COURSE ON LMI OPTIMIZATION
WITH APPLICATIONS IN CONTROL
PART I.3

LMI RELAXATIONS

Didier HENRION
www.laas.fr/∼henrion
henrion@laas.fr

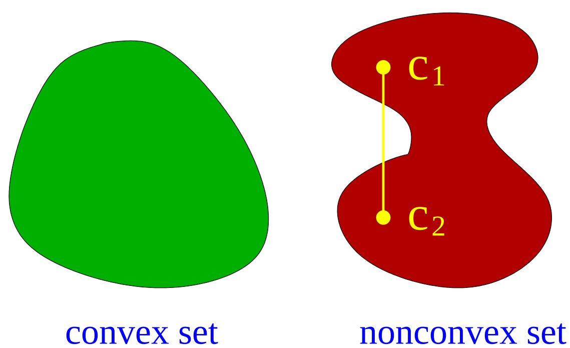
Mme Kupka among verticals (1910-11)
František Kupka (1871-1957)

November 2003

Handling nonconvexity

So far we have studied convex LMI sets

We have seen that additional variables, or liftings can prove useful in describing convex sets with LMIs



But LMI are also frequently used to cope with non-convex sets !

This chapter is dedicated to the joint use of

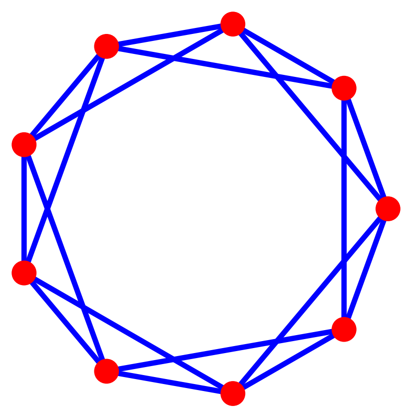
- convex LMI relaxations, and
- additional variables = liftings

Combinatorial optimization

Typical combinatorial optimization problem

$$\begin{array}{ll} \min & x^T Q x \\ \text{s.t.} & x_i \in \{-1, 1\} \end{array}$$

Examples: MAXCUT, knapsack..



Antiweb AW_9^2 graph

Basic non-convex constraints

$$x_i^2 = 1$$

Exponential # of points = NP-hard problem !

LMI relaxation

Basic idea..

For each i replace non-convex constraint

$$x_i^2 = 1$$

with relaxed convex constraint

$$x_i^2 \leq 1$$

which is an LMI constraint

$$\begin{bmatrix} 1 & x_i \\ x_i & 1 \end{bmatrix} \succeq 0$$

Not bad idea, but we can do better..

(Better) LMI relaxation

Replace all non-convex constraints

$$x_i^2 = 1, \quad i = 1, 2, \ldots, n$$

with relaxed LMI constraint

$$X = \begin{bmatrix} 1 & x_1 & x_2 & \cdots & x_n \\ x_1 & 1 & x_{12} & & x_{1n} \\ x_2 & x_{12} & 1 & & x_{2n} \\ \vdots & & & \ddots & \vdots \\ x_n & x_{1n} & x_{2n} & \cdots & 1 \end{bmatrix} \succeq 0$$

where x_{ij} are additional variables = liftings

Always less conservative than previous relaxation because $X \succeq 0$ implies

$$\begin{bmatrix} 1 & x_i \\ x_i & 1 \end{bmatrix} \succeq 0$$

for each $i = 1, 2, \ldots, n$

Rank constrained LMI

In the original problem

$$g^\star = \min \; x^T Q x \quad \text{s.t.} \quad x_i^2 = 1$$

let $X = xx^T$ and then

$$x^T Q x = \operatorname{trace} Q x x^T = \operatorname{trace} Q X$$

and

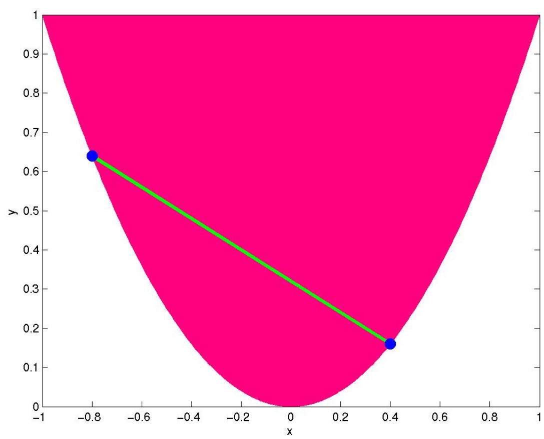
$$x_i^2 = X_{ii} = 1$$

so that the problem can be written as a
rank constrained LMI

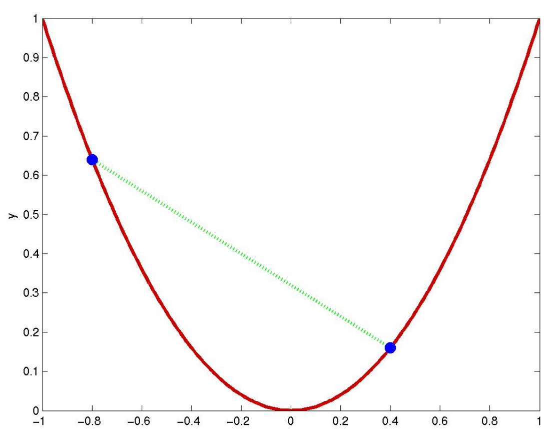
$$\begin{array}{rcll} g^\star & = & \min & \operatorname{trace} QX \\ & & \text{s.t.} & X_{ii} = 1 \\ & & & X \succeq 0 \\ & & & \operatorname{rank} X = 1 \end{array}$$

Example of rank constrained LMI

$$X = \begin{bmatrix} y & x \\ x & 1 \end{bmatrix}$$



Convex set $X \succeq 0$ $(x^2 \leq y)$



Non-convex set $X \succeq 0$, rank $X = 1$ $(x^2 = y)$

Relaxing the rank constraint

All the nonconvexity is concentrated into the rank constraint, so we just drop it !

The obtained LMI relaxation is called Shor's relaxation

$$\begin{array}{rcll} p^\star & = & \min & \operatorname{trace} QX \\ & & \text{s.t.} & X_{ii} = 1 \\ & & & X \succeq 0 \end{array}$$

Naum Zuselevich Shor (Inst Cybernetics, Kiev) in the 1980s was among the first to recognize the relevance of this approach

Since the feasible set is relaxed = enlarged we get a lower bound for the original non-convex optimization problem

$$p^\star \leq g^\star$$

Shor's relaxation

Systematic approach: can be applied to general polynomial optimization problems

Example:

$$x_1^2 x_2 = x_1 \quad \left\{ \begin{array}{c} x_1^2 = x_3 \\ x_3 x_2 = x_1 \end{array} \right. \quad \left\{ \begin{array}{c} X_{11} = X_{30} \\ X_{32} = X_{10} \\ X \succeq 0 \\ \text{rank}\, X = 1 \end{array} \right. \quad \left\{ \begin{array}{c} X_{11} = X_{30} \\ X_{32} = X_{10} \\ X \succeq 0 \end{array} \right.$$

Algorithm:

- introduce lifting variables to reduce polynomials to quadratic and linear terms
- build the rank-one LMI problem
- solve the LMI problem by relaxing the non-convex rank constraint

Relaxed LMI via duality

Consider again the original problem

$$\begin{array}{ll} \min & x^T Q x \\ \text{s.t.} & x_i^2 = 1 \end{array}$$

and build Lagrangian

$$\begin{array}{rcl} L(x,y) & = & x^T Q x - \sum_i y_i (x_i^2 - 1) \\ & = & x^T (Q - Y) x + \operatorname{trace} Y \end{array}$$

where Y is a diagonal matrix and $Q - Y \succeq 0$ must be enforced to ensure that Lagrangian is bounded below

Associated dual problem reads

$$\begin{array}{ll} \max & \operatorname{trace} Y \\ \text{s.t.} & Q - Y \succeq 0 \\ & Y \text{ diagonal} \end{array}$$

This is an LMI problem !

Relaxed LMI via duality

The dual LMI problem

$$\begin{array}{ll} \max & \text{trace}\, Y \\ \text{s.t.} & Q \succeq Y \\ & Y \text{ diagonal} \end{array}$$

has for dual the primal LMI problem

$$\begin{array}{ll} \min & \text{trace}\, QX \\ \text{s.t.} & X_{ii} = 1 \\ & X \succeq 0 \end{array}$$

which is Shor's original LMI relaxation !

More generally it can be shown that

LMI rank dropping
=
Lagrangian relaxation

Example of LMI relaxation

Original nonconvex 0-1 quadratic problem

$$\begin{array}{rcll} g^\star & = & \min & 2x_1x_2 + 4x_1x_3 + 6x_2x_3 \\ & & \text{s.t.} & x_i^2 = 1 \end{array} \qquad Q = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix}$$

Primal and dual LMI solutions

$$X = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \qquad Y = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

yield lower bound

$$p^\star = \operatorname{trace} QX = d^\star = \operatorname{trace} Y = -8$$

(strong duality holds here)

Since $\operatorname{rank} X = 1$ we recover here the optimum

$$x = [1 \;\; 1 \;\; -1]^T$$

such that $X = xx^T$ and hence

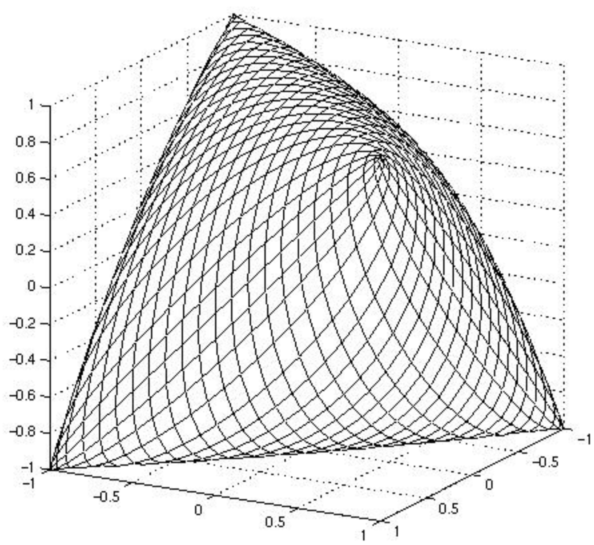
$$g^\star = p^\star = d^\star$$

the relaxation is exact !

Example of LMI relaxation

LMI relaxation of ± 1 constraints

$$X = \begin{bmatrix} 1 & X_{12} & X_{13} \\ X_{12} & 1 & X_{23} \\ X_{13} & X_{23} & 1 \end{bmatrix} \succeq 0\}$$



So we optimize the linear objective function

$$\text{trace}\, QX = 2X_{12} + 4X_{13} + 6X_{23}$$

and the optimal is a vertex $[1 \;\; -1 \;\; -1]$

How good are LMI relaxations ?

We have seen that we can obtain lower bounds for non-convex polynomial minimization with the help of liftings and relaxations

But can we measure the gap between the global optimum and the relaxed optimum ?

In other words

> How much conservative are LMI relaxations ?

Answers only in a (too) few specific cases..

LMI relaxation for MAXCUT

MAXCUT combinatorial optimization problem:

given a graph with arcs (i, j) with weights $a_{ij} \geq 0$
find a partition maximizing total weight of linking arcs

Non-convex quadratic problem

$$\begin{array}{rcl} g^\star & = & \max \ \frac{1}{4}\sum_{i,j} a_{ij}(1 - x_i x_j) \\ & & \text{s.t.} \ x_i^2 = 1 \end{array}$$

with convex LMI relaxation

$$\begin{array}{rcl} d^\star & = & \max \ \frac{1}{4}\sum_{i,j} a_{ij}(1 - X_{ij}) \\ & & \text{s.t.} \ X_{ii} = 1 \\ & & \quad X = X^T \succeq 0 \end{array}$$

With a geometric proof using randomization Goemans and Williamson showed in 1995 that

$$1 \geq \frac{g^\star}{d^\star} \geq 0.8786$$

independently of the data (graph) !

LMI relaxations for quadratic problems

Non-convex quadratic problem

$$\begin{array}{rcl} g^\star & = & \max \; x^T A x \\ & & \text{s.t.} \; x_i^2 = 1 \end{array}$$

with convex LMI relaxation

$$\begin{array}{rcl} d^\star & = & \max \; \text{trace}\, AX \\ & & \text{s.t.} \; X_{ii} = 1 \\ & & \quad\;\; X = X^T \succeq 0 \end{array}$$

For $A \succeq 0$ Nesterov showed recently that

$$1 \geq \frac{g^\star}{d^\star} \geq \frac{2}{\pi} = 0.6366$$

Beyond Shor's relaxation

Recent work (2000) to narrow relaxation gap

- gradually adding lifting variables
- hierarchy of nested LMI relaxations
- theoretical proof of convergence

Dual point of views:

- theory of moments (Lasserre)
- sum-of-squares decompositions (Parrilo)

Tradeoff between conservatism and computational effort

Polynomial multipliers

Polynomial optimization problem

$$\begin{array}{rcll} g^\star & = & \min & g_0(x) \\ & & \text{s.t.} & g_i(x) \geq 0,\ i = 1, \ldots, m \end{array}$$

where $g_i(x)$ are real-valued multivariate polynomials in vector indeterminate $x \in \mathbb{R}^n$

Non-convex problem in general (includes 0-1 or quadratic problems) = NP-hard

Since $g^\star$ is the global optimum, polynomial

$$g_0(x) - g^\star = q_0(x) + \sum_{i=1}^{m} g_i(x) q_i(x) \geq 0$$

is globally positive (non-negative)

Recall Lagrangian when building dual..

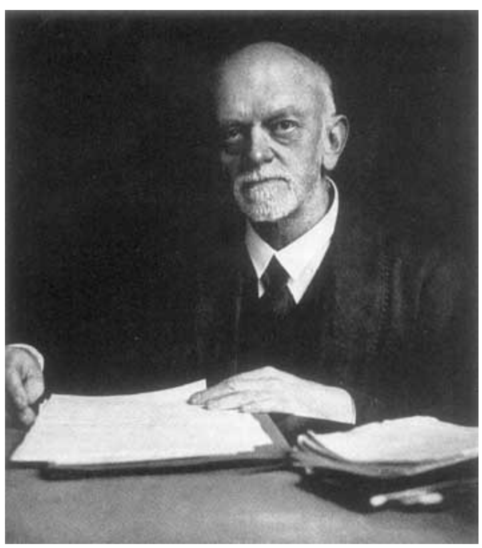
Multipliers $q_i(x)$ are now polynomials !

SOS polynomials

How can we ensure that a real valued polynomial is globally non-negative ?

$$p(x) \geq 0, \; \forall x \in \mathbb{R}^n$$

Hilbert's 17th pb about algebraic sum-of-squares decompositions of rational functions (Intl Congress of Mathematicians, Paris, 1900)



David Hilbert
(1862 Königsberg - 1943 Göttingen)

SOS polynomials

A form is a homogeneous polynomial
= all monomials have same degree

An obvious condition for a polynomial or form $p(x)$ to be non-negative is that is a sum-of-squares (SOS) of other polynomials

$$p(x) = \sum_i q_i^2(x)$$

Unfortunately, not every non-negative polynomial or form is SOS

$$p(x) \text{ SOS} \Longrightarrow p(x) \geq 0$$

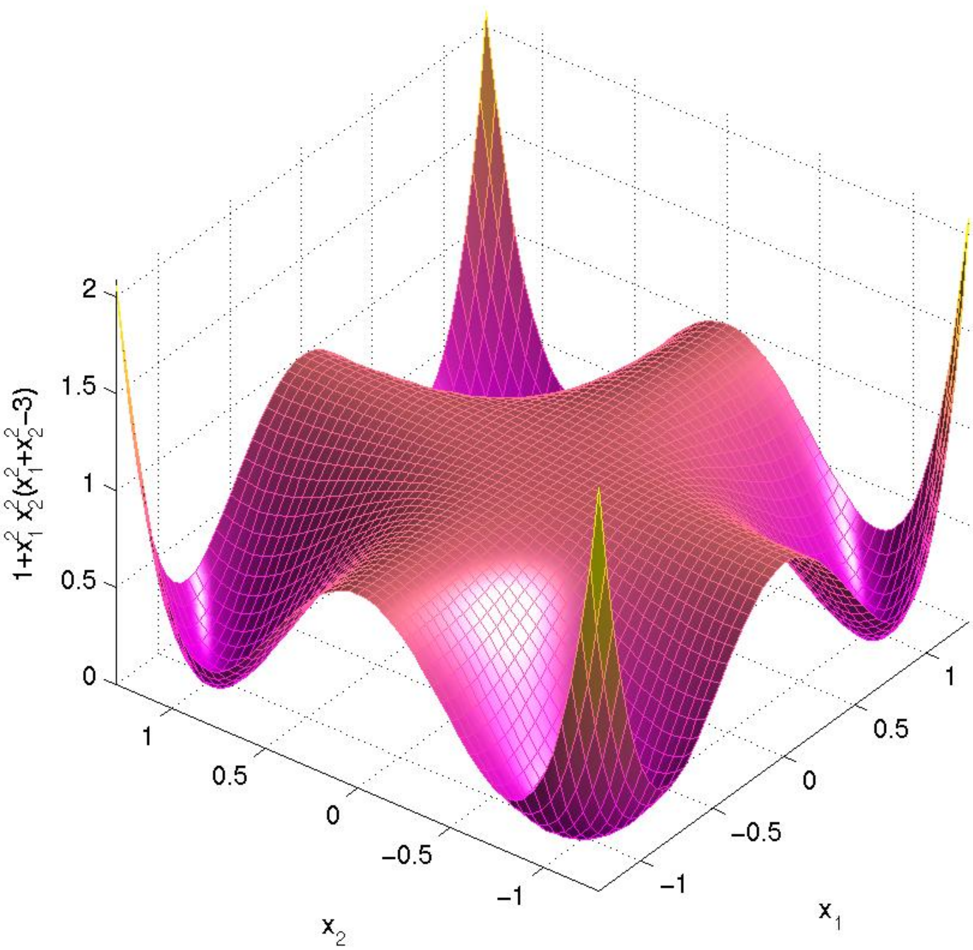
Sufficient non-negativity condition only..

Motzkin's polynomial

Counterexample:

$$p(x) = 1 + x_1^2 x_2^2 (x_1^2 + x_2^2 - 3)$$

cannot be written as an SOS but it is globally non-negative (vanishes at $|x_1| = |x_2| = 1$)



SOS polynomials

Let n denote the number of variables and d the degree

Non-negativity and SOS are sometimes equivalent:

$n = 2$	bivariate forms
	univariate polynomials (dehomogen)
$d = 2$	quadratic forms
$n = 3$, $d = 4$	quartic forms of 3 variables

In all other cases, the set of SOS polynomials (a cone) is a subset of the set of non-negative polynomials

Checking whether a polynomial is non-negative is NP-hard when $d \geq 4$

Note however that the set of SOS polynomials is dense in the set of polynomials nonnegative over the n-dimensional box $[-1, 1]^n$

Most importantly

The cone of SOS polynomials
is LMI representable

as we will see in the sequel..

LMI formulation of SOS polynomials

Polynomial

$$p(x) = \sum_\alpha p_\alpha x^\alpha$$

of degree $|\alpha| \leq 2d$ (α = vector of powers of indeterminates x) is SOS iff

$$p(x) = z^T X z \quad X \succeq 0$$

where z is a vector with all monomials with degree not greater than d

Cholesky factorization

$$X = Q^T Q$$

such that

$$\begin{aligned} p(x) &= z^T Q^T Q z = \|Qz\|_2 = \textstyle\sum_i (Qz)_i^2 \\ &= \textstyle\sum_i q_i^2(x) \end{aligned}$$

Number of squares $q_i^2(x)$ = rank X

LMI formulation of SOS polynomials

Comparing monomial coefficients in expression

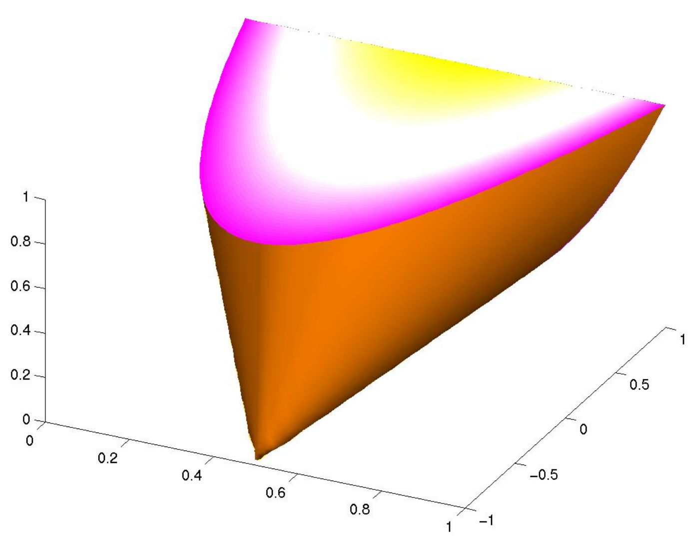
$$p(x) = z^T X z = \sum_\alpha p_\alpha x^\alpha \geq 0$$

we get an LMI

$$\begin{array}{l} \text{trace}\, H_\alpha X = p_\alpha \quad \forall \alpha \\ X \succeq 0 \end{array}$$

where H_α are Hankel-like matrices

SOS polynomials described by an intersection between a subspace and the PSD cone



SOS example

Consider the homogeneous form

$$\begin{aligned} p(x) &= 2x_1^4 + 5x_2^4 + 2x_1^3x_2 - x_1^2x_2^2 \\ &= z^T X z \end{aligned}$$

With monomial vector

$$z = \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1x_2 \end{bmatrix}$$

a general bivariate form of degree 4 reads

$$\begin{aligned} z^T X z &= X_{11}x_1^4 + X_{22}x_2^4 + 2X_{31}x_1^3x_2 \\ &\quad +2X_{32}x_1x_2^3 + (X_{33} + 2X_{21})x_1^2x_2^2 \end{aligned}$$

$p(x)$ SOS iff there exists $X \succeq 0$ such that

$$\begin{array}{ll} X_{11} = 2 & X_{22} = 5 \\ 2X_{31} = 2 & 2X_{32} = 0 \\ X_{33} + 2X_{21} = -1 & \end{array}$$

SOS example

One particular solution is

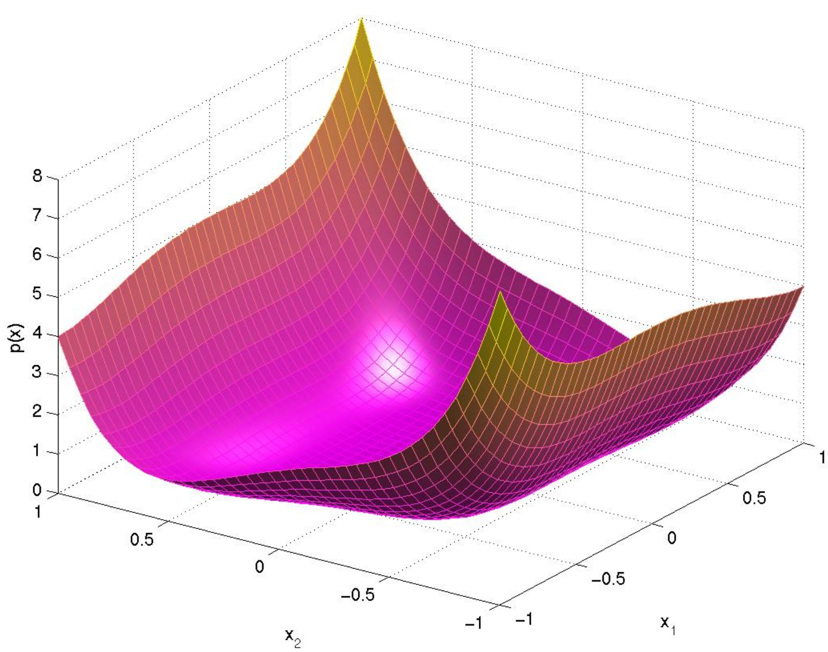
$$\color{red}{X} = \begin{bmatrix} 2 & -3 & 1 \\ -3 & 5 & 0 \\ 1 & 0 & 5 \end{bmatrix} = Q^T Q$$

with Cholesky factor

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

So $p(x)$ is the sum of rank $\color{red}{X} = 2$ squares

$$p(x) = \tfrac{1}{2}(2x_1^2 - 3x_2^2 + x_1 x_2)^2 + \tfrac{1}{2}(x_2^2 + 3x_1 x_2)^2$$



Parametrized SOS

Consider the 4th-degree univariate polynomial

$$p(x) = 1 + 2ax + x^2 + 2bx^3 + x^4$$

parametrized in $a,\ b \in \mathbb{R}^2$

Which values of a and b make $p(x)$ non-negative or equivalently SOS ?

Primal LMI

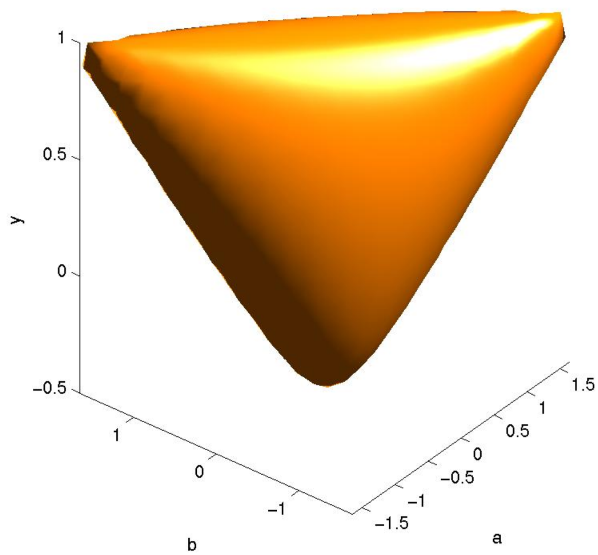
$$\begin{array}{l} \text{trace}\, H_i X = p_i(a,b) \\ X \succeq 0 \end{array}$$

with H_i Hankel matrices, and corresponding reduced LMI (null-space parametrization)

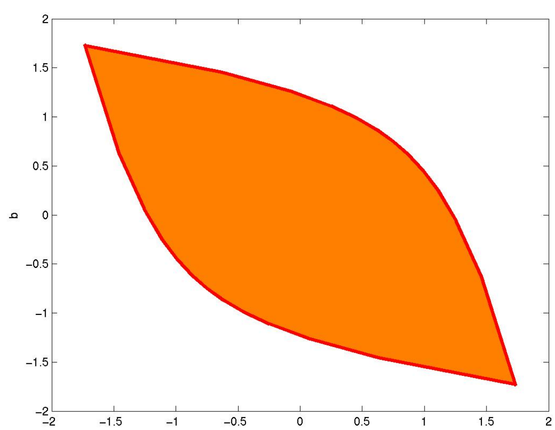
$$\left[\begin{array}{ccc} 1 & a & -y \\ a & 1+2y & b \\ -y & b & 1 \end{array}\right] \succeq 0$$

Parametrized SOS (2)

For $y = 0$, $p(x)$ is SOS iff $a^2 + b^2 \leq 1$



For other values, LMI set in 3D space $(a,\, b,\, y)$



Projection in the plane $(a,\, b)$

Finding polynomial multipliers

Returning to our global optimization problem

$$g^\star = \min \; g_0(x) \quad \text{s.t.} \; g_i(x) \geq 0, \; i = 1, \ldots, m$$

the problem of finding polynomial multipliers $q_i(x)$ such that

$$p(x) = g_0(x) - g^\star = q_0(x) + \sum_{i=1}^{m} g_i(x) q_i(x) \geq 0$$

is SOS can be formulated as an LMI as soon as the degrees of the $q_i(x)$ are fixed

Depending on parity let $\deg p(x) = 2k - 1$ or $2k$ - then the LMI problem of finding an SOS $p(x)$ is referred to as the

LMI relaxation of order k

LMI relaxations: illustration

Non-convex quadratic problem

$$
\begin{array}{ll}
\text{min} & h_0(x) = -2x_1^2 - 2x_2^2 + 2x_1x_2 + 2x_1 + 6x_2 - 10 \\
\text{s.t.} & g_1(x) = -x_1^2 + 2x_1 \geq 0 \\
 & g_2(x) = -x_1^2 - x_2^2 + 2x_1x_2 + 1 \geq 0 \\
 & g_3(x) = -x_2^2 + 6x_2 - 8 \geq 0.
\end{array}
$$

LMI relaxation built by replacing each monomial $x_1^i x_2^j$ with lifting variable y_{ij}

For example, quadratic expression

$$g_2(x) = -x_1^2 - x_2^2 + 2x_1x_2 + 1 \geq 0$$

is replaced with linear expression

$$-y_{20} - y_{02} + 2y_{11} + 1 \geq 0$$

Lifting variables y_{ij} satisfy non-convex relations such as $y_{10}y_{01} = y_{11}$ or $y_{20} = y_{10}^2$

LMI relaxations: illustration (2)

Relax these non-convex relations by enforcing LMI constraint

$$M_1(y) = \left[\begin{array}{c|cc} 1 & y_{10} & y_{01} \\ \hline y_{10} & y_{20} & y_{11} \\ y_{01} & y_{11} & y_{02} \end{array}\right] \succeq 0$$

Moment matrix of first order
relaxing monomials of degree up to 2

You have recognized Shor's relaxation !

First LMI (=Shor's) relaxation of original global optimization problem is given by

$$\begin{array}{ll} \min & -2y_{20} - 2y_{02} + 2y_{11} + 2y_{10} + 6y_{01} - 10 \\ \text{s.t.} & -y_{20} + 2y_{10} \geq 0 \\ & -y_{20} - y_{02} + 2y_{11} + 1 \geq 0 \\ & -y_{02} + 6y_{01} - 8 \geq 0 \\ & M_1(y) \succeq 0 \end{array}$$

LMI relaxations: illustration (3)

To build second LMI relaxation, we must increase size of moment matrix so that it captures expressions of degrees up to 4

Second order moment matrix reads

$$M_2(y) = \left[\begin{array}{c|cc|ccc} 1 & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ \hline y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ \hline y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{array}\right] \succeq 0$$

Constraints are localized on moment matrices, meaning that original constraint

$$g_1(x) = -x_1^2 + 2x_1 \geq 0$$

becomes localizing matrix constraint

$$M_1(g_1 y) = \left[\begin{array}{c|cc} -y_{20}+2y_{10} & -y_{30}+2y_{20} & -y_{21}+2y_{11} \\ \hline -y_{30}+2y_{20} & -y_{40}+2y_{30} & -y_{31}+2y_{21} \\ -y_{21}+2y_{11} & -y_{31}+2y_{21} & -y_{22}+2y_{12} \end{array}\right] \succeq 0$$

LMI relaxations: illustration (3)

Second LMI feasible set included in first LMI feasible set, thus providing a tighter relaxation

$$\begin{array}{ll} \min & -2y_{20} - 2y_{02} + 2y_{11} + 2y_{10} + 6y_{01} - 10 \\ \text{s.t.} & M_1(g_1 y) \succeq 0, \; M_1(g_2 y) \succeq 0, \; M_1(g_3 y) \succeq 0 \\ & M_2(y) \succeq 0 \end{array}$$

Similary, we can build up 3rd, 4th, 5th LMI relaxations..

Hierarchy of LMI relaxations

The LMI relaxation of order k reads

$$
\begin{array}{rcll}
d_k^\star & = & \min & \sum_\alpha (g_0)_\alpha y_\alpha \\
 & & \text{s.t.} & M_k(y) = \sum_\alpha A_\alpha y_\alpha \succeq 0 \\
 & & & M_{k-d_i}(g_i y) = \sum_\alpha A_\alpha^{g_i} y_\alpha \succeq 0 \quad \forall i
\end{array}
$$

with $y_0 = 1$ (normalization)
d_i is half the degree of $g_i(x)$
$M_k(y)$ is the moment matrix
$M_{k-d_i}(g_i y)$ are the localization matrices

The dual LMI

$$
\begin{array}{rcll}
p_k^\star & = & \max & \sum_\alpha \text{trace}\, A_0 X + \sum_i \text{trace}\, A_0^{g_i} X_i \\
 & & \text{s.t.} & \text{trace}\, A_\alpha X \\
 & & & + \sum_i \text{trace}\, A_\alpha^{g_i} X_i = (g_0)_\alpha \quad \forall \alpha \neq 0
\end{array}
$$

corresponds to $p(x)$ SOS

Hierarchy of LMI relaxations

If feasible set $g_i(x) \geq 0$ is compact, and under mild additional assumptions, Lasserre proved in 2000 that

$$p_k^\star = d_k^\star \leq g^\star$$

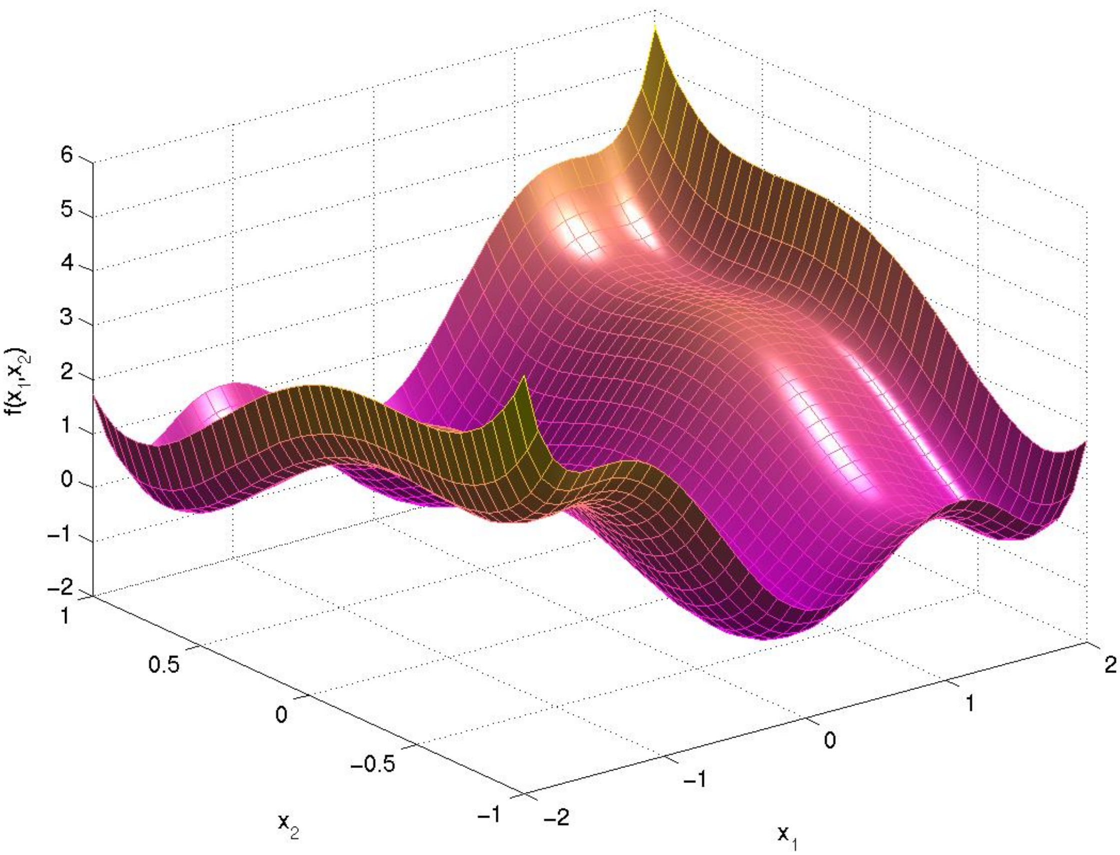
with asymptotic convergence guarantee

$$\lim_{k\to\infty} p_k^\star = g^\star$$

Moreover, in practice, convergence is fast: $p_k^\star$ is very close to $g^\star$ for small k

Camelback function

For the well-known six-hump camelback function



with two global optima and six local optima, the global optimum is reached at the first LMI relaxation ($k = 1$) without any problem splitting

LMI hierarchy: example

Quadratic problem

$$\begin{array}{ll} \min & -2x_1 + x_2 - x_3 \\ \text{s.t.} & x_1(4x_1 - 4x_2 + 4x_3 - 20) + x_2(2x_2 - 2x_3 + 9) \\ & \quad +x_3(2x_3 - 13) + 24 \geq 0 \\ & x_1 + x_2 + x_3 \leq 4, \quad 3x_2 + x_3 \leq 6 \\ & 0 \leq x_1 \leq 2, \quad 0 \leq x_2, \quad 0 \leq x_3 \leq 3. \end{array}$$

Number of LMI variables (M) and size of relaxed LMI problem (N) increase quickly with relaxation order:

Relaxation	LMI opt	M	N
1	-6.0000	9	24
2	-5.6923	34	228
3	-4.0685	83	1200
4	-4.0000	164	4425
5	-4.0000	285	12936
6	-4.0000	454	32144

..yet fourth LMI relaxation solves globally the problem

Complexity

d: overall polynomial degree ($2\delta = d$ or $d+1$)
m: number of polynomial constraints
n: number of polynomial variables
M: number of LMI decision variables
N: size of LMI

$$\begin{aligned} M &= \binom{n+2\delta}{2\delta} - 1 \\ N &= \binom{n+\delta}{\delta} + m\binom{n+\delta-1}{\delta-1} \end{aligned}$$

When n is fixed:

- M grows polynomially in $O(\delta^n)$
- N grows polynomially in $O(m\delta^n)$

LMI relaxations: conclusion

LMI relaxations prove useful to solve general non-convex polynomial optimization problems

Shor's relaxation = rank dropping = Lagrangian relaxation = first order LMI relaxation

Sometimes one can measure the gap between the original problem and its relaxation

A hierarchy of successive LMI relaxations can be built with additional lifting variables and constraints

Theoretical guarantee of asymptotic convergence to global optimum without any problem splitting (no branch and bound scheme)