COURSE ON LMI OPTIMIZATION
WITH APPLICATIONS IN CONTROL
PART II.2

LMIs IN SYSTEMS CONTROL STATE-SPACE METHODS PERFORMANCE ANALYSIS and SYNTHESIS

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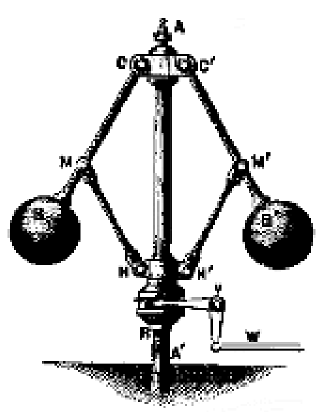

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H_2 space

$\mathcal{H}_2$ is the Hardy space with matrix functions $\hat{f}(s),\ s \in \mathbb{C} \rightarrow \mathbb{C}^n$ analytic in $Re(s) > 0$

$$||\hat{f}||_2 = \left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}^\star(j\omega)\hat{f}(j\omega)d\omega\right)^{1/2} < \infty$$

Paley-Wiener

$$\begin{array}{ccc} \mathcal{L}_2[0\ ,\ +\infty) & \xrightarrow{\mathcal{L}} & \mathcal{H}_2 \\ f(t) & \longrightarrow & \hat{f}(s) = \int_{-\infty}^{+\infty} f(t)e^{-st}dt \end{array}$$

Parseval

$$||f||_2 = ||\hat{f}||_2$$

$\mathcal{RH}_2$ is a subspace of $\mathcal{H}_2$ with all strictly proper and real rational stable transfer matrices

$$\frac{s+1}{(s+2)(s+3)} \in \mathcal{RH}_2 \quad \frac{s+1}{(s+2)(s-3)} \notin \mathcal{RH}_2$$

$$\frac{(s-1)}{(s+1)} \notin \mathcal{RH}_2$$

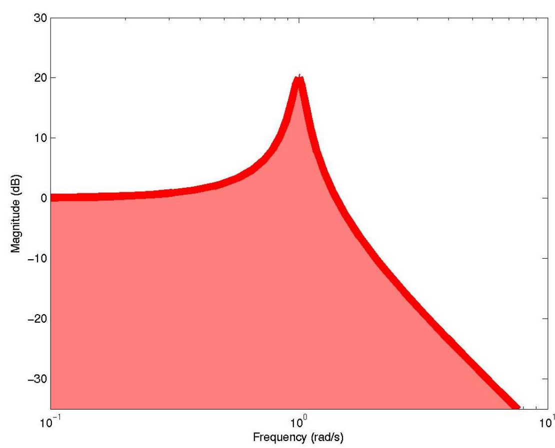
H_2 norm

The H_2 norm of the strictly proper stable LTI system

$$\begin{aligned} \dot{x} &= Ax + Bw \\ z &= Cx \end{aligned}$$

is the energy (l_2 norm) of its impulse response $g(t)$

$$||G||_2^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{tr}(G^\star(j\omega)G(j\omega))d\omega$$
$$||G||_2 = \max_{w_i(t)=\delta} ||z||_2$$



- For MIMO systems, H_2 norm is impulse-to-energy gain or steady-state variance of z in response to white noise
- For MISO systems, H_2 norm is energy-to-peak gain

Computing the H_2 norm

Let $G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & \mathbf{0} \end{array}\right]$

Defining the controllability Grammian and the observability Grammian

$$P_c = \int_0^\infty e^{At} B B' e^{A't} dt \quad P_o = \int_0^\infty e^{A't} C' C e^{At} dt$$

solutions to the Lyapunov equations

$$A' P_o + P_o A + C' C = \mathbf{0}$$

$$A P_c + P_c A' + B B' = \mathbf{0}$$

and hence

$$||G||_2^2 = \text{tr}\,[C P_c C'] = \text{tr}\,[B' P_o B]$$

(A, C) observable iff $P_o \succ 0$
(A, B) controllable iff $P_c \succ 0$

LMI computation of the H_2 norm

Dual Lyapunov equations formulated as dual LMIs

The following statements are equivalent

- $\|G\|_2^2 < \gamma^2$
- $\exists\ P \in \mathbb{S}_n^{++}$

  $$A'P + PA + C'C \preceq \mathbf{0} \quad \text{tr}\, B'PB < \gamma^2$$

- $\exists\ Q \in \mathbb{S}_n^{++}$

  $$AQ + QA' + BB' \preceq \mathbf{0} \quad \text{tr}\, CQC' < \gamma^2$$

- $\exists\ X \in \mathbb{S}_n^{++}$ and $Z \in \mathbb{R}^{r\times r}$

  $$\begin{bmatrix} A'X + XA & XB \\ B'X & -\mathbf{1} \end{bmatrix} \prec \mathbf{0} \quad \begin{bmatrix} X & C' \\ C & Z \end{bmatrix} \succ \mathbf{0} \quad \text{tr}\ Z < \gamma^2$$

- $\exists\ Y \in \mathbb{S}_n^{++}$ and $T \in \mathbb{R}^{m\times m}$

  $$\begin{bmatrix} AY + YA' & YC' \\ CY & -\mathbf{1} \end{bmatrix} \prec \mathbf{0} \quad \begin{bmatrix} Y & B \\ B' & T \end{bmatrix} \succ \mathbf{0} \quad \text{tr}\ T < \gamma^2$$

H_∞ space

$\mathcal{H}_\infty$ is the Hardy space with matrix functions $\hat{f}(s),\ s \in \mathbb{C} \rightarrow \mathbb{C}^{n\times m}$ analytic in $Re(s) > 0$

$$||\hat{f}||_\infty = \sup_{Re(s)>0} \overline{\sigma}(\hat{f}(s)) = \sup_{\omega\in\mathbb{R}} \overline{\sigma}(\hat{f}(j\omega)) < \infty$$

$\mathcal{RH}_\infty$ is a real rational subset of $\mathcal{H}_\infty$ with all proper and real rational stable transfer matrices

$$\frac{s+1}{(s+2)(s-3)} \notin \mathcal{RH}_\infty \qquad \frac{(s-1)}{(s+1)} \in \mathcal{RH}_\infty$$

Godfrey Harold Hardy
(1877 Cranleigh - 1947 Cambridge)

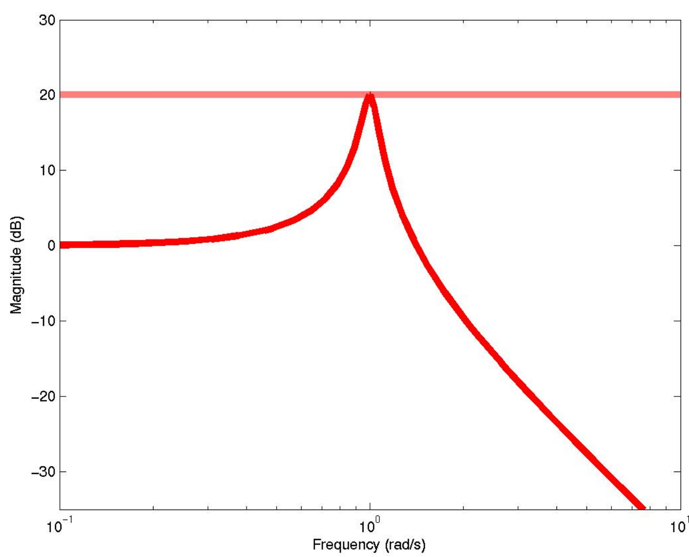
H_∞ norm

Let the proper stable LTI system $G(s) = C(sI - A)^{-1}B + D$

$$
\begin{aligned}
\dot{x} &= Ax + Bw \\
z &= Cx + Dw
\end{aligned}
$$

The H_∞ norm is the induced energy-to-energy gain (l_2 to l_2)

$$
\|G\|_\infty = \sup_{\|w\|_2=1} \|Gw\|_2 = \sup_{\|w\|_2=1} \|z\|_2 = \sup_{\omega} \overline{\sigma}(G(j\omega))
$$



It is the worst-case gain

Computing the H_∞ norm

In contrast with the H_2 norm, computation of the H_∞ norm requires a search over ω or an iterative algorithm

A- Set up a fine grid of frequency points $\{\omega_1, \cdots, \omega_l\}$

$$\|G\|_\infty \sim \max_{1 \le k \le l} \overline{\sigma}(G(j\omega_k))$$

B- $\|G(s)\|_\infty < \gamma$ iff $R = \gamma^2 \mathbf{1} - D'D \succ 0$ and the Hamiltonian matrix

$$\begin{bmatrix} A + BR^{-1}D'C & BR^{-1}B' \\ -C'(\mathbf{1} + DR^{-1}D')C & -(A + BR^{-1}D'C)' \end{bmatrix}$$

has no eigenvalues on the imaginary axis

Bisection algorithm - γ-iterations

We can design a bisection algorithm
with guaranteed quadratic convergence
to find the minimum value of γ such that
the Hamiltonian has no imaginary eigenvalues

1- Select $[\gamma_l \quad \gamma_u]$ with $\gamma_l > \overline{\sigma}(D)$

2- If $(\gamma_u - \gamma_l)/\gamma_l \leq \epsilon$ stop;

$$\|G\|_\infty \sim (\gamma_u + \gamma_l)/2$$

otherwise go to the next step;

2- Set $\gamma = 1/2(\gamma_l + \gamma_u)$ and compute H_γ

3- Compute the eigenvalues of H_γ

If $\Lambda(H_\gamma) \cap \mathbb{C}^0$ set $[\gamma_l \quad \gamma]$ and go back to step 2

else set $[\gamma \quad \gamma_{max}]$ and and go back to step 2

LMI computation of the H_∞ norm

Refer to the part of the course on norm-bounded uncertainty

$$\sup_{\|z\|_2=1} \|w\| = \|\Delta\| < \gamma^{-1}$$

The following statements are equivalent

- $\|G\|_\infty < \gamma$
- $\exists\ P \in \mathbb{S}_n^{++}$

$$\begin{bmatrix} A'P + PA + C'C & PB + C'D \\ B'P + D'C & D'D - \gamma^2 \mathbf{1} \end{bmatrix} \prec \mathbf{0}$$

- $\exists\ P \in \mathbb{S}_n^{++}$

$$\begin{bmatrix} A'P + PA & PB & C' \\ B'P & -\gamma\mathbf{1} & D' \\ C & D & -\gamma\mathbf{1} \end{bmatrix} \prec \mathbf{0}$$

State-feedback stabilization

Open-loop continuous-time LTI system

$$\dot{x} = Ax + Bu$$

with state-feedback controller

$$u = Kx$$

produces closed-loop system

$$\dot{x} = (A + BK)x$$

Applying Lyapunov LMI stability condition

$$(A + BK)'P + P(A + BK) \prec \mathbf{0} \quad P \succ \mathbf{0}$$

we get bilinear terms...

Bilinear Matrix Inequalities (BMIs) are non-convex in general !

State-feedback design:
linearizing change of variables

Project BMI onto $P^{-1} \succ \mathbf{0}$

$$(A+BK)'P + P(A+BK) \prec \mathbf{0}$$
$$\Longleftrightarrow$$
$$P^{-1}\left[(A+BK)'P + P(A+BK)\right]P^{-1} \prec \mathbf{0}$$
$$\Longleftrightarrow$$
$$P^{-1}A' + P^{-1}K'B' + AP^{-1} + BKP^{-1} \prec \mathbf{0}$$

Denoting

$$Q = P^{-1} \quad Y = KP^{-1}$$

we derive a state-feedback design LMI

$$AQ + QA' + BY + Y'B' \prec \mathbf{0} \quad Q \succ \mathbf{0}$$

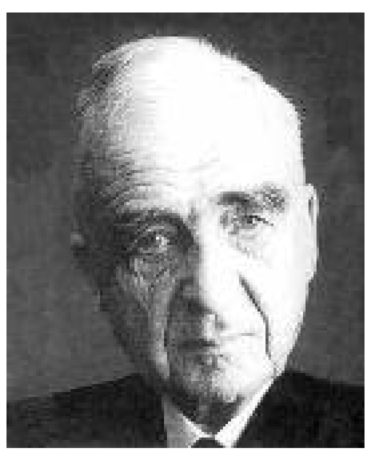
We obtained an LMI thanks to a one-to-one linearizing change of variables

Finsler's theorem

Recall Finsler's theorem, already seen in the first chapter of this course...

The following statements are equivalent

1. $x'Ax > 0$ for all $x \neq 0$ s.t. $Bx = 0$
2. $\tilde{B}'A\tilde{B} \succ \mathbf{0}$ where $B\tilde{B} = \mathbf{0}$
3. $A + \lambda B'B \succ \mathbf{0}$ for some scalar λ
4. $A + XB + B'X' \succ \mathbf{0}$ for some matrix X



Paul Finsler
(1894 Heilbronn - 1970 Zurich)

State-feedback design: null-space projection

Item 2 of Finsler's theorem may be used by projecting onto the (full column rank) null-space $\tilde{B}$ of B'

$$B'\tilde{B} = \mathbf{0}$$

so that BMI

$$A'P + PA + K'B'P + PBK \prec \mathbf{0}$$

is equivalent to the projected LMI

$$\tilde{B}'(AQ + QA')\tilde{B} \prec \mathbf{0} \quad Q \succ \mathbf{0}$$

Feedback K can be recovered from Lyapunov matrix Q as

$$K = -\frac{1}{2}B'Q^{-1}$$

Here we obtained an LMI thanks to a projection onto a null-space

State-feedback design: Riccati inequality

We can also use item 3 of Finsler's theorem to convert BMI

$$A'P + PA + K'B'P + PBK \prec \mathbf{0}$$

into

$$A'P + PA - \lambda PBB'P \prec \mathbf{0}$$

where $\lambda \geq 0$ is an unknown scalar

Now replacing P with λP we get

$$A'P + PA - PBB'P \prec \mathbf{0}$$

which is related to the Riccati equation

$$A'P + PA - PBB'P + Q = \mathbf{0}$$

for some matrix $Q \succ \mathbf{0}$

Shows equivalence between state-feedback LMI stabilizability and the linear quadratic regulator (LQR) problem

Robust state-feedback design for polytopic uncertainty

LTI system $\dot{x} = Ax + Bu$ affected by polytopic uncertainty

$$(A, B) \in \text{co}\,\{(A_1, B_1), \dots, (A_N, B_N)\}$$

and search for a robust state-feedback law $u = Kx$

Start with analysis conditions

$$(A_i + B_i K)' P + P(A_i + B_i K) \prec \mathbf{0} \ \forall\ i \quad Q \succ \mathbf{0}$$

and we obtain the quadratic stabilizability LMI

$$A_i Q + Q A_i' + B_i Y + Y' B_i' \prec \mathbf{0} \ \forall\ i \quad Q \succ \mathbf{0}$$

with the linearizing change of variables

$$Q = P^{-1} \quad Y = K P^{-1}$$

State-feedback H_2 control

Let the continuous-time LTI system

$$\begin{aligned} \dot{x} &= Ax + B_w w + B_u u \\ z &= C_z x + D_{zw} w + D_{zu} u \end{aligned}$$

with state-feedback controller

$$u = Kx$$

Closed-loop system is given by

$$\begin{aligned} \dot{x} &= (A + B_u K)x + B_w w \\ z &= (C_z + D_{zu} K)x + D_{zw} w \end{aligned}$$

with transfer function

$$G(s) = D_{zw} + (C_z + D_{zu}K)(sI - A - B_u K)^{-1} B_w$$

between performance signals w and z

H_2 performance specification

$$\|G(s)\|_2 < \gamma$$

We must have $D_{zw} = 0$ (finite gain)

H_2 design LMIs

As usual, start with analysis condition:

$\exists\ K$ such that $\|G(s)\|_2 < \gamma$ iff

$$\text{tr}\,(C_z + D_{zu}K)Q(C_z + D_{zu}K)' < \gamma$$

$$(A + B_uK)Q + Q(A + B_uK) + BB' \prec \mathbf{0}$$

Remember equivalent statements about H_2 analysis and obtain the overall LMI formulation

$$\text{tr}\ Z < \gamma^2$$

$$\begin{bmatrix} Z & C_zX + D_{zu}R \\ XC_z' + R'D_{zu}' & X \end{bmatrix} \succ \mathbf{0}$$

$$AX + XA' + B_uR + R'B_u' + B_wB_w' \prec \mathbf{0}$$

with resulting H_2 suboptimal state-feedback

$$K = RX^{-1}$$

Optimal H_2 control: minimize γ^2

Quadratic H_2 design LMIs

Let the polytopic uncertain LTI system

$$M = \begin{bmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \end{bmatrix} \in \mathrm{co}\,\{M_1, \cdots, M_N\}$$

$$\Gamma_q^* = \min \gamma^2$$

$$\mathrm{tr}\, Z < \gamma^2$$

$$\begin{bmatrix} Z & C_z^i X + D_{zu}^i R \\ X C_z^{i'} + R' D_{zu}^{i'} & Q \end{bmatrix} \succ \mathbf{0}$$

$$\begin{bmatrix} A^i X + X A^{i'} + B_u^i R + R' B_u^{i'} & B_w^i \\ B_w^{i'} & \mathbf{1} \end{bmatrix} \prec \mathbf{0}$$

with resulting robust H_2 suboptimal state-feedback

$$K = R X^{-1}$$

$$\|G\|_{2w.c.} \le \sqrt{\Gamma_q^*}$$

State-feedback H_∞ control

Similarly, with H_∞ performance specification

$$\|G(s)\|_\infty < \gamma$$

on transfer function between w and z we obtain

$$\begin{bmatrix} AQ + QA' + B_u Y + Y'B_u' & \star & \star \\ C_z Q + D_{zu} Y & -\gamma^2 \mathbf{1} & \star \\ B_w' & D_{zw}' & -\mathbf{1} \end{bmatrix} \prec \mathbf{0}$$

$$Q \succ \mathbf{0}$$

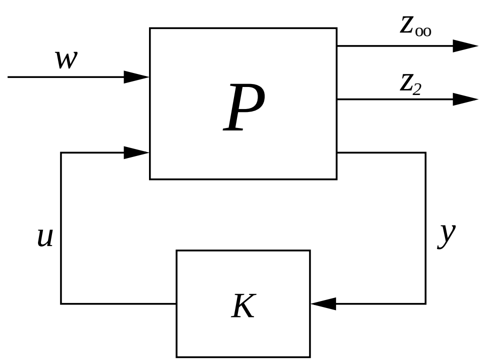
with resulting H_∞ suboptimal state-feedback

$$K = YQ^{-1}$$

Optimal H_∞ control: minimize γ

Mixed H_2/H_∞ control

$$P(s) := \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_\infty & D_{\infty w} & D_{\infty u} \\ C_2 & \mathbf{0} & D_{2u} \end{array}\right]$$



H_2/H_∞ problem

For a given admissible H_∞ performance level γ, find an admissible feedback, $K \in \mathcal{K}$, s.t.:

$$\alpha^* = \inf_{K \in \mathcal{K}} \quad ||G_2(K)||_2$$

$$\text{s.t.} \quad ||G_\infty(K)||_\infty \leq \gamma$$

Mixed H_2/H_∞ control (2)

- $K_2^* = \arg\left[\inf_{K\in\mathcal{K}} \ \ ||G_2||_2 = \alpha_2^*\right]$

- $\gamma_2 = \|G_\infty(K_2^*)\|_\infty$

- $K_\infty^* = \arg\left[\inf_{K\in\mathcal{K}} \ \ ||G_\infty||_\infty = \gamma_\infty^*\right]$

Note that

- For $\gamma < \gamma_\infty^*$, the mixed problem has no solution

- For $\gamma_2 \leq \gamma$, the solution of the mixed problem is given by (α_2^*, K_2^*) and the H_∞ constraint is redundant

- For $\gamma_\infty^* \leq \gamma < \gamma_2$, the pure mixed problem is a non trivial infinite dimension optimization problem

Mixed H_2/H_∞ control (3)

- Open problem without analytical solution nor general numerical one

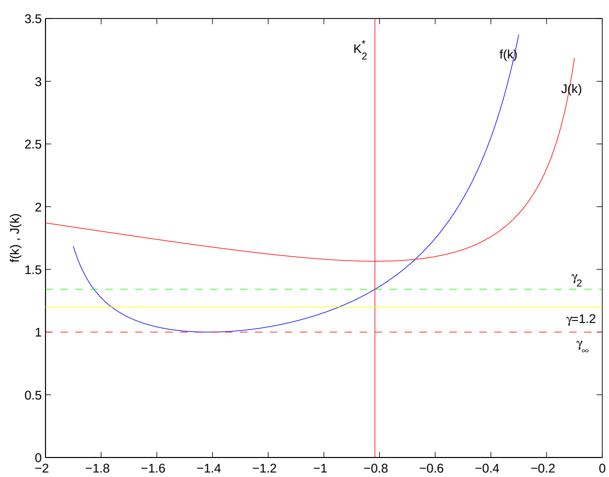
- Trade-off between nominal performance and robust stability constraint

$$\min_k \quad J(k) = \sqrt{-\frac{2+3k^2}{2k}}$$

under

$$k < 0$$

$$f(k) = \frac{2}{\sqrt{k^2(4-k^2)}} \leq \gamma$$



Mixed H_2/H_∞ control via LMIs

Formulation of H_∞ constraint

$$\begin{bmatrix} AQ_\infty + Q_\infty A' + B_u Y_\infty + Y_\infty{}' B_u' & \star & \star \\ C_z Q_\infty + D_{\infty u} Y_\infty & -\gamma^2 \mathbf{1} & \star \\ B_w' & D_{\infty w}' & -\mathbf{1} \end{bmatrix} \prec \mathbf{0}$$

$$Q_\infty \succ \mathbf{0}$$

and formulation of H_2 constraint

$$\operatorname{tr} Z < \alpha$$

$$\begin{bmatrix} Z & C_2 X_2 + D_{2u} R_2 \\ X_2 C_2' + R_2{}' D_{2u}' & X_2 \end{bmatrix} \succ \mathbf{0}$$

$$AX_2 + X_2 A' + B_u R_2 + R_2{}' B_u' + B_w B_w' \prec \mathbf{0}$$

Problem:

We cannot linearize simultaneously !

$$K = Y_\infty Q_\infty^{-1} = R_2 X_2^{-1}$$

Mixed H_2/H_∞ control via LMIs (2)

Remedy: Lyapunov Shaping Paradigm

Enforce $X_2 = Q_\infty = Q$!

Trade-off: Conservatism/tractability

Resulting mixed H_2/H_∞ design LMI

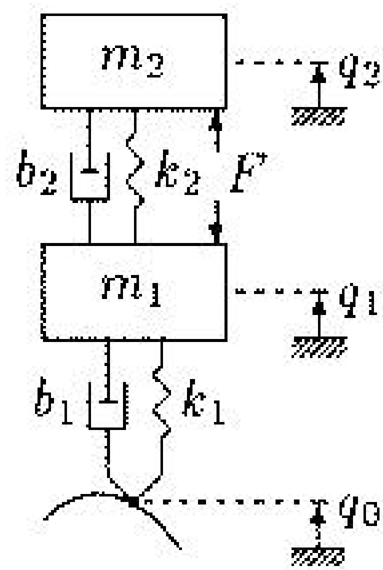
$$
\begin{aligned}
\Gamma_l^* = \;& \min\ \alpha \\
& \mathrm{tr}\, Z < \alpha \\
& \begin{bmatrix} Z & C_2 Q + D_{2u} Y \\ Q C_2' + Y' D_{2u}' & Q \end{bmatrix} \succ \mathbf{0} \\
& AQ + QA' + B_u Y + Y' B_u' + B_w B_w' \prec \mathbf{0} \\
& \begin{bmatrix} AQ + QA' + B_u Y + Y' B_u' & \star & \star \\ C_z Q + D_{\infty u} Y & -\gamma^2 \mathbf{1} & \star \\ B_w' & D_{\infty w}' & -\mathbf{1} \end{bmatrix} \prec \mathbf{0} \\
& Q \succ \mathbf{0}
\end{aligned}
$$

Guaranteed cost mixed H_2/H_∞:

$$\alpha^* \leq \sqrt{\Gamma_l^*}$$

Mixed H_2/H_∞ control: example

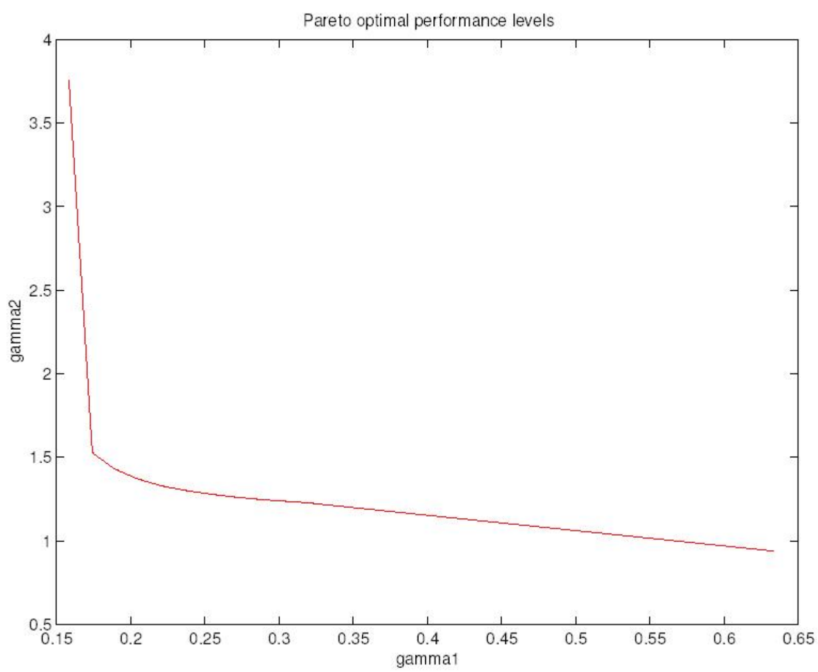
Active suspension system (Weiland)



$$m_2\ddot{q}_2 + b_2(\dot{q}_2 - \dot{q}_1) + k_2(q_2 - q_1) + F = 0$$
$$m_1\ddot{q}_1 + b_2(\dot{q}_1 - \dot{q}_2) + k_2(q_1 - q_2)$$
$$+k_1(q_1 - q_0) + b_1(\dot{q}_1 - \dot{q}_0) + F = 0$$

$$z = \begin{bmatrix} q_1 - q_0 \\ F \\ \ddot{q}_2 \\ q_2 - q_1 \end{bmatrix} \; y = \begin{bmatrix} \ddot{q}_2 \\ q_2 - q_1 \end{bmatrix} \; w = q_0 \;\; u = F$$

$G_\infty(s)$ from q_0 to $[q_1 - q_0 \quad F]$
$G_2(s)$ from q_0 to $[\ddot{q}_2 \quad q_2 - q_1]$



Trade-off between $\|G_\infty\|_\infty \le \gamma_1$ and $\|G_2\|_2 \le \gamma_2$

Dynamic output-feedback

Continuous-time LTI open-loop system

$$\begin{aligned} \dot{x} &= Ax + B_w w + B_u u \\ z &= C_z x + D_{zw} w + D_{zu} u \\ y &= C_y x + D_{yw} w \end{aligned}$$

with dynamic output-feedback controller

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c y \\ u &= C_c x_c + D_c y \end{aligned}$$

Denote closed-loop system as

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{A}\tilde{x} + \tilde{B}w \\ z &= \tilde{C}\tilde{x} + \tilde{D}w \end{aligned} \quad \text{with } \tilde{x} = \begin{bmatrix} x \\ x_c \end{bmatrix} \text{ and}$$

$$\tilde{A} = \begin{bmatrix} A + B_u D_c C_y & B_u C_c \\ B_c C_y & A_c \end{bmatrix} \qquad \tilde{B} = \begin{bmatrix} B_w + B_u D_c D_{yw} \\ B_c D_{yw} \end{bmatrix}$$

$$\tilde{C} = \begin{bmatrix} C_z + D_{zu} D_c C_y & D_{zu} C_c \end{bmatrix} \qquad \tilde{D} = D_{zw} + D_{zu} D_c D_{yw}$$

Affine expressions on controller matrices

H_2 output feedback design

H_2 design conditions

$$\text{tr}\, Z < \alpha$$
$$\begin{bmatrix} Z & \tilde{C}\tilde{Q} \\ \star & \tilde{Q} \end{bmatrix} \succ \mathbf{0}$$
$$\begin{bmatrix} \tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}' & \tilde{B} \\ \tilde{B}' & -\mathbf{1} \end{bmatrix} \prec \mathbf{0}$$

linearized with a specific change of variables

Denote

$$\tilde{Q} = \begin{bmatrix} Q & \bar{Q}' \\ \bar{Q} & \times \end{bmatrix} \qquad \tilde{P} = \tilde{Q}^{-1} = \begin{bmatrix} P & \bar{P} \\ \bar{P}' & \times \end{bmatrix}$$

so that $\bar{P}$ and $\bar{Q}$ can be obtained from P and Q via relation

$$PQ + \bar{P}\bar{Q} = \mathbf{1}$$

Always possible when controller has same order than the open-loop plant

Linearizing change of variables for H_2 output-feedback design

Then define

$$\begin{bmatrix} X & U \\ Y & V \end{bmatrix} = \begin{bmatrix} \bar{P} & PB_u \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} \bar{Q} & \mathbf{0} \\ C_y Q & \mathbf{1} \end{bmatrix} + \begin{bmatrix} P \\ \mathbf{0} \end{bmatrix} A \begin{bmatrix} Q & \mathbf{0} \end{bmatrix}$$

which is a one-to-one affine relation with converse

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} = \begin{bmatrix} \bar{P}^{-1} & -\bar{P}^{-1}PB_u \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} X - PAQ & U \\ Y & V \end{bmatrix} \begin{bmatrix} \bar{Q}^{-1} & \mathbf{0} \\ -C_y Q \bar{Q}^{-1} & \mathbf{1} \end{bmatrix}$$

We derive the following H_2 design LMI

$$\operatorname{tr} Z < \alpha$$

$$D_{zw} + D_{zu} V D_{yw} = \mathbf{0}$$

$$\begin{bmatrix} Z & C_z Q + D_{zu} Y & C_z + D_{zu} V C_y \\ \star & Q & \mathbf{1} \\ \star & \star & P \end{bmatrix} \succ \mathbf{0}$$

$$\begin{bmatrix} AQ + B_u Y + (\star) & A + B_u V C_y + X' & B_w + B_u V D_{yw} \\ \star & PA + UC_y + (\star) & PB_w + UD_{yw} \\ \star & \star & -\mathbf{1} \end{bmatrix} \prec \mathbf{0}$$

in decision variables Q, P, W (Lyapunov) and X, Y, U, V (controller)

Controller matrices are obtained via the relation

$$PQ + \bar{P}\bar{Q} = \mathbf{1}$$

(tedious but straightforward linear algebra)

H_∞ output-feedback design

Similarly two-step procedure for full-order H_∞ output-feedback design:

- solve LMI for Lyapunov variables Q, P, W and controller variables X, Y, U, V
- retrieve controller matrices via linear algebra

Alternative LMI formulation via projection onto null-spaces (recall elimination lemma)

$$N' \begin{bmatrix} AQ + QA' & QC_z' & B_w \\ \star & -\gamma\mathbf{1} & D_{zw} \\ \star & \star & -\gamma\mathbf{1} \end{bmatrix} N \prec \mathbf{0}$$

$$M' \begin{bmatrix} A'P + PA & PB_w & C_z' \\ \star & -\gamma\mathbf{1} & D_{zw}' \\ \star & \star & -\gamma\mathbf{1} \end{bmatrix} M \prec \mathbf{0}$$

$$\begin{bmatrix} Q & \mathbf{1} \\ \mathbf{1} & P \end{bmatrix} \succeq \mathbf{0}$$

where N and M are null-space basis

$$\begin{bmatrix} B_u' & D_{zu}^\star & \mathbf{0} \end{bmatrix} N = \mathbf{0} \qquad \begin{bmatrix} C_u & D_{yw} & \mathbf{0} \end{bmatrix} M = \mathbf{0}$$

Reduced-order controller

For reduced-order controller of order $n_c < n$ there exists a solution $\bar{P}$,$\bar{Q}$ to the equation

$$PQ + \bar{P}\bar{Q} = \mathbf{1}$$

iff

$$\text{rank}\,(PQ - \mathbf{1}) = n_c$$
$$\Longleftrightarrow$$
$$\text{rank} \begin{bmatrix} Q & \mathbf{1} \\ \mathbf{1} & P \end{bmatrix} = n + n_c$$

Static output feedback iff $PQ = \mathbf{1}$

Difficult rank constrained LMI problem or BMI problem !