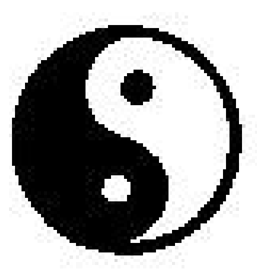
COURSE ON LMI OPTIMIZATION
WITH APPLICATIONS IN CONTROL
PART I.2

# Lagrangian and SDP duality


Didier HENRION
www.laas.fr/∼henrion
henrion@laas.fr

Denis ARZELIER
www.laas.fr/∼arzelier
arzelier@laas.fr

May 2004

# Duality

- Versatile notion
- Theoritical results and numerical methods
- Certificates of infeasibility

Lagrangian duality has many applications and interpretations (price or tax, game, geometry...)

Applications of SDP duality:

- numerical solvers design
- problems reduction
- new theoretical insights into control problems

In the sequel we will recall some basic facts about Lagrangian duality and SDP duality

## Lagrangian duality

Let the primal problem

$$
\begin{array}{lll}
p^\star = & \min_{x \in \mathbb{R}^n} & f_0(x) \\
& \text{s.t.} & f_i(x) \leq 0 \quad i = 1, \cdots, m \\
& & h_i(x) = 0 \quad i = 1, \cdots, p
\end{array}
$$

Define Lagrangian $L(.,.,.)\ \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$

$$L(x, \lambda, \mu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \mu_i h_i(x)$$

where $\lambda$, $\mu$ are Lagrange multipliers vectors or dual variables

Let the Lagrange dual function

$$g(\lambda, \mu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \mu)$$

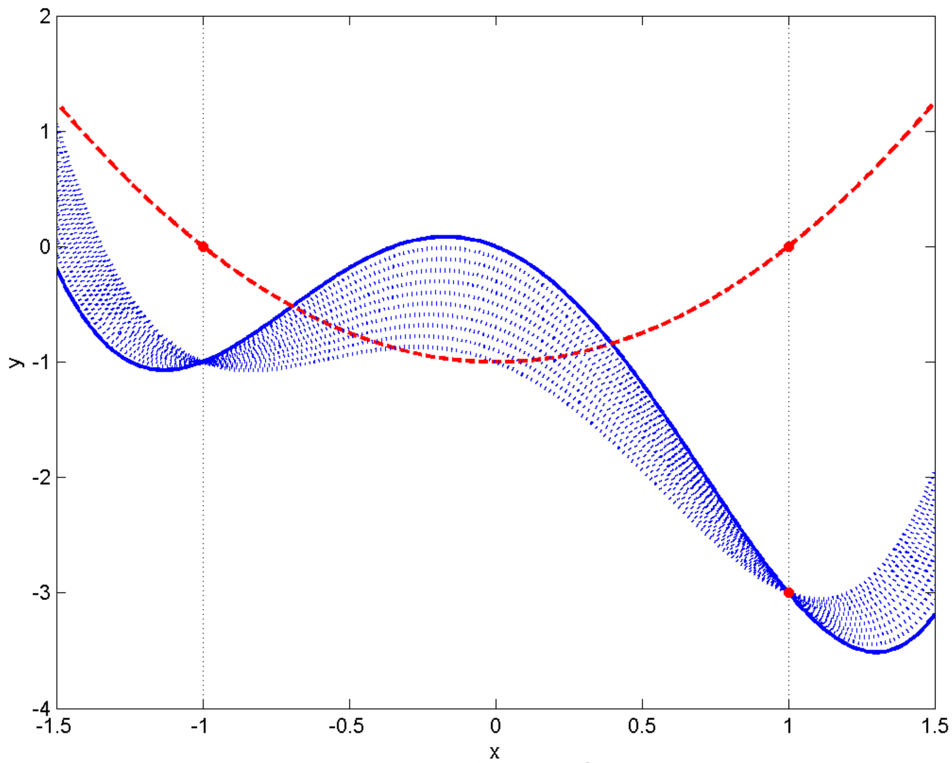
- $g$ is always concave
- $g(\lambda, \mu) = -\infty$ if there is no finite infimum

## Lagrangian duality (2)

A pair $(\lambda, \mu)$ s.t. $\lambda \succeq \mathbf{0}$ and $g(\lambda, \mu) > -\infty$ is dual feasible

For any primal feasible $x$ and dual feasible pair $(\lambda, \mu)$

$$g(\lambda, \mu) \leq p^* \leq f_0(x)$$




$$\min_x \quad x^4 - 3x^2 - x$$
$$\text{under} \quad x^2 - 1 \leq 0$$

# Lagrangian duality (3)

Lagrange dual problem

$$d^\star = \max_{\lambda,\mu} \; g(\lambda,\mu) \quad \text{s.t.} \quad \lambda \succeq \mathbf{0}$$

The Lagrange dual problem is a convex optimization problem

| Primal | Dual |
|---|---|
| $\inf_{x\in\mathbb{R}^n} \; \sup_{\lambda,\mu} \; L(x,\lambda,\mu)$ s.t. $\lambda \succeq \mathbf{0}$ | $\sup_{\lambda,\mu} \; \inf_{x\in\mathbb{R}^n} \; L(x,\lambda,\mu)$ s.t. $\lambda \succeq \mathbf{0}$ |

A Lagrangian relaxation consists in solving the dual problem instead of the primal problem

## Weak and strong duality

Weak duality (max-min inequality):

$$p^\star \geq d^\star$$

because

$$g(\lambda, \mu) \leq f_0(x) + \sum_{i=1}^{m} \lambda_i \underbrace{f_i(x)}_{\leq 0} + \sum_{i=1}^{p} \mu_i \underbrace{h_i(x)}_{=0} \leq f_0(x)$$

for any primal feasible $x$ and dual feasible $\lambda$, $\mu$

The difference $p^\star - d^\star \geq 0$ is called duality gap

Strong duality (saddle-point property):

$$p^\star = d^\star$$

Sometimes, constraint qualifications ensure that strong duality holds
Example: Slater's condition = strictly feasible convex primal problem

# Geometric interpretation of duality

Consider the primal optimization problem

$$\begin{array}{rcll} p^\star & = & \min\limits_{x \in \mathbb{R}} & f_0(x) \\ & & \text{s.t.} & f_1(x) \leq 0 \end{array}$$

with Lagrangian and dual function

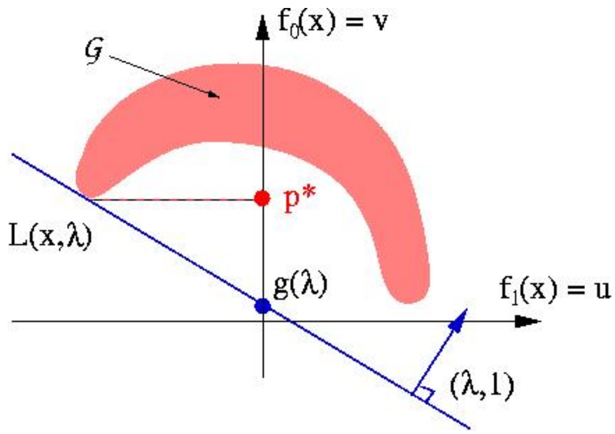
$$L(x, \lambda) = f_0(x) + \lambda f_1(x) \quad g(\lambda) = \inf_x L(x, \lambda)$$

The dual problem:

$$\begin{array}{rcll} d^\star & = & \max\limits_{\lambda} & g(\lambda) \\ & & \text{s.t.} & \lambda \succeq \mathbf{0} \end{array}$$

## Geometric interpretation of duality (2)

Set of values $\mathcal{G} = (f_1(x),\ f_0(x)),\ \forall\ x \in \mathcal{D}$



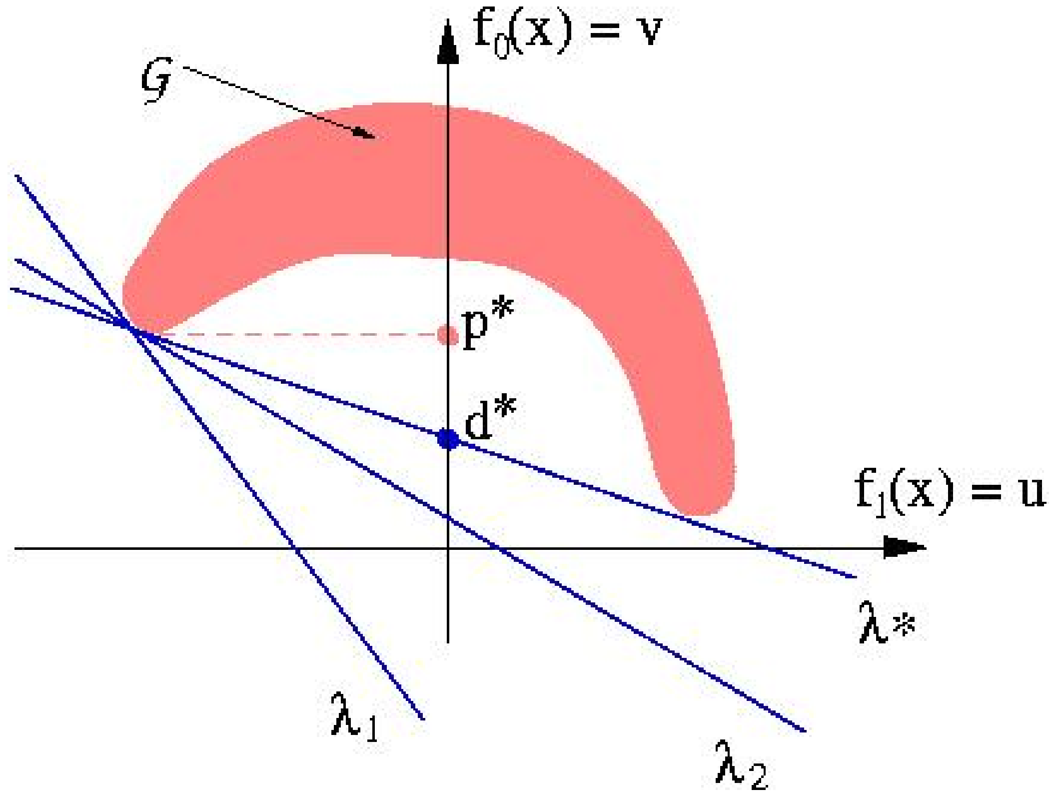

$$L(x,\lambda) = f_0(x) + \lambda f_1(x) = \begin{bmatrix} \lambda & 1 \end{bmatrix} \begin{bmatrix} f_1(x) \\ f_0(x) \end{bmatrix}$$

$$g(\lambda) = \inf_{x \in \mathcal{D}} L(\lambda, x) = \inf_{x \in \mathcal{D}} \left\{ \begin{bmatrix} \lambda & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \ (u,v) \in \mathcal{G} \right\}$$

Supporting hyperplane with slope $-\lambda$

$$\begin{bmatrix} \lambda & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \geq g(\lambda)\ (u,v) \in \mathcal{G}$$

## Geometric interpretation of duality (3)




Three supporting hyperplanes, including the optimum $\lambda^\star$ yielding $d^\star < p^\star$

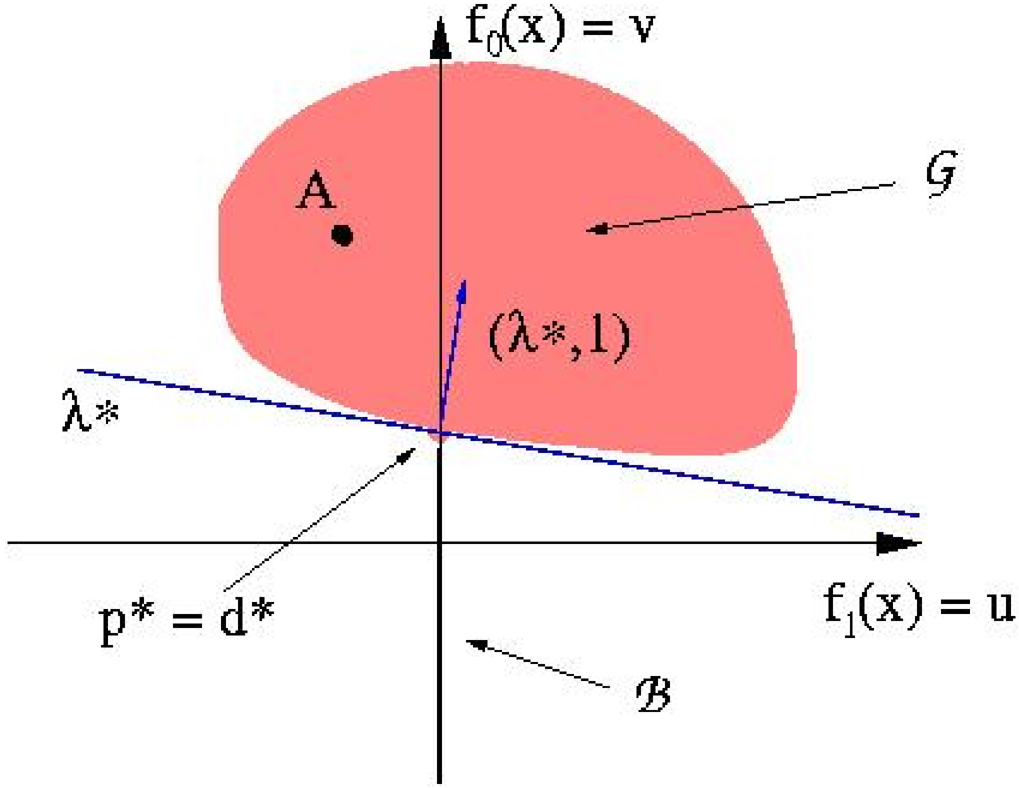
No strong duality here

$$p^* - d^* > 0$$

Duality gap $\neq 0$

# Geometric interpretation of duality (4)

$$\mathcal{B} = \{(0, s) \in \mathbb{R} \times \mathbb{R} \ : \ s < p^*\}$$




- Separating hyperplane theorem for $\mathcal{G}$ and $\mathcal{B}$
- The separating hyperplane is a supporting hyperplane to $\mathcal{G}$ in $(0, p^*)$
- Slater's condition ensures the hyperplane is non vertical

## Optimality conditions

Suppose that strong duality holds, let $x^\star$ be primal optimal and $(\lambda^\star, \mu^\star)$ be dual optimal,

$$
\begin{aligned}
f_0(x^\star) &= g(\lambda^\star, \mu^\star) \\
&= \inf_x \left( f_0(x) + \sum_{i=1}^m \lambda_i^\star f_i(x) + \sum_{i=1}^p \mu_i^\star h_i(x) \right) \\
&\le f_0(x^\star) + \sum_{i=1}^m \lambda_i^\star f_i(x^\star) + \sum_{i=1}^p \mu_i^\star h_i(x^\star) \\
&\le f_0(x^\star)
\end{aligned}
$$

$$\lambda_i^\star f_i(x^\star) = 0 \quad i = 1, \cdots, m$$

This is complementary slackness condition

$$\lambda_i^\star > 0 \;\Rightarrow\; f_i(x^\star) = 0 \quad \text{or} \quad f_i(x^\star) < 0 \;\Rightarrow\; \lambda_i^\star = 0$$

In words, the $i$th optimal Lagrange multiplier is zero unless the $i$th constraint is active at the optimum

# LP duality

Primal LP (standard form):

$$p^\star = \min_{x \in \mathbb{R}^n} \; c'x \quad \text{s.t.} \quad Ax = b \quad b \in \mathbb{R}^p, \quad x \succeq \mathbf{0}$$

Lagrange dual function:

$$g(\lambda, \mu) = \inf_{x \in \mathcal{D}} (c'x + \mu'(b - Ax) - \lambda' x)$$

$$= \begin{cases} b'\mu & \text{if } c - A'\mu - \lambda = \mathbf{0} \\ -\infty & \text{otherwise} \end{cases}$$

Lagrange dual problem:

$$\max_{\lambda \in \mathbb{R}^n} \; g(\lambda, \mu) = \begin{cases} b'\mu & \text{if } c - A'\mu - \lambda = \mathbf{0} \\ -\infty & \text{otherwise} \end{cases} \quad \text{s.t.} \quad \lambda \succeq \mathbf{0}$$

# LP duality (2)

Dual LP:

$$
\begin{aligned}
d^\star \;=\; & \max_{\mu \in \mathbb{R}^p} \quad b'\mu \\
& \text{s.t.} \quad \lambda = c - A'\mu \succeq \mathbf{0}
\end{aligned}
$$

Complementary slackness:

$$(x^\star)'\lambda^\star = 0$$

If primal (dual) is feasible then strong duality holds

Strong duality fails for LPs when both dual and primal are infeasible

$$
\begin{aligned}
& \min_{x} \quad x \\
& \text{s.t.} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} x \preceq \begin{bmatrix} -1 \\ 1 \end{bmatrix}
\end{aligned}
$$

## KKT optimality conditions

$f_i$, $h_i$ are differentiable and strong duality holds

$$
\begin{gathered}
h_i(x^\star) = 0,\ i = 1, \cdots, p, \quad \text{(primal feasible)} \\
f_i(x^\star) \leq 0,\ i = 1, \cdots, m,\ \text{(primal feasible)} \\
\lambda_i^\star \succeq \mathbf{0},\ i = 1, \cdots, m,\ \text{(dual feasible)} \\
\lambda_i^\star f_i(x^\star) = 0,\ i = 1, \cdots, m,\ \text{(complementary)} \\
\nabla f_0(x^\star) + \sum_{i=1}^{p} \lambda_i^\star \nabla f_i(x^\star) + \sum_{i=1}^{p} \mu_i^\star \nabla h_i(x^\star) = 0
\end{gathered}
$$

Necessary Karush-Kuhn-Tucker conditions satisfied by any primal and dual optimal pair $x^\star$ and $(\lambda^\star, \mu^\star)$

For convex problems, KKT conditions are also sufficient

# History of KKT conditions

"Nonlinear programming" paper written jointly by Albert W. Tucker and Harold W. Kuhn (Princeton Univ) launched the theory of NLP in 1950

Later on, it turned out that this theorem had been proved already:

• First in 1939 in a MSc thesis by William Karush supervised by Lawrence M. Graves (Univ Chicago)

• Second in 1948 by Fritz John in a paper rejected by the Duke Math J, later on published in a collection of essays for Richard Courant's 60th birthday

## Feasibility of inequalities

$$\exists\ x \in \mathbb{R}^n \ : \ \begin{cases} f_i(x) \leq 0 & i = 1, \cdots, m \\ h_i(x) = 0 & i = 1, \cdots, p \end{cases}$$

Dual function: $g(.,.) \ : \ \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$

$$g(\lambda, \mu) = \inf_{x \in \mathcal{D}} \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \mu_i h_i(x)$$

The dual feasibility problem is

$$\exists\ (\lambda, \mu) \in \mathbb{R}^m \times \mathbb{R}^p \ : \ \begin{cases} g(\lambda, \mu) > 0 \\ \lambda \succeq \mathbf{0} \end{cases}$$

Theorem of weak alternatives

At most, one of the two (primal and dual) is feasible

## Feasibility of inequalities (2)

If $f_i$ are convex functions, $h_i$ are affine functions and some type of constraint qualification holds:

Theorem of strong alternatives

Exactly one of the two alternative holds

A dual feasible pair $(\lambda, \mu)$ gives a certificate (proof) of infeasibility of the primal

Example of Farkas'lemma

$$\exists\ x \in \mathbb{R}^n \ : \ \begin{cases} Ax \preceq \mathbf{0} \\ c'x < 0 \end{cases}$$

$$\exists\ \lambda \in \mathbb{R}^m \ : \ \begin{cases} A'\lambda + c = 0 \\ \lambda \succeq \mathbf{0} \end{cases}$$

# Conic duality

Let the primal:

$$
\begin{array}{rcll}
p^\star & = & \min\limits_{x \in \mathbb{R}^n} & f_0(x) \\
& & \text{s.t.} & f_i(x) \preceq_{\mathcal{K}_i} \mathbf{0} \quad i = 1, \cdots m
\end{array}
$$

Lagrange dual function: $g(.) \ : \ \mathbb{R}^m \to \mathbb{R}$

$$
g(\lambda) = \inf_{x \in \mathcal{D}} f_0(x) + \sum_{i=1}^{m} \lambda_i' f_i(x)
$$

Lagrange dual problem:

$$
\begin{array}{rcll}
d^\star & = & \max\limits_{\lambda \in \mathbb{R}^m} & g(\lambda) \\
& & \text{s.t.} & \lambda_i \succeq_{\mathcal{K}_i^*} \mathbf{0}, \quad i = 1, \cdots, m
\end{array}
$$

# Conic duality (2)

- Weak duality
- Strong duality:

- if primal is s.f. with finite $p^\star$ then $p^\star = d^\star$ reached by dual

- if dual is s.f. with finite $d^\star$ then $p^\star = d^\star$ reached by primal

- if primal and dual are s.f. then $p^\star = d^\star$

- Complementary slackness:

$$\begin{array}{l} \lambda_i^{\star\prime} f_i(x^\star) = 0 \\ \lambda_i^\star \succ_{K_i^\star} \mathbf{0} \Rightarrow f_i(x^\star) = 0 \\ f_i(x^\star) \prec_{K_i} \mathbf{0} \Rightarrow \lambda_i^\star = \mathbf{0} \end{array}$$

- KKT conditions:

$$\begin{array}{l} f_i(x^\star) \preceq_{\mathcal{K}_i} \mathbf{0} \\ \lambda_i^\star \succeq_{\mathcal{K}_i^\star} \mathbf{0} \\ \nabla f_0(x^\star) + \sum_{i=1}^{m} \nabla f_i(x^\star)' \lambda_i^\star = \mathbf{0} \end{array}$$

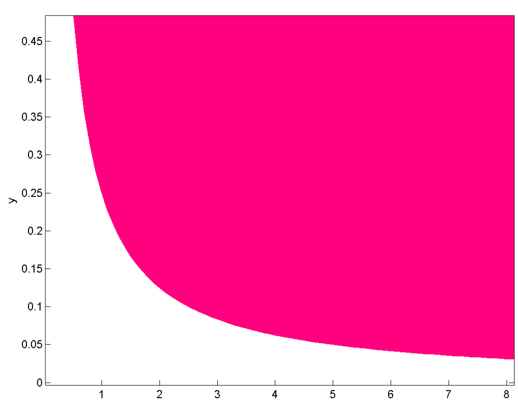
# Example of conic duality

**Example**
Consider the primal conic program

$$\begin{aligned} \min \quad & x_1 \\ \text{s.t.} \quad & \begin{bmatrix} x_1 - x_2 \\ 1 \\ x_1 + x_2 \end{bmatrix} \succeq_{\mathbb{L}_3} \mathbf{0} \Leftrightarrow \begin{array}{l} x_1 + x_2 > 0 \\ 4x_1x_2 \geq 1 \end{array} \end{aligned}$$

with dual

$$\begin{aligned} \max \quad & -\lambda_2 \\ \text{s.t.} \quad & \begin{cases} \lambda_1 + \lambda_3 = 1 \\ -\lambda_1 + \lambda_3 = 0 \\ \lambda \in \mathbb{L}_3 \end{cases} \Leftrightarrow \begin{array}{l} \lambda_1 = \lambda_3 = 1/2 \\ 1/2 \geq \sqrt{1/2 + \lambda_2{}^2} \end{array} \end{aligned}$$




The primal is strictly feasible and bounded below with $p^\star = 0$ which is not reached since dual problem is infeasible $d^\star = -\infty$

## SDP duality

Primal SDP:

$$\begin{array}{rcll} p^\star & = & \min_{x \in \mathbb{R}^n} & c'x \\ & & \text{s.t.} & F_0 + \sum_{i=1}^n x_i F_i \preceq \mathbf{0} \end{array}$$

Lagrange dual function:

$$\begin{array}{rcl} g(Z) & = & \inf_{x \in \mathcal{D}} \left( c'x + \text{tr}\, ZF(x) \right) \\ & = & \begin{cases} \text{tr}\, F_0 Z & \text{if tr}\, F_i Z + c_i = 0 \quad i = 1, \cdots, n \\ -\infty & \text{otherwise} \end{cases} \end{array}$$

Dual SDP:

$$\begin{array}{rcll} d^\star & = & \max_{Z \in \mathbb{S}_m} & \text{tr}\, F_0 Z \\ & & \text{s.t.} & \text{tr}\, F_i Z + c_i = 0 \quad i = 1, \cdots, n \\ & & & Z \succeq \mathbf{0} \end{array}$$

Complementary slackness:

$$\text{tr}\, F(x^\star) Z^\star = 0 \iff F(x^\star) Z^\star = Z^\star F(x^\star) = \mathbf{0}$$

# Example of SDP duality gap

**Example**
Consider the primal semidefinite program

$$\begin{array}{ll} \min & x_1 \\ \text{s.t.} & \begin{bmatrix} 0 & x_1 & 0 \\ x_1 & -x_2 & 0 \\ 0 & 0 & -1-x_1 \end{bmatrix} \preceq \mathbf{0} \end{array}$$

with dual

$$\begin{array}{ll} \max & -z_6 \\ \text{s.t.} & \begin{bmatrix} z_1 & (1-z_6)/2 & z_4 \\ (1-z_6)/2 & 0 & z_5 \\ z_4 & z_5 & z_6 \end{bmatrix} \succeq 0 \end{array}$$

In the primal necessarily $x_1 = 0$ ($x_1$ appears in a row with zero diagonal entry) so the primal optimum is

$$x_1^\star = 0$$

Similarly, in the dual necessarily $(1 - z_6)/2 = 0$ so the dual optimum is

$$z_6^\star = 1$$

There is a nonzero duality gap here $(p^\star = 0) > (d^\star = -1)$

# Conic theorem of alternatives

$$f_i(x) \preceq_{\mathcal{K}_i} \mathbf{0} \qquad \mathcal{K}_i \subseteq \mathbb{R}^{k_i}$$

Lagrange dual function

$$g(\lambda) = \inf_{x \in \mathcal{D}} \sum_{i=1}^{m} \lambda_i' f_i(x) \quad \lambda_i \in \mathbb{R}^{k_i}$$

Weak alternatives:

$$1- \quad f_i(x) \preceq_{\mathcal{K}_i} \mathbf{0} \quad i = 1, \cdots, m$$

$$2- \quad \lambda_i \succeq_{\mathcal{K}_i^\star} \mathbf{0} \qquad g(\lambda) > 0$$

Strong alternatives:

$f_i$ $\mathcal{K}_i$-convex and $\exists\ x \in$ relint$\mathcal{D}$

$$1- \quad f_i(x) \prec_{\mathcal{K}_i} \mathbf{0} \quad i = 1, \cdots, m$$

$$2- \quad \lambda_i \succeq_{\mathcal{K}_i^\star} \mathbf{0} \qquad g(\lambda) \geq 0$$

## Theorem of alternatives for LMIs

For the LMI feasible set

$$F(x) = F_0 + \sum_i x_i F_i \prec \mathbf{0}$$

Exactly one statement is true
1- $\exists\ x$ s.t. $F(x) \prec 0$
2- $\exists\ \mathbf{0} \neq Z \succeq 0$ s.t.
trace $F_0 Z \succeq \mathbf{0}$ and trace $F_i Z = \mathbf{0}$ for $i = 1, \cdots, n$

Useful for giving certificate of infeasibility of LMIs

Rich literature on theorems of alternatives for generalized inequalities, e.g. nonpolyhedral convex cones

Elegant proofs of standard results (Lyapunov, ARE) in linear systems control, see later..

## S-procedure

S-procedure: also frequently useful in robust and nonlinear control, also an outcome of the theorem of alternatives

if there exist real numbers $\lambda_i \geq 0$ such that

$$\sum_{i=1}^{p} \lambda_i A_i \prec 0$$

then

$$\exists\ x \neq 0 \in \mathbb{C}^n \text{ s.t. } x' A_i x \geq 0,\ i = 1, \ldots, p$$

The converse also holds (no duality gap)

- when $p = 1$ for real quadratic forms (from the theorem of alternatives)
- when $p = 2$ for complex quadratic forms

# Finsler's theorem

Finsler's theorem: a very useful trick in robust control, directly follows from the theorem of alternatives

The following statements are equivalent

$x^\star A x > 0$ for all $x \neq 0$ s.t. $Hx = 0$
$\tilde{H}^\star A \tilde{H} \succ 0$ where $H\tilde{H} = 0$
$A + \lambda H^\star H \succ 0$ for some scalar $\lambda$
$A + XH + H^\star X^\star \succ 0$ for some matrix $X$

1 − $\exists\ \tau \in \mathbb{R} \mid \tau H H^* + A \succ \mathbf{0}$
2 − $\exists\ Z \in \mathbb{S}^n_+$ : $\mathrm{tr}(H^* Z H) = 0$ and $\mathrm{tr}(AZ) \leq 0$

3 − $\exists\ X \in \mathbb{C}^{m\times n} \mid HX + (XH)^* + A \succ \mathbf{0}$
4 − $\exists\ Z \in \mathbb{S}^n_+$ : $ZH = 0$ and $\mathrm{tr}(AZ) \geq 0$

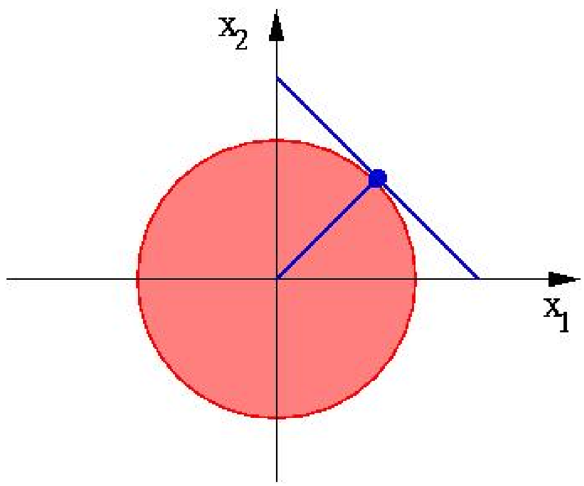
# Reformulations

Linear LMI constraint = projection in subspace

Using explicit subspace basis, more efficient formulations (less decision variables) can be obtained

Example: original problem

$$\begin{array}{ll} \max & 2x_1 + 2x_2 \\ \text{s.t.} & \left[\begin{array}{cc} 1+x_1 & x_2 \\ x_2 & 1-x_1 \end{array}\right] \succeq 0 \end{array}$$




with dual

$$\begin{array}{ll} \min & \text{trace} \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right] Z \\ \text{s.t.} & \text{trace} \left[\begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array}\right] Z = 2 \\ & \text{trace} \left[\begin{array}{cc} 0 & -1 \\ -1 & 0 \end{array}\right] Z = 2 \\ & Z \succeq 0 \end{array}$$

# Reformulations (2)

Denoting

$$Z = \begin{bmatrix} z_{11} & z_{21} \\ z_{21} & z_{22} \end{bmatrix}$$

the linear trace constraints on $Z$ can be written

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} z_{11} \\ z_{21} \\ z_{22} \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Particular solution and explicit null-space basis

$$\begin{bmatrix} z_{11} \\ z_{21} \\ z_{22} \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \bar{z}$$

so we obtain the equivalent dual problem
with less variables

$$\begin{array}{ll} \min & 2\bar{z} \\ \text{s.t.} & \begin{bmatrix} \bar{z}-1 & -1 \\ -1 & \bar{z}+1 \end{bmatrix} \succeq 0 \end{array}$$

and primal

$$\begin{array}{ll} \min & \text{trace} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \bar{X} \\ \text{s.t.} & \text{trace} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bar{X} = 2 \\ & \bar{X} \succeq 0 \end{array}$$