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M-FUNCTIONS AND PARALLEL ASYNCHRONOUS ALGORITHMS*

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Abstract. The solution of nonlinear systems of equations $Fx = z$ via parallel asynchronous algorithms is considered. It is shown that when F is continuous, off-diagonally antitone, and strictly diagonally isotone, then point asynchronous iterations converge monotonically to a solution of the problem from supersolutions and subsolutions. A global convergence result for asynchronous iterations, when F is a continuous, surjective M-function is also presented.

Key words. nonlinear systems of equations, M-functions, parallel computation, asynchronous iterations

AMS(MOS) subject classifications. 65W05, 65H10, 65N20

1. Introduction. There is a variety of parallel iterative methods for nonlinear problems (see Baudet [1], Schendel [12, App. 2], Sloboda [13]). In this paper we consider asynchronous relaxation methods for nonlinear systems of equations. There is now considerable understanding of the convergence properties of parallel asynchronous iterations for a broad class of problems including some linear and nonlinear systems of equations, network flow problems, and dynamic programming. First, Chazan and Miranker [4] have formulated a model of parallel asynchronous algorithms. They have shown that parallel asynchronous iterations converge to the solution of a linear system of equations $Ax = b$ if and only if A is an H-matrix. Donnelly [5] has given convergence results for overrelaxed periodic schemes in the linear case. Miellou [6] and Baudet [1] have extended the results of Chazan and Miranker to nonlinear fixed-point problems by proving the convergence of parallel asynchronous algorithms for P-contraction mappings. Concurrently, Miellou [7] has shown that asynchronous iterations converge monotonically from supersolutions and subsolutions for continuous, isotone fixed-point mappings. Bertsekas [2] has also shown the monotone convergence of a distributed asynchronous algorithm for a broad class of dynamic programming problems. In a recent paper [3], Bertsekas and El Baz obtained the same result for single commodity convex network flow problems. In these last two papers the convergence is based on the property of isotonicity of the fixed-point mappings. Finally, Miellou [8] has considered the nonlinear system of equations $Fx = z$ and some corresponding fixed-point mapping G for block asynchronous iterations, and has shown that when F is a continuous, surjective M-function, then G is isotone. Moreover, if G is continuous, then asynchronous iterations converge monotonically from supersolutions and subsolutions.

In this paper we concentrate on point asynchronous iterations. We show that when F is continuous, off-diagonally antitone, and strictly diagonally isotone, then asynchronous iterations converge monotonically from supersolutions and subsolutions. We show also that any asynchronous iteration converges to the unique solution of $Fx = z$, whatever the value of z, and for any starting point when F is a continuous surjective M-function. The results presented in this paper extend to asynchronous iteration convergence results for underrelaxed Gauss–Seidel and Jacobi iterations proved by Rheinboldt [10, § 3]. For other extensions of the convergence results in [10], in particular to block processes, the reader is referred to Rheinboldt [11, § 6].

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The class of problems considered in this study is broad. Off-diagonally antitone mappings and M-functions occur in the discretization of certain boundary value problems and in the study of nonlinear network flows (see [10], [11]).

In § 2 we introduce a fixed-point problem associated with the nonlinear system of equations $Fx = z$ and study the properties of the fixed-point mapping when F is continuous, off-diagonally antitone, and strictly diagonally isotone. In § 3 we present convergence results for asynchronous iterations.

2. Preliminaries. We consider the solution of nonlinear systems of equations

$$Fx \equiv \begin{pmatrix} f_1(x_1, \cdots, x_n) \\ \cdots \\ f_n(x_1, \cdots, x_n) \end{pmatrix} = \begin{pmatrix} z_1 \\ \cdots \\ z_n \end{pmatrix},$$

where $x_1, \cdots, x_n$ denote the components of vector x element of the n-dimensional real linear space R^n. The natural partial ordering on R^n is defined by

$$\text{For } x, y \in R^n, \quad x \leqq y \quad \text{if and only if } x_i \leqq y_i, \quad i = 1, \cdots, n.$$

LEMMA 2.1. *Let $F: D \subset R^n \to R^n$ be continuous, off-diagonally antitone, and strictly diagonally isotone, and suppose that for some $z \in R^n$ there exist points $x^0, y^0 \in D$ such that*

$$x^0 \leqq y^0, \quad D' = \{x \in R^n \mid x^0 \leqq x \leqq y^0\} \subset D, \quad Fx^0 \leqq z \leqq Fy^0.$$

Then, for any $x \in D'$ there exists a unique vector $\hat{x} \in D'$ with components $\hat{x}_i$ for which

$$f_i(x_1, \cdots, \hat{x}_i, \cdots, x_n) = z_i, \qquad i = 1, \cdots, n. \tag{2.1}$$

Proof. Suppose that for $x \in D'$ and $i \in \{1, \cdots, n\}$, $f_i(x) < z_i$. Since F is off-diagonally antitone and $x \leqq y^0$, it follows that (see [10, Def. 2.7])

$$f_i(x) < z_i \leqq f_i(y^0) \leqq f_i(x_1, \cdots, y_i^0, \cdots, x_n). \tag{2.2}$$

By the continuity and strict diagonal isotonicity of F (see [10, Def. 2.7]), (2.2) implies the existence of a unique $\hat{x}_i$ for which

$$x_i^0 \leqq \hat{x}_i \leqq y_i^0 \quad \text{and} \quad f_i(x_1, \cdots, \hat{x}_i, \cdots, x_n) = z_i.$$

For further details about off-diagonally antitone and strictly diagonally isotone mappings the reader is referred to Ortega and Rheinboldt [9, § 13.5] and Rheinboldt [10, § 2], [11, § 2].

Now suppose that $f_i(x) \geqq z_i$. From the off-diagonal antitonicity of F it follows that

$$f_i(x_1, \cdots, x_i^0, \cdots, x_n) \leqq f_i(x^0) \leqq z_i \leqq f_i(x). \tag{2.3}$$

By the continuity and strict diagonal isotonicity of F, (2.3) implies the existence of a unique $\hat{x}_i$ for which

$$x_i^0 \leqq \hat{x}_i \leqq y_i^0 \quad \text{and} \quad f_i(x_1, \cdots, \hat{x}_i, \cdots, x_n) = z_i.$$

We introduce the fixed-point mapping $G: D' \subset R^n \to D'$ defined by

$$Gx = \hat{x}, \quad \text{where } \hat{x} \text{ is defined by (2.1).} \tag{2.4}$$

Clearly, G is well defined; moreover, x^* is a fixed point of G if and only if $Fx^* = z$.

LEMMA 2.2. *Under the hypothesis of Lemma 2.1, the fixed-point mapping G, which is defined by (2.4), is continuous and isotone on D'. Moreover, $x^0 \leqq Gx^0$, $Gy^0 \leqq y^0$ for starting points x^0 and y^0 for which $Fx^0 \leqq z \leqq Fy^0$.*

This result is derived from the proof of Theorem 6.3 of [11]. x^0 and y^0 are a so-called subsolution and supersolution, respectively.

In the notational conventions of this paper a subscript denotes a component index and a superscript denotes an iteration index.

3. Convergence of asynchronous iterations. We consider asynchronous iterations for the solution of systems of n equations $Fx = z$.

In brief, an asynchronous iteration relative to the solution of $Fx = z$, the starting point x^0, the sequence of delays $\{k^p = (k_1^p, \cdots, k_n^p)\}$, and the sequence of nonempty subsets of $\{1, \cdots, n\}$ denoted by $\{h^p\}$ is a sequence of points $\{x^p\}$ defined recursively by

$$x_i^{p+1} = x_i^p \quad \text{if } i \notin h^p,$$

$$f_i(x_1^{p-k_1^p}, \cdots, x_i^{p+1}, \cdots, x_n^{p-k_n^p}) = z_i \quad \text{if } i \in h^p,$$

where for each $i = 1, \cdots, n$:

i occurs infinitely often in the sequence $\{h^p\}$, k_i^p is a nonnegative integer, $p = 0, 1, \cdots$, the function $d_i(p) = p - k_i^p$ is isotone, and $\lim_{p\to\infty} d_i(p) = +\infty$.

For an analysis and examples of asynchronous iterations, reference is made to Baudet [1]. The following theorem states a sufficient condition for the monotone convergence of certain asynchronous iterations.

THEOREM 3.1. *Suppose that the conditions of Lemma 2.1 hold. Then the asynchronous iterations $\{x^p\}$ and $\{y^p\}$ corresponding to the same sequences $\{h^p\}$, $\{k^p\}$ and starting from x^0 and y^0, respectively, are uniquely defined and satisfy*

$$x^0 \leqq x^p \leqq x^{p+1} \leqq y^{p+1} \leqq y^p \leqq y^0, \qquad p = 0, 1, \cdots,$$

$$\lim_{p\to\infty} x^p = x^* \leqq y^* = \lim_{p\to\infty} y^p, \qquad Fx^* = Fy^* = z.$$

Proof. We recall that by Lemma 2.2 the fixed-point mapping G, which is defined by (2.4), is continuous and isotone on D'. Theorem 3.1 then follows from a convergence result of Miellou mentioned in the Introduction (see [7, Prop. 1]).

LEMMA 3.2. *Suppose that the conditions of Lemma 2.1 hold. Then, $Fx^p \leqq z \leqq Fy^p$, $p = 0, 1, \cdots$.*

Proof. Since F is off-diagonally antitone and strictly diagonally isotone, it follows from Theorem 3.1 that for $p = 0, 1, \cdots$, and $i \in h(p)$

$$z_i = f_i(y_1^{p-k_1^p}, \cdots, y_i^{p+1}, \cdots, y_n^{p-k_n^p}) \leqq f_i(y_1^p, \cdots, y_i^{p+1}, \cdots, y_n^p) \leqq f_i(y^p).$$

For $p = 0, 1, \cdots$, and $i \notin h(p)$, it follows also that

$$z_i \leqq f_i(y^0) \leqq f_i(y_1^p, \cdots, y_i^{p+1}, \cdots, y_n^p) = f_i(y^p),$$

if y_i^{p+1}, and

$$z_i \leqq f_i(y_1^p, \cdots, y_i^{p+1}, \cdots, y_n^p) = f_i(y^p),$$

if $y_i^{p+1} = y_i^{q+1}$, $q + 1 \leqq p$, $\quad z_i = f_i(y_1^{q-k_1^q}, \cdots, y_i^{q+1}, \cdots, y_n^{q-k_n^q})$.

Analogously, we can show that $Fx^p \leqq z$, $p = 0, 1, \cdots$.

We now consider a particular class of off-diagonally antitone and strictly diagonally isotone mappings: M-functions. The reader is referred to Rheinboldt [10, § 2], [11, § 2] for a complete study of M-functions. Theorem 3.1 and Lemma 3.2 apply in particular to continuous M-functions. We now state a global convergence result for continuous surjective M-functions.

THEOREM 3.3. *Let $F: R^n \to R^n$ be a continuous, surjective M-function. Then, for any $z \in R^n$, any asynchronous iteration $\{x^p\}$ converges to the unique solution x^* of $Fx = z$ for any starting point $x^0 \in R^n$.*

Proof. For given $x^0, z \in R^n$ define the vectors $\underline{z}$, $\bar{z}$, $\underline{x}^0$, $\bar{x}^0 \in R^n$ by

$$(3.1)\qquad \begin{aligned} &\underline{z}_i = \min(f_i(x^0), z_i), \quad \bar{z}_i = \max(f_i(x^0), z_i), \quad i = 1, \cdots, n, \\ &\underline{x}^0 = F^{-1}\underline{z}, \qquad \bar{x}^0 = F^{-1}\bar{z}. \end{aligned}$$

Let $\{\underline{x}^p\}$, $\{\bar{x}^p\}$, and $\{x^p\}$ denote the asynchronous iterations relative to the same problem and to the same sequences $\{h^p\}$, $\{k^p\}$, and that start from $\underline{x}^0$, $\bar{x}^0$, and x^0, respectively. Since a continuous surjective M-function is surjectively diagonally isotone (see [10, Def. 2.7, Thm. 2.10]), it follows that for $p = 0, 1, \cdots$, and $i \in h^p$ the solutions $\underline{x}_i^{p+1}$, $\bar{x}_i^{p+1}$, and x_i^{p+1} of the equations

$$\begin{aligned} f_i(\underline{x}_1^{p-k_1^p}, \cdots, \underline{x}_i^{p+1}, \cdots, \underline{x}_n^{p-k_n^p}) &= z_i, \\ f_i(\bar{x}_1^{p-k_1^p}, \cdots, \bar{x}_i^{p+1}, \cdots, \bar{x}_n^{p-k_n^p}) &= z_i, \\ f_i(x_1^{p-k_1^p}, \cdots, x_i^{p+1}, \cdots, x_n^{p-k_n^p}) &= z_i, \end{aligned}$$

exist and are unique. It follows also that the asynchronous iterations $\{\underline{x}^p\}$, $\{\bar{x}^p\}$, and $\{x^p\}$ are well defined.

First, we show by induction that

$$(3.2)\qquad \underline{x}^p \leqq x^p \leqq \bar{x}^p, \qquad p = 0, 1, \cdots.$$

From (3.1), $F\underline{x}^0 \leqq Fx^0 \leqq F\bar{x}^0$. Since F is inverse isotone, we have (see [10, Def. 2.2]) $\underline{x}^0 \leqq x^0 \leqq \bar{x}^0$. From (3.1) and the inverse isotonicity of F it follows also that $F\underline{x}^0 \leqq z \leqq F\bar{x}^0$, $\underline{x}^0 \leqq x^* \leqq \bar{x}^0$.

Suppose that for some $p \geqq 0$

$$(3.3)\qquad \underline{x}^k \leqq x^k \leqq \bar{x}^k \quad \text{for } 0 \leqq k \leqq p.$$

Then, if $i \notin h^p$, it is straightforward that

$$\underline{x}_i^{p+1} = \underline{x}_i^p, \quad \bar{x}_i^{p+1} = \bar{x}_i^p, \quad x_i^{p+1} = x_i^p, \quad \underline{x}_i^{p+1} \leqq x_i^{p+1} \leqq \bar{x}_i^{p+1}.$$

If $i \in h^p$ from (3.3) and the off-diagonal antitonicity of F it follows that

$$\begin{aligned} f_i(\underline{x}_1^{p-k_1^p}, \cdots, \underline{x}_i^{p+1}, \cdots, \underline{x}_n^{p-k_n^p}) = z_i &= f_i(x_1^{p-k_1^p}, \cdots, x_i^{p+1}, \cdots, x_n^{p-k_n^p}) \\ &\leqq f_i(\underline{x}_1^{p-k_1^p}, \cdots, x_i^{p+1}, \cdots, \underline{x}_n^{p-k_n^p}). \end{aligned}$$

Then, by the strict diagonal isotonicity of F, $\underline{x}_i^{p+1} \leqq x_i^{p+1}$. Analogously, we can show that $x^{p+1} \leqq \bar{x}^{p+1}$. This completes the induction. By Theorem 3.1 we have

$$(3.4)\qquad \lim_{p \to \infty} \underline{x}^p = \lim_{p \to \infty} \bar{x}^p = x^* = F^{-1}z.$$

Then, from (3.2) and (3.4), it follows that $\lim_{p \to \infty} x^p = x^*$.

The results of Theorems 3.1 and 3.3 can be extended without difficulties to underrelaxed asynchronous iterations.

REFERENCES

[1] G. M. BAUDET, *Asynchronous iterative methods for multiprocessors*, J. Assoc. Comput. Mach., 25 (1978), pp. 226–244.

[2] D. P. BERTSEKAS, *Distributed dynamic programming*, IEEE Trans. Automat. Control, 27 (1982), pp. 610–616.

[3] D. P. BERTSEKAS AND D. EL BAZ, *Distributed asynchronous relaxation methods for convex network flow problems*, SIAM J. Control Optim., 25 (1987), pp. 74-85.

[4] D. CHAZAN AND W. MIRANKER, *Chaotic relaxation*, Linear Algebra Appl., 2 (1969), pp. 199-222.

[5] J. D. P. DONNELLY, *Periodic chaotic relaxation*, Linear Algebra Appl., 4 (1971), pp. 117-128.

[6] J. C. MIELLOU, *Algorithmes de relaxation chaotique à retards*, RAIRO, R-1 (1975), pp. 55-82.

[7] ———, *Itérations chaotiques à retards, étude de la convergence dans le cas d'espaces partiellement ordonnés*, C.R. Acad. Sci. Paris Sér. 1 Math., 280 (1975), pp. 233-236.

[8] ———, *Asynchronous iterations and order intervals*, in Parallel Algorithms and Architectures, North-Holland, Amsterdam, New York, 1986.

[9] J. M. ORTEGA AND W. C. RHEINBOLDT, *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, New York, 1970.

[10] W. C. RHEINBOLDT, *On M-functions and their application to nonlinear Gauss-Seidel iterations and to network flows*, J. Math. Anal. Appl., 32 (1970), pp. 274-307.

[11] ———, *On classes of n-dimensional nonlinear mappings generalizing several types of matrices*, in Numerical Solution of Partial Differential Equations-II, B. Hubbard, ed., Academic Press, New York, 1971, pp. 501-546.

[12] U. SCHENDEL, *Introduction to Numerical Methods for Parallel Computers*, Ellis-Horwood, Chichester, U.K., 1984.

[13] F. SLOBODA, *Nonlinear iterative methods and parallel computation*, Apl. Mat., 21 (1976), pp. 252-262.