynomial

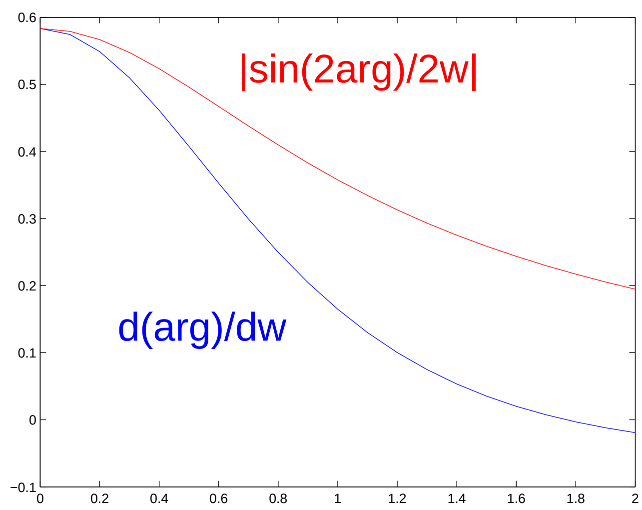
- stability segments (eigenvalue criterion)
- stability boxes (Kharitonov's theorem)
- stability polytopes (Edge theorem)

There exists other results, such as Rantzer's growth condition: a polynomial $g(s)$ is a convex direction iff

$$\frac{d}{d\omega}\arg g(j\omega) \leq \left|\frac{\sin 2 \arg g(j\omega)}{2\omega}\right|, \quad \omega > 0$$

It means that given any stable $f(s)$ such that $f(s)+g(s)$ is stable then the whole segment $[f(s),\, g(s)]$ is stable

Example
$g(s) = 24 + 14s - 13s^2 - 2s^3 + s^4$ is a growth direction



Stability polytopes

Largest hyper-rectangle around a nominally stable polynomial

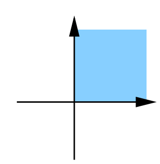
$$p(s) + r \sum_{i=0}^{n} [-\varepsilon_i, \varepsilon_i] s^i$$

obtained with the eigenvalue criterion applied on the 4 Kharitonov polynomials

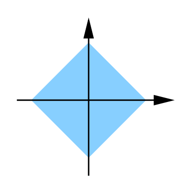
In general, there is no systematic way to obtain more general stability polytopes, namely because of computational complexity
(no analytic formula for the volume of a polytope)

Well-known candidates:

- ct LHP: outer approximation (necessary stab cond)
  positive cone $p_i > 0$



- dt unit disk: inner approximation (sufficient stab cond)
  diamond $|p_0| + |p_1| + \cdots + |p_{n-1}| < 1$

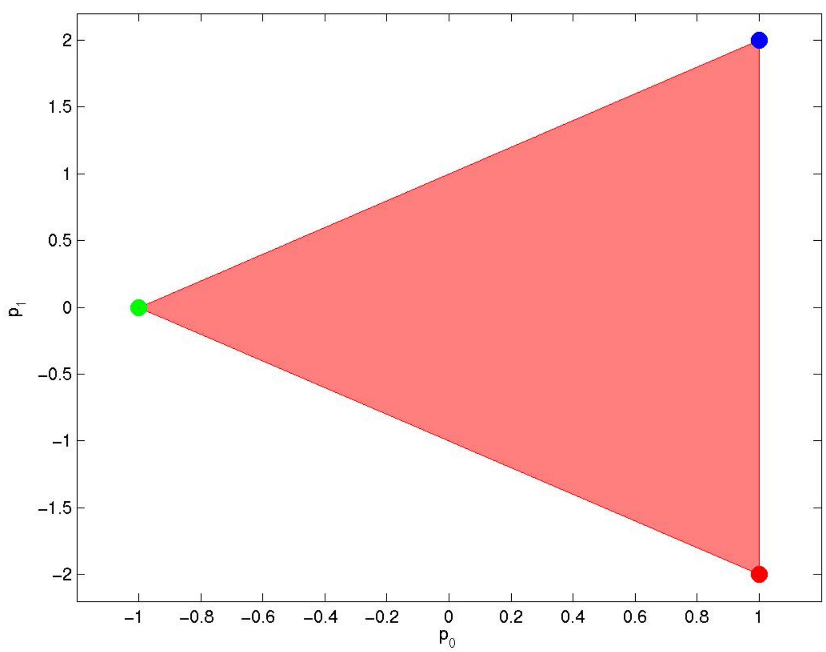


Stability region (second degree)

Necessary stab cond in dt: convex hull of stability domain is a polytope whose $n+1$ vertices are polynomials with roots +1 or -1

Example
When $n = 2$: triangle with vertices

$$\begin{aligned}(z+1)(z+1) &= 1+2z+z^2\\ (z+1)(z-1) &= -1+z^2\\ (z-1)(z-1) &= 1-2z+z^2\end{aligned}$$

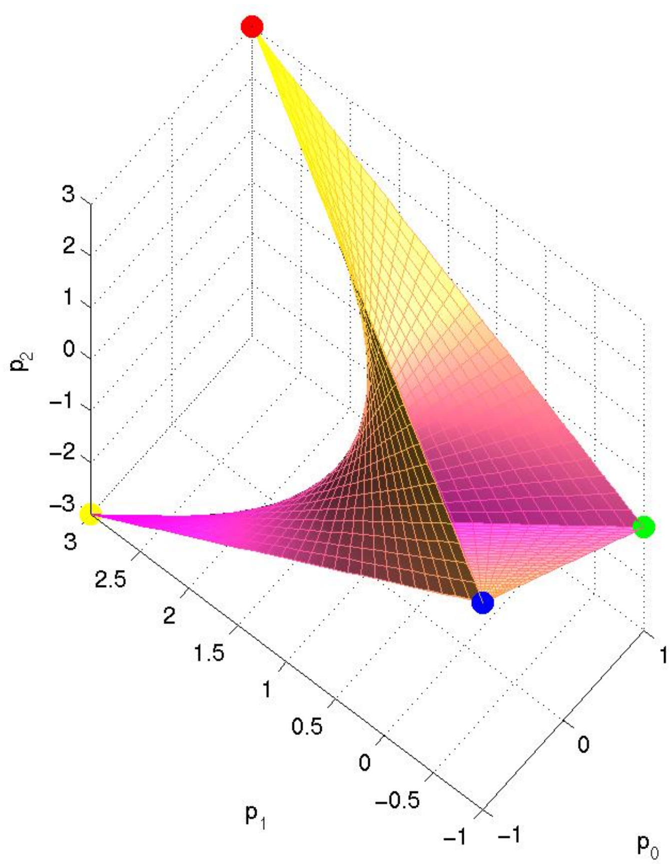


Stability region (third degree)

Example

Third degree dt polynomial: two hyperplanes and a non-convex hyperbolic paraboloid with a saddle point at $p(z) = p_0 + p_1 z + p_2 z^2 + z^3 = z(1 + z^2)$

$$\begin{aligned}
(z+1)(z+1)(z+1) &= 1 + 3z + 3z^2 + z^3 \\
(z+1)(z+1)(z-1) &= -1 - z + z^2 + z^3 \\
(z+1)(z-1)(z-1) &= 1 - z - z^2 + z^3 \\
(z-1)(z-1)(z-1) &= -1 + 3z - 3z^2 + z^3
\end{aligned}$$



Stability hyper-spheres

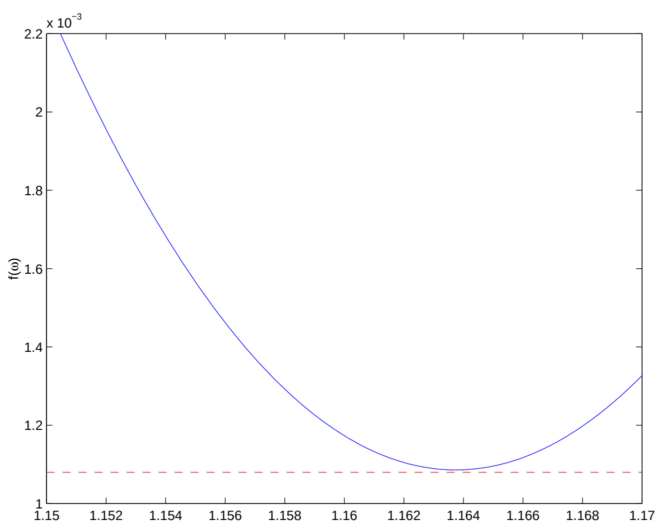
Largest hyper-sphere around a nominally stable polynomial

$$p(s) + \sum_{i=0}^{n} q_i s^i, \ \|q\| \le r$$

has radius

$$r_{\max} = \min\{|p_0|, |p_n|, \inf_{\omega>0} \sqrt{\frac{(\mathrm{Re}\, p(j\omega))^2}{1+w^4+w^8..} + \frac{(\mathrm{Im}\, p(j\omega))^2}{w^2+w^6+..}}\}$$

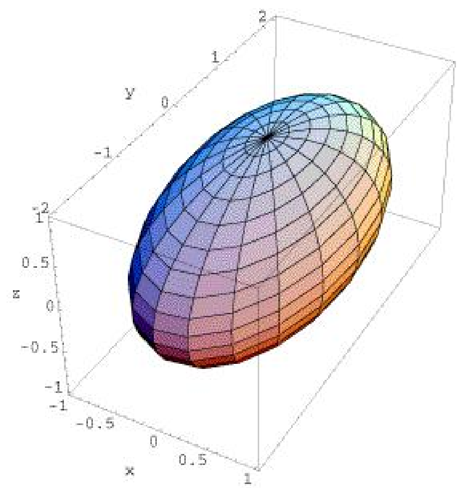
Example
$(2+q_0)+(1.4+q_1)s+(1.5+q_2)s^2+(1+q_3)s^3, \ \|q\| \le r$



$r_{\max} = \min\{2, \ 1, \ \inf_{\omega>0} f(w)\} = 1.08 \cdot 10^{-3}$

Stability ellipsoids

A weighted and rotated hyper-sphere is an ellipsoid



We are interested in inner ellipsoidal approximations of stability domains

$$E = \{p \, : \, (p - \bar{p})^\star P (p - \bar{p}) \leq 1\}$$

where

p	coef vector of polynomial $p(s)$
$\bar{p}$	center of ellipsoid
P	positive definite matrix

Hermite stability criterion

Charles Hermite (1822 Dieuze - 1901 Paris)

The polynomial $p(s) = p_0 + p_1 s + \cdots + p_n s^n$ is stable if and only if

$$H(x) = \sum_i \sum_j p_i p_j H_{ij} \succ 0$$

where matrices H_{ij} are given and depend on the root clustering region only

Examples for $n = 3$:

continuous-time stability

$$H(p) = \begin{bmatrix} 2p_0p_1 & 0 & 2p_0p_3 \\ 0 & 2p_1p_2 - 2p_0p_3 & 0 \\ 2p_0p_3 & 0 & 2p_2p_3 \end{bmatrix}$$

discrete-time stability

$$H(p) = \begin{bmatrix} p_3^2 - p_0^2 & p_2p_3 - p_0p_1 & p_1p_3 - p_0p_2 \\ p_2p_3 - p_0p_1 & p_2^2 + p_3^2 - p_0^2 - p_1^2 & p_2p_3 - p_0p_1 \\ p_1p_3 - p_0p_2 & p_2p_3 - p_0p_1 & p_3^2 - p_0^2 \end{bmatrix}$$

Inner ellipsoidal approximation

Our objective is then to find $\bar{p}$ and P such that the ellipsoid

$$E = \{p : (p - \bar{p})^\star P (p - \bar{p}) \leq 1\}$$

is a convex inner approximation of the actual non-convex stability region

$$S = \{p : H(p) \succ 0\}$$

that is to say

$$\boxed{E \subset S}$$

Naturally, we will try to enlarge the volume of the ellipsoid as much as we can

The Hermite matrix defining stability region S can be written as

$$H(p) = (I_n \otimes p^\star)\tilde{H}(I_n \otimes p) \succ 0$$

where the big and sparse matrix $\tilde{H}$ depends on the stability region only

Preliminaries

The quadratic inequality defining stability ellipsoid E can be written as

$$\boxed{P(p) = p^\star \tilde{P} p \geq 0}$$

where

$$\begin{array}{rcl} \tilde{P} & = & \left[\begin{array}{cc} P_{11} & P_{12} \\ P_{12}^\star & P_{22} \end{array}\right] \quad P_{22} \quad \text{scalar} \\ \bar{p} & = & -P_{11}^{-1} P_{12} \\ P & = & -P_{11}/(P_{22} - P_{12}^\star P_{11}^{-1} P_{12}) \end{array}$$

Ellipsoid E is non-empty and bounded iff

$$P_{11} \prec 0 \quad P_{22} - P_{12}^\star P_{11}^{-1} P_{12} > 0$$

the later inequality is non-convex so we

- either specify a vector p_0 in E

$$\boxed{p_0^\star \tilde{P} p_0 = 1}$$

- or specify the center $\bar{p}$ of E

$$\boxed{P_{11} = -P \prec 0 \quad P_{12} = P\bar{p} \quad P_{22} = 1 - \bar{p}^\star P \bar{p}}$$

LMI inner ellipsoidal approximation

If there exists some symmetric block matrix

$$S = \begin{bmatrix} 0 & S_{21}^\star & \cdots & S_{n1}^\star \\ S_{21} & 0 & & S_{n2}^\star \\ \vdots & & \ddots & \vdots \\ S_{n1} & S_{n2} & \cdots & 0 \end{bmatrix}$$

made up of skew-symmetric blocks $S_{ij} = -S_{ij}^\star$ and a positive scalar λ such that the LMI

$$\boxed{\lambda \tilde{H} \succ I_n \otimes \tilde{P} + S}$$

is satisfied, then

$$\begin{array}{rcl} \lambda H(x) & = & \lambda (I_n \otimes p)^\star \tilde{H} (I_n \otimes p) \\ & \succ & (I_n \otimes p)^\star (I_n \otimes \tilde{P} + S)(I_n \otimes p) \\ & = & I_n \otimes P(p) \end{array}$$

therefore

$H(p) \succ 0$ for all p such that $P(p) \geq 0$

and the inclusion $E \subset S$ is ensured

Example

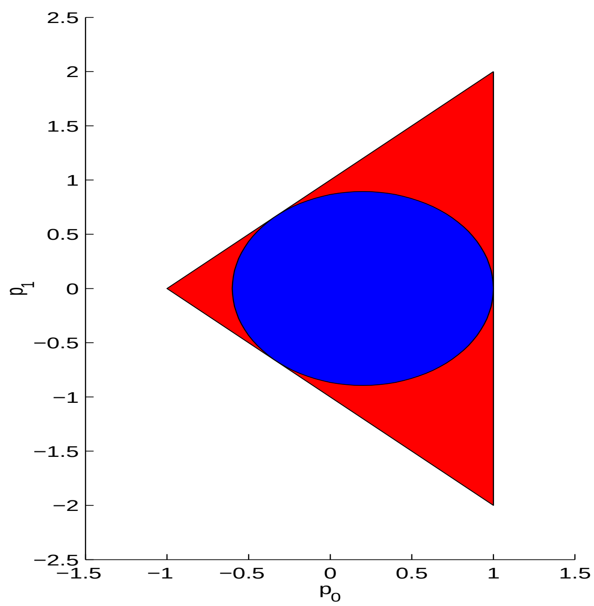
Discrete-time second degree polynomial

$$p(z) = p_0 + p_1 z + z^2$$

We solve the LMI problem and we obtain

$$P = \begin{bmatrix} 1.5625 & 0 \\ 0 & 1.2501 \end{bmatrix} \quad \bar{p} = \begin{bmatrix} 0.2000 \\ 0 \end{bmatrix}$$

which describes an ellipse E inscribed in the exact triangular stability domain S



Stability ellipsoids

Example

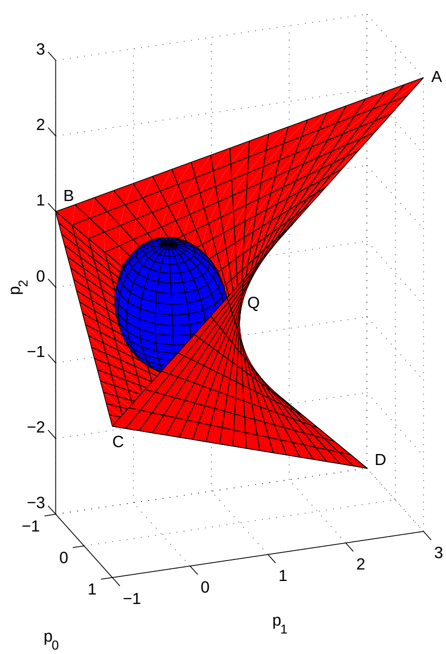
Discrete-time third degree polynomial

$$p(z) = p_0 + p_1 z + p_2 z^2 + z^3$$

We solve the LMI problem and we obtain

$$P = \begin{bmatrix} 2.3378 & 0 & 0.5397 \\ 0 & 2.1368 & 0 \\ 0.5397 & 0 & 1.7552 \end{bmatrix} \qquad \bar{x} = \begin{bmatrix} 0 \\ 0.1235 \\ 0 \end{bmatrix}$$

which describes a convex ellipse E inscribed in the exact stability domain S delimited by the non-convex hyperbolic paraboloid



Very simple scalar convex sufficient stability condition

$$2.4166 p_0^2 + 2.2088 p_1^2 + 1.8143 p_2^2 - 0.5458 p_1 + 1.1158 p_0 p_2 \leq 1$$

Volume of stability ellipsoid

In the discrete-time case, the well-known sufficient stability condition defines a diamond
$D = \{p : |p_0| + |p_1| + \cdots + |p_{n-1}| < 1\}$

For different values of degree n, we compared volumes of exact stability domain S, ellipsoid E and diamond D

	$n = 2$	$n = 3$	$n = 4$	$n = 5$
Stability domain S	4.0000	5.3333	7.1111	7.5852
Ellipsoid E	2.2479	1.4677	0.7770	0.3171
Diamond D	2.0000	1.3333	0.6667	0.2667

E is "larger" than D, yet very small wrt S

In the last part of this course, we will propose better LMI inner approximations of the stability domain