Links Between Robust and Quadratic Stability of Uncertain Discrete-Time Polynomials[1]

DIDIER HENRION[2,3]
MICHAEL ŠEBEK[4,5]
VLADIMÍR KUČERA[4,5]

Abstract

An uncertain polynomial is robustly stable, or stable in the sense of Kharitonov, if it is stable for any admissible value of the uncertainty, provided the uncertainty is not varying. The same polynomial is quadratically stable, or stable in the sense of Lyapunov, if it is stable for any admissible value of the uncertainty, regardless of whether the uncertainty is varying or not. In this paper, relationships between robust and quadratic stability of discrete-time uncertain polynomials are studied.

1 Introduction

Most results pertaining to the robust control literature can be cast into the two following categories:

- Kharitonov-like results, for systems with parametric or structured uncertainty. Launched by the seminal Kharitonov theorem for polynomials with interval uncertainty, these results generally apply to systems described in a transfer function setting [3, 5].
- Lyapunov-like results, for systems with frequency-domain or unstructured uncertainty. Originating from the work of Lyapunov on the stability of motion, these results generally apply to systems described in a state-space setting [6, 22, 21].

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[2]Corresponding author. E-mail: henrion@laas.fr. Fax: +33 5 61 33 69 69.
[3]LAAS-CNRS, 7 Avenue du Colonel Roche, 31 077 Toulouse, Cedex 4, France.
[4]Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, Pod vodárenskou věží 4, 182 08 Prague 8, Czech Republic.
[5]Trnka Laboratory for Automatic Control, Faculty of Electrical Engineering, Czech University of Technology, Technická 2, 166 27 Prague 6, Czech Republic.

Despite this apparently neat separation, several results tend to show that both categories are actually connected. For example, an early Lyapunov-based proof of the Routh-Hurwitz stability criterion was unveiled in 1962 by Parks [16]. In the same vein, Mansour and Anderson demonstrated the theorem of Kharitonov via the second method of Lyapunov [15]. Further relationships between quadratic and robust stability were pointed out recently using game theory and convex optimization [7]. Finally, efficient Linear Matrix Inequality (LMI) optimization techniques, traditionally relevant in a state-space setting [6, 22, 21], were used in [9, 10] as relaxation procedures when pursuing a polynomial approach [12, 13] to control system design.

This paper aims at further reinforcing the above mentioned links. Thanks to the theory of Schur-Cohn-Fujiwara matrices [11, 17] and to recent achievements by Amato *et al.* on bounded rate parameters [1], we enlighten some relationships between robust (or Kharitonov) stability and quadratic (or Lyapunov) stability in the simple case of a discrete-time polynomial affected by a single real uncertain parameter. Most of the material in this paper is based on well established results. However, we believe that our main contribution is in the way these results are combined to lend new insights into a key feature of robustness theory.

The paper is organized as follows. In Section 2, we define the robust stability of an uncertain polynomial and the associated real stability radius. A new polynomial matrix eigenvalue technique is proposed for computing the real stability radius, based on the Schur-Cohn-Fujiwara stability criterion. In Section 3, the quadratic stability of an uncertain polynomial is introduced, together with the quadratic stability radius. An LMI technique is then described to compute the quadratic stability radius. In Section 4, new links between both kinds of stability are investigated. For this purpose, we combine the Schur-Cohn-Fujiwara criterion and recently published results on piecewise-constant Lyapunov functions. Following an illustrating numerical example, the paper ends with some concluding remarks.

2 Robust Stability

Throughout the paper, we will study the discrete-time uncertain polynomial

$$p(z,r) = p_0(z) + rp_1(z)$$

where $p_0(z)$ is a Schur monic polynomial, $p_1(z)$ is a polynomial such that $\deg p_1(z) < \deg p_0(z)$ and r is an uncertain parameter belonging to a real symmetric interval, i.e.

$$|r| \leq r_{\max}. \tag{1}$$

Several kinds of stability can be defined with respect to uncertain polynomial $p(z,r)$. In this section, we first focus on the definition most frequently encountered in the parametric approach to control systems [3, 5].

Definition 1 *Uncertain polynomial $p(z,r)$ is robustly stable if it is stable for any fixed value of r such that (1) holds.*

Definition 2 *The real stability radius of $p(z,r)$ is the smallest absolute value of r that destabilizes $p(z,r)$, i.e.*

$$r_R = \min r_{\max} \text{ s.t. } p(z,r) \text{ is not Schur.}$$

There are several ways to evaluate the real stability radius of a polynomial. All of them stem from the concept of a guardian map [18]. Well-known methods for computing r_R are the Schur matrix eigenvalue criterion [3, §17.9] and the Sylvester resultant eigenvalue criterion. The latter has the advantage of being easily extended to polynomial matrices [19]. In the sequel, we propose a new, alternative method for evaluating r_R. It is based on a straightforward use of the Schur-Cohn-Fujiwara matrix of a polynomial.

Definition 3 *[11, 17] Let $a(z) = a_0 + a_1 z + \ldots + a_n z^n$ be a discrete-time polynomial such that $a_n > 0$. The (i,j)th entry of the $n \times n$ symmetric Schur-Cohn-Fujiwara (SCF) matrix P_a associated with $a(z)$ reads*

$$[P_a]_{ij} = \sum_{k=1}^{\min(i,j)} (a_{n-i+k} a_{n-j+k} - a_{i-k} a_{j-k}).$$

For $n = 3$ one has, for example,

$$P_a = \begin{bmatrix} a_3^2 - a_0^2 & a_2 a_3 - a_1 a_0 & a_1 a_3 - a_2 a_0 \\ a_2 a_3 - a_1 a_0 & a_3^2 + a_2^2 - a_0^2 - a_1^2 & a_2 a_3 - a_1 a_0 \\ a_1 a_3 - a_2 a_0 & a_2 a_3 - a_1 a_0 & a_3^2 - a_0^2 \end{bmatrix}.$$

Based on the above matrix is the Schur-Cohn-Fujiwara stability criterion, the quadratic counterpart to Schur determinantal criterion.

Theorem 1 *[11, 17] Polynomial $a(z)$ is Schur if and only if SCF matrix P_a is positive definite.*

When considering stability of uncertain polynomials, we shall also use compound SCF matrices, defined as follows.

Definition 4 *Let $a(z) = a_{\text{even}}(z^2) + z a_{\text{odd}}(z^2)$ and $b(z) = b_{\text{even}}(z^2) + z b_{\text{odd}}(z^2)$. Define $c(z) = a_{\text{even}}(z^2) + z b_{\text{odd}}(z^2)$. The Schur-Cohn-Fujiwara matrix of $a(z)$ and $b(z)$ is defined as*

$$P_{ab} = P_c.$$

Using Definition 4, Theorem 1 and the results on compound SCF matrices developed in [9], we can state the following

Lemma 1 *The real stability radius r_R of polynomial $p(z,r)$ is the real zero of quadratic polynomial matrix*

$$P_p(r) = P_{p_0} + r(P_{p_0 p_1} + P_{p_1 p_0}) + r^2 P_{p_1}.$$

that lies nearest to the origin, i.e.

$$r_R = \min |r| \ \textit{s.t.} \ \det P_p(r) = 0 \ \textit{and} \ \operatorname{Im} r = 0.$$

Zeros of polynomial matrices can be computed for instance with the eigenvalue method proposed in [14]. Note that, although not mentioned in [18], the SCF matrix introduced in Definition 3 also belongs to the family of guardian maps for discrete-time polynomials. The continuous-time counterpart of the SCF matrix was recently used in [9, 10] to derive robust controller design methods through the polynomial approach. We shall make use of the SCF matrix later on in Section 4.

3 Quadratic Stability

In this section, we introduce a new kind of stability for uncertain polynomial $p(z,r)$, originally defined for linear systems in a state-space setting [6, 22, 21].

For this purpose, suppose that uncertain parameter r is time-varying and denote by $r(k)$ the value taken by r at sample times $k = 0, 1, 2, \ldots$. Let $p(z, r(k))$ be the characteristic polynomial appearing at the denominator of the transfer function of a discrete-time uncertain linear system

$$x_{k+1} = A(r(k))x_k, \tag{2}$$

i.e. it holds

$$p(z, r(k)) = \det(zI - A(r(k))).$$

When entries of state-space matrix $A(r)$ depend affinely on uncertain parameter r, then the affine dependence of $p(z,r)$ with respect to r is preserved if $A(r)$ is affected by rank-one perturbations [3, §19.4]. One possible choice is to select $A(r)$ as the companion matrix of $p(z,r)$, viz.

$$A(r) = \underbrace{\begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ -p_{0_0} & -p_{0_1} & \cdots & -p_{0_{n-1}} \end{bmatrix}}_{A_0} + r \underbrace{\begin{bmatrix} 0 & 0 & & \\ & & \ddots & \\ & & & 0 \\ -p_{1_0} & -p_{1_1} & \cdots & -p_{1_{n-1}} \end{bmatrix}}_{A_1}. \tag{3}$$

In order to study uncertain linear system (2), Barmish [2] introduced the notion of quadratic stability. It can readily be extended to cover stability of uncertain polynomial $p(z,r)$.

Definition 5 *Uncertain polynomial $p(z,r)$ is quadratically stable if and only if there exists a symmetric positive definite matrix P such that*

$$A'(r(k))PA(r(k)) - P < 0 \tag{4}$$

for any $k = 0, 1, 2, \ldots$

Quadratic stability has launched an entire area of research in the late eighties, based on the simple but remarkable fact that, under some assumptions on the uncertainty, the problem of finding a unique Lyapunov matrix valid on the whole uncertainty set can be cast into a convex optimization problem [4]. Later on, these problems where coined as LMI optimization problems and shown to be solved efficiently via semidefinite programming, a generalization of linear programming to the cone of positive definite matrices [6].

With notation (3), LMI (4) can be written as a polynomial matrix inequality linear in P and quadratic in r

$$(A_0'PA_0 - P) + r(A_1'P + PA_1) + r^2 A_1'PA_1 < 0,$$

or, equivalently, using a standard Schur complement argument, as an LMI eigenvalue problem in r

$$\begin{bmatrix} A_0'PA_0 - P & 0 \\ 0 & -P \end{bmatrix} + r \begin{bmatrix} A_1'P + PA_1 & A_1'P \\ PA_1 & 0 \end{bmatrix} < 0. \tag{5}$$

As a result, LMI (4) is convex in r: the inequality holds for any $r(k)$ within the interval $[-r_{\max}, r_{\max}]$ if and only if it holds at both extrema.

Lemma 2 *Uncertain polynomial $p(z,r)$ is quadratically stable if and only if there exists a symmetric positive definite matrix P such that*

$$\begin{array}{rcl} A'(-r_{\max})PA(-r_{\max}) - P & < & 0 \\ A'(r_{\max})PA(r_{\max}) - P & < & 0. \end{array} \tag{6}$$

A noteworthy property of quadratic stability is that the uncertain parameter $r(k)$ can vary arbitrary fast without threatening stability of polynomial $p(z, r(k))$. We shall elaborate further on this point in the next section.

Definition 6 *The quadratic stability radius of $p(z,r)$ is the smallest absolute value of r such that $p(z,r)$ is not quadratically stable, i.e.*

$$r_Q = \min r_{\max} \text{ s.t. (6) is infeasible.}$$

The next lemma follows from Lemma 2 and inequality (5).

Lemma 3 *The quadratic stability radius r_Q of polynomial $p(z,r)$ can be obtained by solving for a symmetric positive definite matrix P the eigenvalue LMI optimization problem*

$$
\begin{aligned}
r_Q = \ & \max r \\
s.t. \ & \begin{bmatrix} A_0'PA_0 - P & 0 \\ 0 & -P \end{bmatrix} + r \begin{bmatrix} A_1'P + PA_1 & A_1'P \\ PA_1 & 0 \end{bmatrix} < 0 \\
& \begin{bmatrix} A_0'PA_0 - P & 0 \\ 0 & -P \end{bmatrix} - r \begin{bmatrix} A_1'P + PA_1 & A_1'P \\ PA_1 & 0 \end{bmatrix} < 0.
\end{aligned}
$$

4 Links Between Robust and Quadratic Stability

In this section, we focus on establishing links between the previously defined notions of robust and quadratic stability of polynomial $p(z,r)$.

Note that robust stability implies that $p(z,r)$ is stable for any fixed value of r such that $|r| \leq r_R$ but does not imply anything about stability of $p(z,r)$ when r varies. Conversely, quadratic stability precisely ensures that $p(z,r)$ remains stable for any arbitrarily fast varying r, provided $|r| \leq r_Q$. Obviously, it holds

$$r_Q \leq r_R.$$

In a recent publication [1], Amato and co-workers studied linear sytems with stability radii located between r_Q and r_R. Motivated by their results, we first propose a straightforward reformulation of their main theorem. Then, we show that the Schur-Cohn-Fujiwara matrix defined in Section 2 will prove useful to unveil relationships between robust and quadratic stability of uncertain polynomials.

As in Section 3, we suppose that uncertain parameter r is time-varying and denote by $r(k)$ its value at time $k = 0, 1, 2, \ldots$. The variation of r at time k is defined as $\Delta_k = |r(k+1) - r(k)|$. We say that r varies at rate Δ if

$$\Delta = \max_{k=0,1,2,\ldots} \Delta_k.$$

Moreover, suppose that the uncertainty interval $[-r_{\max}, r_{\max}]$ is split into N sub-intervals $[r_{i-1}, r_i]$ such that $r_0 = -r_{\max}$, $r_N = r_{\max}$ and

$$r_i - r_{i-1} \geq \Delta \tag{7}$$

holds for any $i = 1, 2, \ldots, N$.

Theorem 2 *Given a partition of the uncertainty interval into N sub-intervals $[r_{i-1}, r_i]$, $i = 1, 2, \ldots, N$, uncertain polynomial $p(z,r)$ is stable with r varying at rate Δ if there exists a family of symmetric positive definite matrices $P_1, P_2, \ldots P_N$ such that*

$$A'(r_j)P_iA(r_j) - P_k < 0 \tag{8}$$

for $i = 1, 2, \ldots, N, \quad j = i-1, i, \quad k = i-1, i, i+1$ and $1 \leq k \leq N$.

The proof, developed in [1], follows by considering the first difference of the Lyapunov function associated to linear system (2). Note that Theorem 2 is only a sufficient condition of stability dependent upon the partitioning of the uncertainty interval. Associated to Theorem 2 is the following stability radius.

Definition 7 *Given a partition of the uncertainty interval, the bounded-rate stability radius of $p(z,r)$ is defined as*

$$r_B = \min r_N \text{ s.t. (8) is infeasible.}$$

The following result is the counterpart to Lemma 3.

Lemma 4 *Given a partition of the uncertainty interval into N sub-intervals $[r_{i-1}, r_i]$, $i = 1, 2, \ldots, N$, the bounded-rate stability radius r_B of polynomial $p(z,r)$ can be obtained by solving for symmetric positive definite matrices $P_1, P_2, \ldots, P_N$ the eigenvalue LMI optimization problem*

$$\begin{array}{rl} r_B = & \max r_N \\ s.t. & \left[\begin{array}{cc} A_0'P_iA_0 - P_k & 0 \\ 0 & -P_i \end{array}\right] + r_j \left[\begin{array}{cc} A_1'P_i + P_iA_1 & A_1'P_i \\ P_iA_1 & 0 \end{array}\right] < 0 \end{array}$$

for $i = 1, 2, \ldots, N$, $\quad j = i-1, i, \quad k = i-1, i, i+1$ and $1 \leq k \leq N$.

It must be underlined that the maximum rate of variation Δ, the stability radius r_B and the partition of the uncertainty interval are strongly intercorrelated parameters.

Theorem 2 therefore provides a piecewise-constant Lyapunov matrix that proves stability of uncertain polynomial $p(z,r)$ given a partition of the uncertainty interval and provided $r(k)$ varies sufficiently slowly. Our main observation then follows from the study of the two following extreme cases:

Corollary 1 *When $N = 1$, there is no partition of the uncertainty interval. Theorem 2 reduces to Lemma 3 and becomes a sufficient condition of quadratic stability, i.e.*

$$r_B = r_Q.$$

Proof Lemma 3 ensures stability of polynomial $p(z,r)$ for arbitrary fast varying uncertainty r, hence it ensures bounded-rate stability of $p(z,r)$ for arbitrary variation rate Δ. $\square$

Corollary 2 *When $N \to \infty$, parameter r tends to vary continuously within the uncertainty intervals. The piecewise-constant Lyapunov matrix tends to vary continuously with r and can be considered as parameter dependent. Theorem 2 becomes a sufficient condition of robust stability, i.e.*

$$r_B = r_R.$$

Proof As shown in [17, §2.10], the SCF matrix introduced in Definition 3 is actually a special choice of a parameter dependent Lyapunov matrix. More precisely, if $P(r)$ is the SCF matrix of polynomial $p(z,r)$ as in Lemma 1 and $A(r)$ is the companion matrix defined in equation (3), it can be shown that

$$x_k'[A'(r(k))P(r(k))A(r(k)) - P(r(k))]x_k < 0$$

for any non-zero vector x_k in the trajectory of linear system (2). Since $\min_i(r_i - r_{i-1})$ tends to 0, the rate of variation Δ also tends to 0 in view of relation (7). As a result, stability of uncertain polynomial $p(z,r)$ is only ensured when r is constant. □

5 Example

We consider the example given in [1] where

$$\begin{aligned} p_0(z) &= z^2 - 1.9979z + 0.9980 \\ p_1(z) &= -8 \cdot 10^{-5}. \end{aligned}$$

First we compute the real stability radius with the help of Lemma 1. The quadratic SCF matrix reads

$$\begin{aligned} P(r) = \; & 10^{-3}\begin{bmatrix} 3.9960 & -3.9958 \\ -3.9958 & 3.9960 \end{bmatrix} + 10^{-4}r\begin{bmatrix} 1.5968 & -1.5983 \\ -1.5983 & 1.5968 \end{bmatrix} + \\ & 10^{-9}r^2\begin{bmatrix} -6.4000 & 0 \\ 0 & -6.4000 \end{bmatrix}. \end{aligned}$$

Using the Polynomial Toolbox for MATLAB[1] [20], we found that the zeros of this polynomial matrix are located at $4.9951 \cdot 10^4$, -25.004, -24.996 and 1.2500. Hence

$$r_R = 1.2500.$$

The quadratic stability radius is derived by carrying out a one-dimensional search on the LMI optimization problem of Lemma 3 where

$$A_0 = \begin{bmatrix} 0 & 1 \\ -0.9980 & 1.9979 \end{bmatrix} \quad A_1 = \begin{bmatrix} 0 & 0 \\ 8 \cdot 10^{-5} & 0 \end{bmatrix}.$$

With the LMI Control Toolbox for MATLAB [8], we got

$$r_Q = 0.2483.$$

Then, we considered bounded-rate stability radius r_B for a partition of the uncertainty interval into N sub-intervals $[r_{i-1}, r_i]$, $i = 1, 2, \ldots, N$ of equal lengths, i.e. $r_i = (2i - N)r/N$. Using Lemma 4, we computed values of r_B as a function of N. Our results are reported in Table 1.

[1] MATLAB is a trademark of The MathWorks, Inc.

N	1	2	3	4	5	10
r_B	0.2483	0.2504	0.2519	0.2539	0.2557	0.2653
N	15	20	25	30	35	40
r_B	0.2749	0.2845	0.2942	0.3040	0.3136	0.3180

Table 1: Stability radius r_B versus number of partitions N.

Practically, the size of the LMI systems to be solved precluded us from computing r_B for higher values of N. Theoretically, when N tends to infinity, radius r_B tends to r_R as pointed out in Corollary 2. A special choice of a parameter dependent Lyapunov matrix is the SCF matrix $P(r)$ given above.

6 Conclusion

In this note, we combined several results of the robust control literature to provide new insights into the relationships existing between Kharitonov and Lyapunov approaches to robustness. Using a piecewise-constant Lyapunov function defined on a partitioned uncertainty interval, we showed that, for a discrete-time polynomial featuring a single uncertain parameter,

- quadratic stability corresponds to a partition into a single interval and a constant Lyapunov function, whereas
- robust stability corresponds to a partition into an infinite number of intervals and a parameter-dependent Lyapunov function. One possible choice of Lyapunov matrix is the Schur-Cohn-Fujiwara matrix.

Several open research directions underline the fact that the results proposed in this note are only preliminary. Relevant topics include extensions of our work to continuous-time uncertain polynomials, to polynomials with several uncertain parameters and to robust stability of polynomial matrices.

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