COURSE ON LMI OPTIMIZATION WITH APPLICATIONS IN CONTROL PART I.3

# GEOMETRY OF LMI SETS


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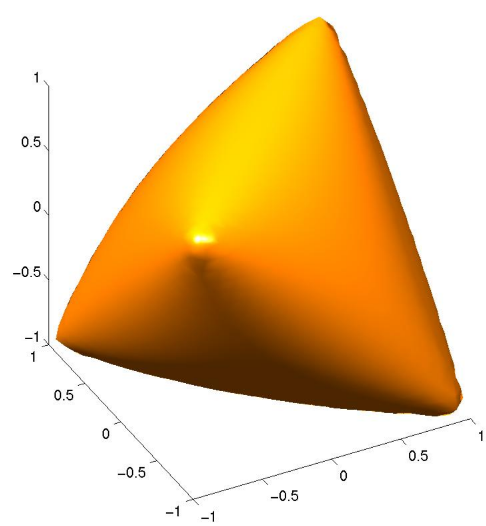


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## Geometry of LMI sets

Given $F_i \in \mathbb{S}^m$ we want to characterize the shape in $\mathbb{R}^n$ of the LMI set

$$\mathcal{S} = \{x \in \mathbb{R}^n \ : \ F(x) = F_0 + \sum_{i=1}^{n} x_i F_i \succeq 0\}$$

Matrix $F(x)$ is PSD iff its diagonal minors $f_i(x)$ are nonnegative

Diagonal minors are multivariate polynomials of indeterminates $x_i$

So the LMI set can be described as

$$\mathcal{S} = \{x \in \mathbb{R}^n \ : \ f_i(x) \geq 0, \ i = 1, \ldots, n\}$$

which is a semialgebraic set

Moreover, it is a convex set

## Example of 2D LMI feasible set

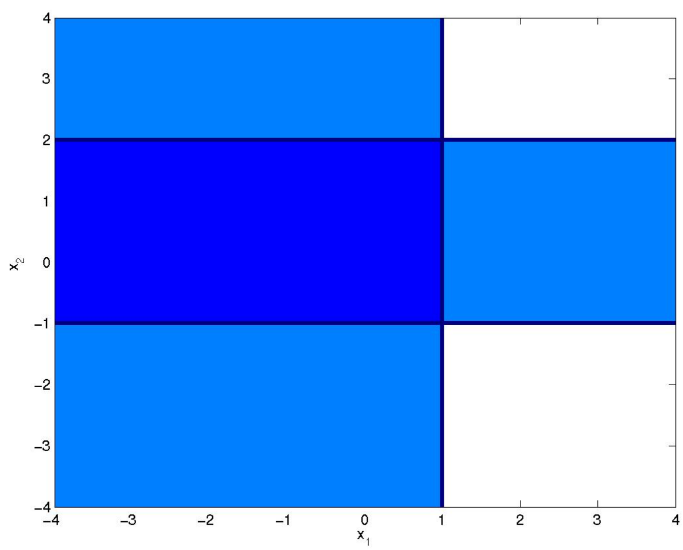
$$F(x) = \begin{bmatrix} 1 - x_1 & x_1 + x_2 & x_1 \\ x_1 + x_2 & 2 - x_2 & 0 \\ x_1 & 0 & 1 + x_2 \end{bmatrix} \succeq 0$$

Feasible iff all principal minors nonnegative

System of polynomial inequalities $f_i(x) \geq 0$

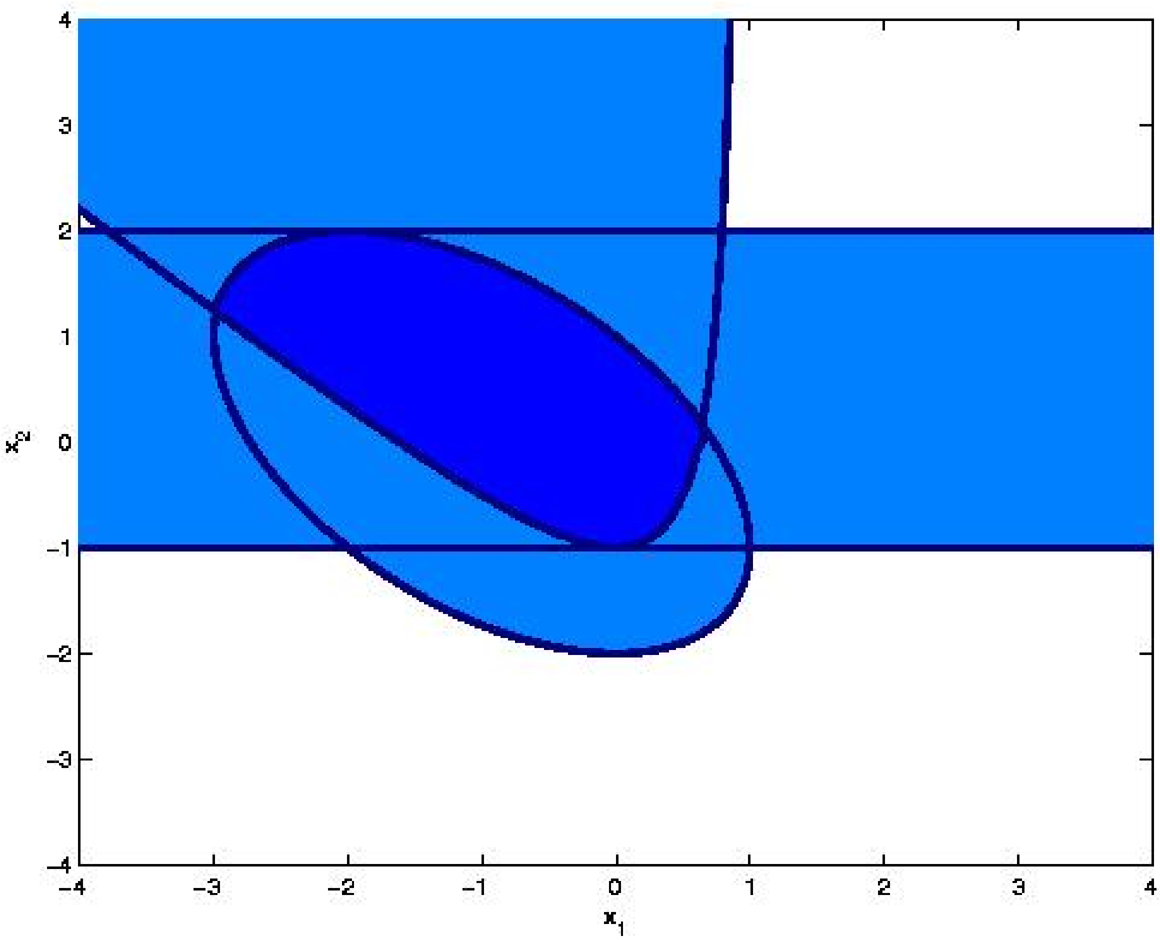
1st order minors

$$f_1(x) = 1 - x_1 \geq 0$$
$$f_2(x) = 2 - x_2 \geq 0$$
$$f_3(x) = 1 + x_2 \geq 0$$



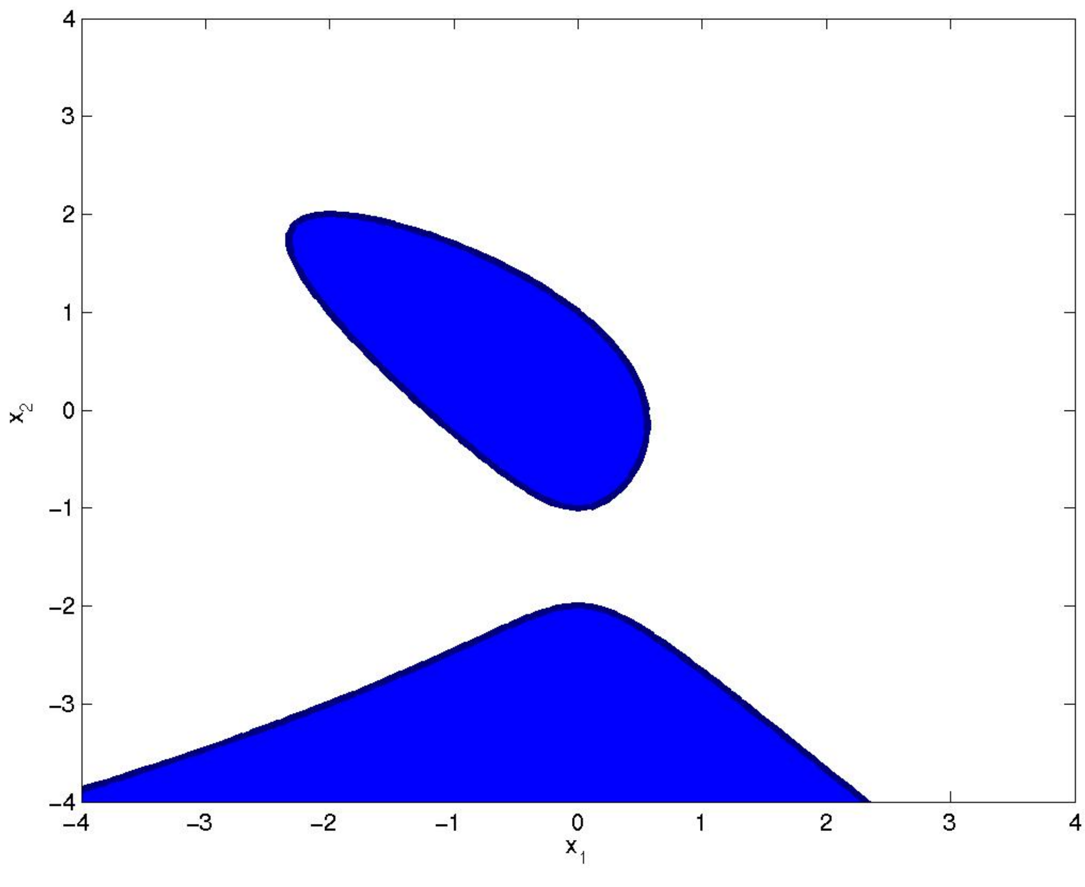
2nd order minors

$$f_4(x) = (1 - x_1)(2 - x_2) - (x_1 + x_2)^2 \geq 0$$
$$f_5(x) = (1 - x_1)(1 + x_2) - x_1^2 \geq 0$$
$$f_6(x) = (2 - x_2)(1 + x_2) \geq 0$$

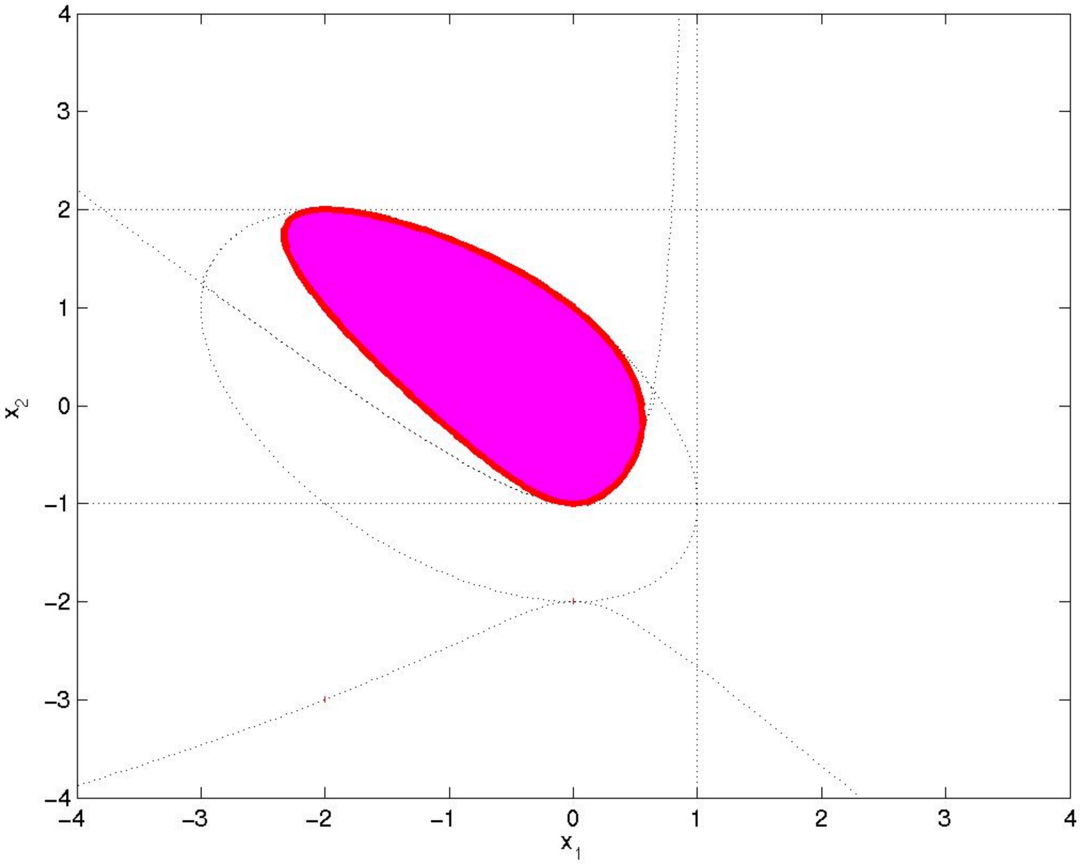


3rd order minor

$$
\begin{aligned}
f_7(x) \;=\; & (1+x_2)((1-x_1)(2-x_2)-(x_1+x_2)^2) \\
& -x_1^2(2-x_2) \geq 0
\end{aligned}
$$



LMI feasible set = intersection of semialgebraic sets $f_i(x) \geq 0$ for $i = 1, \ldots, 7$

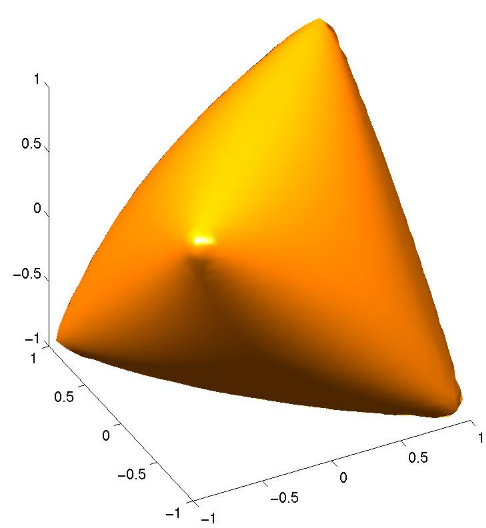


## Example of 3D LMI feasible set

LMI set

$$\mathcal{S} = \{x \in \mathbb{R}^3 \; : \; \begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & x_3 \\ x_2 & x_3 & 1 \end{bmatrix} \succeq 0\}$$

arising in SDP relaxation of MAXCUT



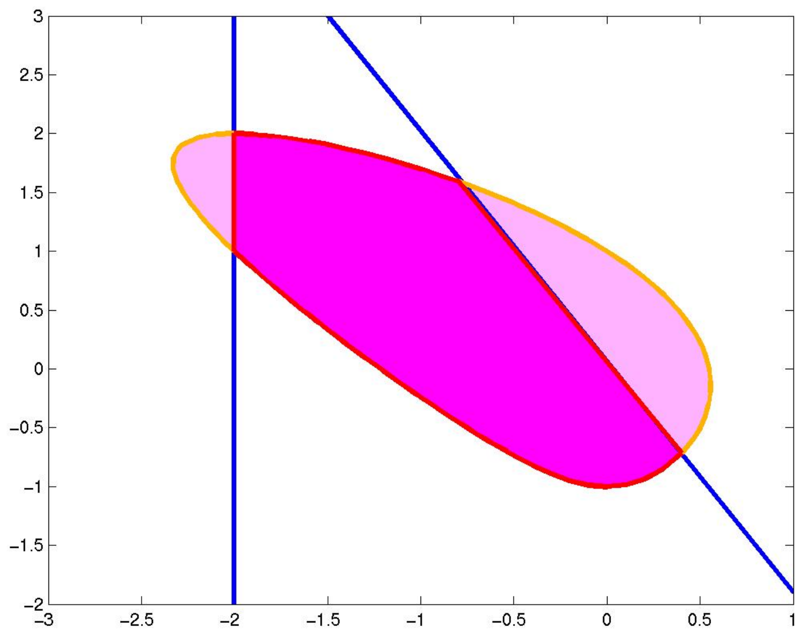

Semialgebraic set

$$\mathcal{S} = \{x \in \mathbb{R}^3 \; : \; 1 + 2x_1x_2x_3 - (x_1^2 + x_2^2 + x_3^2) \geq 0, \\ x_1^2 \leq 1, \; x_2^2 \leq 1, \; x_3^2 \leq 1\}$$

## Intersection of LMI sets

Intersection of LMI feasible sets

$F(x) \succeq 0 \quad x_1 \geq -2 \quad 2x_1 + x_2 \leq 0$




is also an LMI

$$\begin{bmatrix} F(x) & 0 & 0 \\ 0 & x_1 + 2 & 0 \\ 0 & 0 & -2x_1 - x_2 \end{bmatrix} \succeq 0$$

## Conic representability

LMI sets are convex semialgebraic sets.. but are all convex semialgebraic sets representable by LMIs ?

A set $X \subset \mathbb{R}^n$ is conic quadratic representable (CQR) if there exist $N$ affine mappings $F_i(x,u)$ s.t.

$$x \in X \iff \begin{array}{l} \exists\ u\ :\ F_i(x,u) = A_i \begin{bmatrix} x \\ u \end{bmatrix} - b_i \succeq_{\mathbb{L}^{m_i}} 0 \\ i = 1, \cdots, N \end{array}$$

A convex function $f\ :\ \mathbb{R}^n \to \mathbb{R}$ is CQR if its epigraph

$$\mathcal{E}pi = \{(x,\, t) \in \mathbb{R}^n \times \mathbb{R}\ :\ f(x) \leq t\}$$

is CQR

# SDP/LMI representability (1)

We say that a convex set $X \subset \mathbb{R}^n$ is SDP representable if there exists an affine mapping $F(x,u)$ such that

$$\boxed{x \in X \iff \exists\, u \;:\; F(x,u) \succeq 0}$$

In words, if $X$ is the projection of the solution set of the LMI $F(x,u) \succeq 0$ onto the $x$-space and $u$ are additional, or lifting variables

We say that a convex set $X \subset \mathbb{R}^n$ is LMI representable if there exists an affine mapping $F(x)$ such that

$$\boxed{x \in X \iff F(x) \succeq 0}$$

In other words, additional variables $u$ are not allowed

Similarly, a convex function $f : \mathbb{R}^n \to \mathbb{R}$ is SDP or LMI representable if its epigraph

$$\mathcal{E}_{pi} = \{(x,\, t) \in \mathbb{R}^n \times \mathbb{R} \;:\; f(x) \leq t\}$$

is an SDP or LMI representable set

## SDP/LMI representability (2)
## CQR and SDP representability

The Lorentz, or ice-cream cone

$$\mathbb{L}^{n+1} = \left\{ \begin{bmatrix} x \\ t \end{bmatrix} \in \mathbb{R}^{n+1} \; : \; \|x\|_2 \le t \right\}$$

is SDP representable as

$$\mathbb{L}^{n+1} = \left\{ \begin{bmatrix} x \\ t \end{bmatrix} \; : \; \begin{bmatrix} tI_n & x \\ x' & t \end{bmatrix} \succeq 0 \right\}$$

As a result, all (convex quadratic) conic representable sets are also SDP representable

$$\mathbb{L}^n \subset \mathbb{S}^n_+$$

In the sequel we first give a list of conic representable sets (following Ben-Tal and Nemirovski 2000)

## SDP/LMI representability (3)
## Quadratic forms

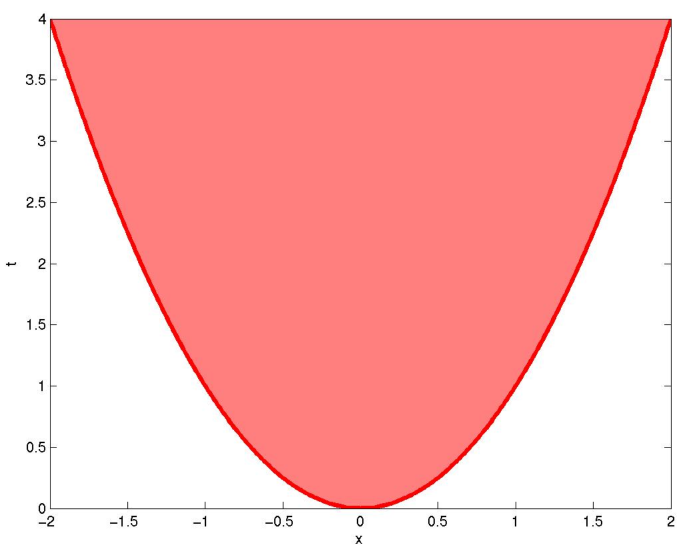
The Euclidean norm $\{x, t \in \mathbb{R}^n \times \mathbb{R} \,:\, \|x\|_2 \leq t\}$ is CQR by definition

The squared Euclidean norm

$$\left\{(x, t) \in \mathbb{R}^n \times \mathbb{R} \,:\, x'x \leq t\right\}$$

is CQR as

$$\boxed{\left\| \begin{bmatrix} x \\ \frac{t-1}{2} \end{bmatrix} \right\|_2 \leq \frac{t+1}{2}}$$



## SDP/LMI representability (4)
## Quadratic forms (2)

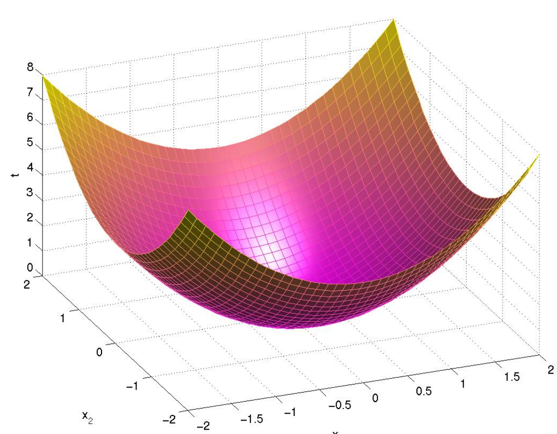
More generally, the convex quadratic set

$$\left\{ x \in \mathbb{R}^n, t \in \mathbb{R} \ : \ x'Ax + b'x + c \leq 0 \right\}$$

with $A = A' \succeq 0$ is CQR as

$$\boxed{\left\| \left[ \begin{array}{c} Dx \\ \frac{t+b'x+c}{2} \end{array} \right] \right\|_2 \leq \frac{t-b'x-c}{2}}$$

where $D$ is the Cholesky factor of $A = D'D$



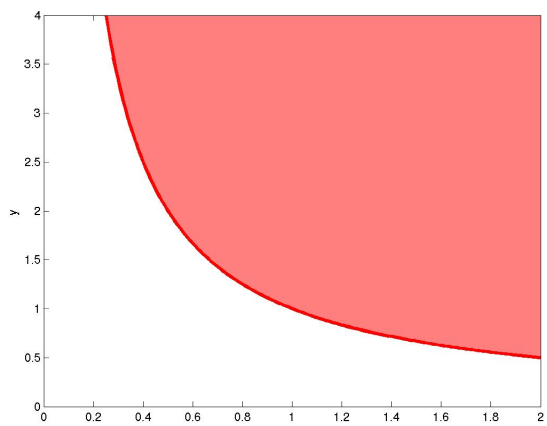
## SDP/LMI representability (5)
## Hyperbola

The branch of hyperbola

$$\left\{(x,y) \in \mathbb{R}^2 \,:\, xy \geq 1,\, x > 0\right\}$$

is CQR as

$$\boxed{\left\|\begin{bmatrix} \frac{x-y}{2} \\ 1 \end{bmatrix}\right\|_2 \leq \frac{x+y}{2}}$$



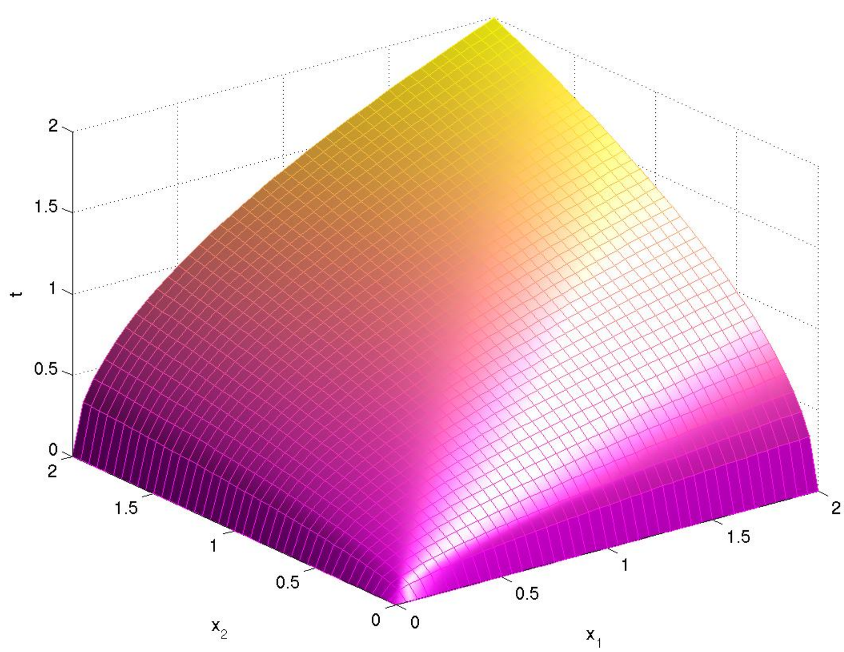
## SDP/LMI representability (6)
## Geometric mean of two variables

The hypograph of the geometric mean
of 2 variables

$$\left\{(x_1, x_2, t) \in \mathbb{R}^3 \;:\; x_1, x_2 \geq 0,\; \sqrt{x_1 x_2} \geq t\right\}$$

is CQR as

$$\exists\; u \;:\; u \geq t,\; \left\| \left[ \begin{array}{c} u \\ \frac{x_1 - x_2}{2} \end{array} \right] \right\|_2 \leq \frac{x_1 + x_2}{2}$$



## SDP/LMI representability (7)
## Geometric mean of several variables

The hypograph of the geometric mean of $2^k$ variables

$$\left\{(x_1,\ldots,x_{2^k},t)\in\mathbb{R}^{2^k+1}\ :\ x_i\geq 0,\ (x_1\cdots x_{2^k})^{1/2^k}\geq t\right\}$$

is also CQR

Proof: Iterate the previous construction

Example with $k = 3$:

$$(x_1x_2\cdots x_8)^{1/8}\geq t$$

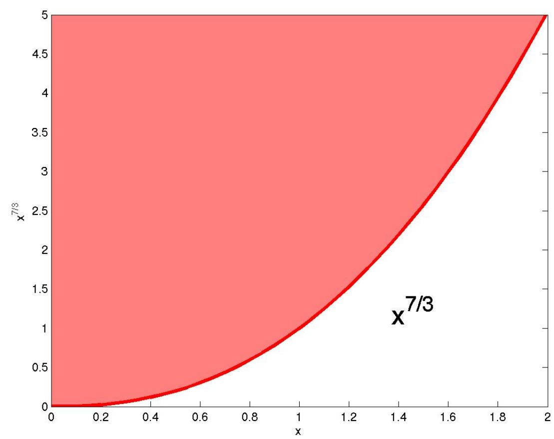
$$\left.\begin{array}{rcl}\sqrt{x_{01}x_{02}} & \geq & x_{11}\\ \sqrt{x_{03}x_{04}} & \geq & x_{12}\\ \sqrt{x_{05}x_{06}} & \geq & x_{13}\\ \sqrt{x_{07}x_{08}} & \geq & x_{14}\end{array}\right\}\ \left.\begin{array}{rcl}\sqrt{x_{11}x_{12}} & \geq & x_{21}\\ \sqrt{x_{13}x_{14}} & \geq & x_{22}\end{array}\right\}\ \sqrt{x_{21}x_{22}}\geq x_{31}\geq t$$

Useful idea in other SDP representability problems

## SDP/LMI representability (8)
## Rational functions (1)

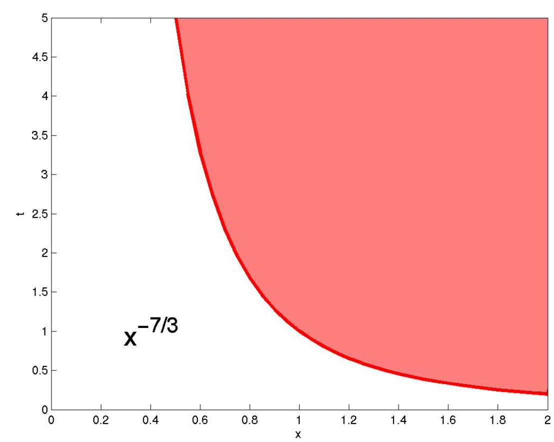
Using similar ideas, we can show that the increasing rational power functions

$$f(x) = x^{p/q}, \quad x \geq 0$$




with rational $p/q \geq 1$, as well as the decreasing

$$g(x) = x^{-p/q}, \quad x \geq 0$$




with rational $p/q \geq 0$, are both CQR

# SDP/LMI representability (9)
# Rational functions (2)

Example:

$$\left\{(x,t) \in \mathbb{R}^2 \;:\; x \geq 0,\; x^{7/3} \leq t\right\}$$

Start from conic representable

$$\hat{t} \leq (\hat{x}_1 \cdots \hat{x}_8)^{1/8}$$

and replace

$$\begin{array}{l} \hat{t} = \hat{x}_1 = x \geq 0 \\ \hat{x}_2 = \hat{x}_3 = \hat{x}_4 = t \geq 0 \\ \hat{x}_5 = \hat{x}_6 = \hat{x}_7 = \hat{x}_8 = 1 \end{array}$$

to get

$$\begin{array}{rcl} x & \leq & x^{1/8}t^{3/8} \\ x^{7/8} & \leq & t^{3/8} \\ x^{7/3} & \leq & t \end{array}$$

Same idea works for any rational $p/q \geq 1$

- lift = use additional variables, and
- project in the space of original variables

## SDP/LMI representability (10)
## Even power monomial (1)

The epigraph of even power monomial

$$\mathcal{E}_{pi} = \left\{x, t \,:\, x^{2p} \leq t\right\}$$

where $p$ is a positive integer, is CQR

Note that

$$\{x, t \,:\, x^{2p} \leq t\}$$

$$\Longleftrightarrow$$

$$\{x, y, t \,:\, x^2 \leq y\}$$
$$\{x, y, t \,:\, y \geq 0,\ y^p \leq t\}$$

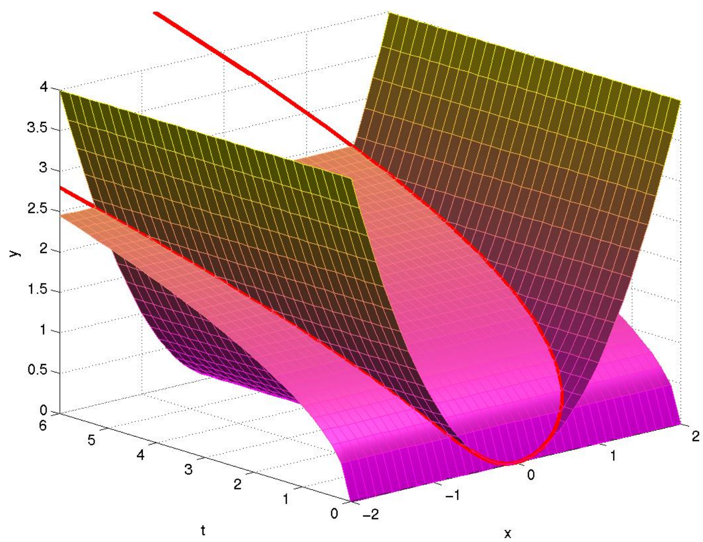
both CQR

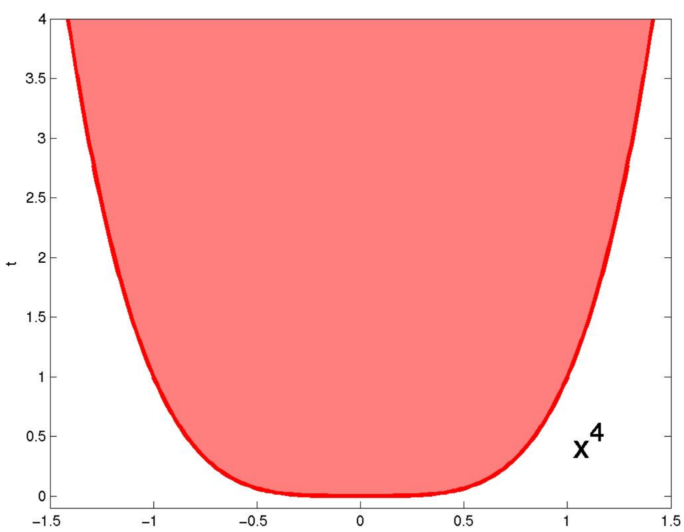
Use lifting $y$ and project back onto $x, t$

Similarly, even power polynomials are CQR (combinations of monomials)

# SDP/LMI representability (11)
# Even power monomial (2)

$$\mathcal{E}_{pi} = \left\{ x, t \;:\; x^4 \leq t \right\}$$



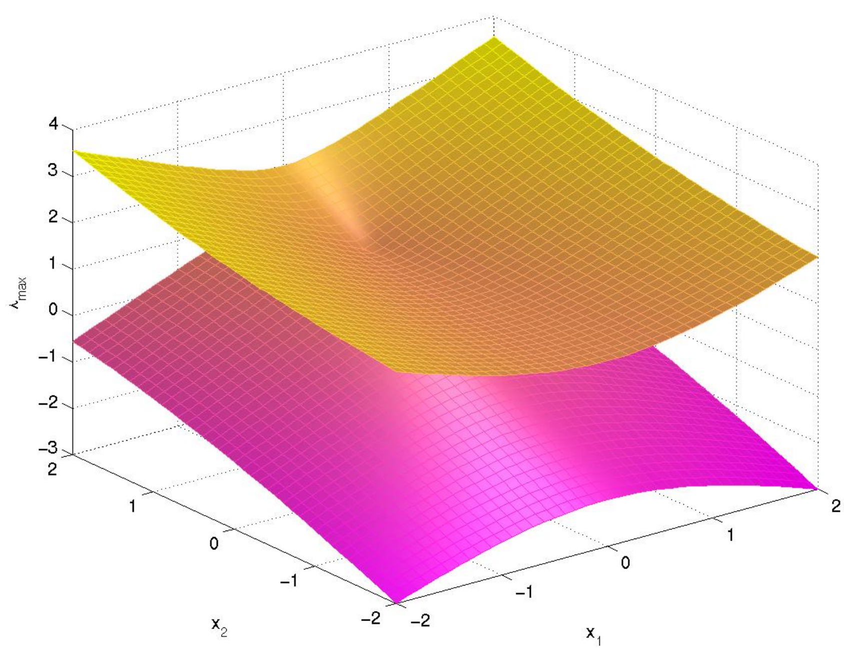
## SDP/LMI representability (12)
## Largest eigenvalue

The epigraph of the function largest eigenvalue of a symmetric matrix

$$\left\{X = X' \in \mathbb{R}^{n\times n}, t \in \mathbb{R} \;:\; \lambda_{\mathsf{max}}(X) \le t\right\}$$

is SDP (LMI) representable as

$$\boxed{X \preceq tI_n}$$




Eigenvalues of matrix $\begin{bmatrix} 1 & x_1 \\ x_1 & x_2 \end{bmatrix}$

## SDP/LMI representability (13)
## Sums of largest eigenvalues

Let

$$S_k(X) = \sum_{i=1}^{k} \lambda_i(X), \quad k = 1, \ldots, n$$

denote the sum of the $k$ largest eigenvalues of $X \in \mathbb{S}^n$

The epigraph

$$\left\{ X \in \mathbb{S}^{n \times n}, t \in \mathbb{R} \; : \; S_k(X) \leq t \right\}$$

is SDP representable as

$$\begin{array}{c} t - ks - \text{trace } Z \succeq 0 \\ Z \succeq 0 \\ Z - X + sI_n \succeq 0 \end{array}$$

where $Z$ and $s$ are additional variables

# Determinant of a PSD matrix

The determinant

$$\det(X) = \prod_{i=1}^{n} \lambda_i(X)$$

is not a convex function of $X$, but the function

$$f_q(X) = -\det^q(X), \quad X = X' \succeq 0$$

is convex when $q \in [0,\ 1/n]$ is rational

The epigraph

$$\{f_q(X) \leq t\}$$

is SDP representable as

$$\boxed{\begin{array}{c} \begin{bmatrix} X & \Delta \\ \Delta' & \text{diag}\,\Delta \end{bmatrix} \succeq 0 \\ t \leq (\delta_1 \cdots \delta_n)^q \end{array}}$$

since we know that the latter constraint (hypograph of a concave monomial) is conic representable

Here $\Delta$ is a lower triangular matrix of additional variables with diagonal entries $\delta_i$

## Application: extremal ellipsoids

Various representations of an ellipsoid in $\mathbb{R}^n$

$$\begin{aligned} E &= \{x \in \mathbb{R}^n \;:\; x'Px + 2x'q + r \leq 0\} \\ &= \{x \in \mathbb{R}^n \;:\; (x - x_c)'P(x - x_c) \leq 1\} \\ &= \{x = Qy + x_c \in \mathbb{R}^n \;:\; y'y \leq 1\} \\ &= \{x \in \mathbb{R}^n \;:\; \|Rx - x_c\| \leq 1\} \end{aligned}$$

where $Q = R^{-1} = P^{-1/2} \succ 0$

Volume of ellipsoid $E = \{Qy + x_c \;:\; y'y \leq 1\}$

$$\boxed{\mathrm{vol}\, E = k_n \det Q}$$

where $k_n$ is volume of $n$-dimensional unit ball

$$k_n = \begin{cases} \frac{2^{(n+1)/2}\pi^{(n-1)/2}}{n(n-2)!!} & \text{for } n \text{ odd} \\ \frac{2\pi^{n/2}}{n(n/2-1)!} & \text{for } n \text{ even} \end{cases}$$

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $k_n$ | 2.00 | 3.14 | 4.19 | 4.93 | 5.26 | 5.17 | 4.72 | 4.06 |

Unit ball has maximum volume for $n = 5$ !

## Outer and inner ellipsoidal approximations

Let $S \subset \mathbb{R}^n$ be a solid = a closed bounded convex set with nonempty interior

- the largest volume ellipsoid $E_{\text{in}}$ contained in $S$ is unique and satisfies

$$\boxed{E_{\text{in}} \subset S \subset nE_{\text{in}}}$$

- the smallest volume ellipsoid $E_{\text{out}}$ containing $S$ is unique and satisfies

$$\boxed{E_{\text{out}}/n \subset S \subset E_{\text{out}}}$$

These are Löwner-John ellipsoids

Factor $n$ reduces to $\sqrt{n}$ if $S$ is symmetric

How can these ellipsoids be computed ?

## Ellipsoids and polytopes (1)

Let the intersection of hyperplanes

$$S = \left\{x \in \mathbb{R}^n \; : \; a_i' x \leq b_i, \; i = 1, \ldots, m\right\}$$

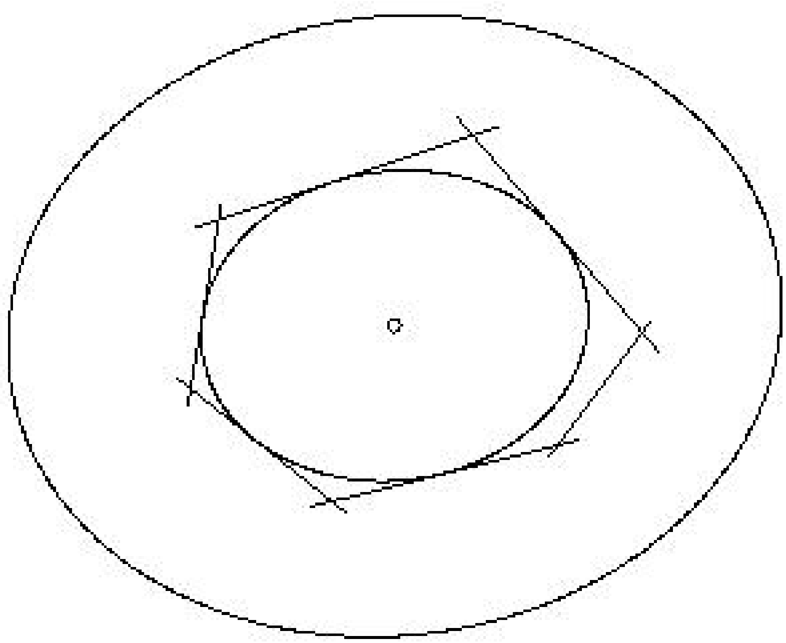
describe a polytope

The largest volume ellipsoid contained in $S$ is

$$E = \left\{Qy + x_c \; : \; y'y \leq 1\right\}$$

where $Q$, $x_c$ are optimal solutions of the LMI

$$\begin{array}{ll} \max & \det^{1/n} Q \\ & Q \succeq 0 \\ & \|Q a_i\|_2 \leq b_i - a_i' x_c, \quad i = 1, \ldots, m \end{array}$$



## Ellipsoids and polytopes (2)

Let the convex hull of vertices

$$S = \text{co}\ \{x_1, \dots, x_m\}$$

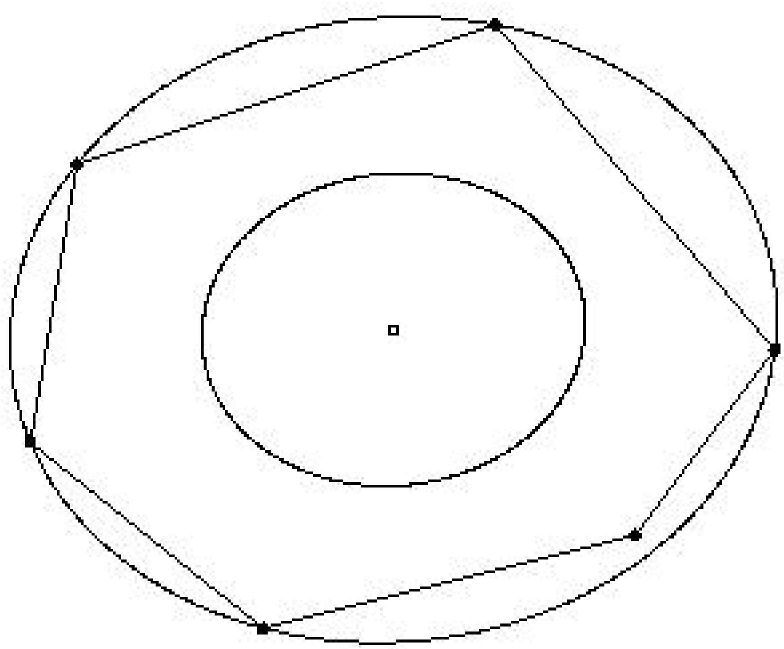
describe a polytope

The smallest volume ellipsoid containing $S$ is

$$E = \{x\ :\ (x - x_c)' P (x - x_c) \leq 1\}$$

where $P$, $x_c = -P^{-1}q$ are optimal solutions of the LMI

$$\begin{array}{ll} \max & t \\ & t \leq \det^{1/n} P \\ & \left[\begin{array}{cc} P & q \\ q' & r \end{array}\right] \succeq 0 \\ & x_i' P x_i + 2x_i' q + r \leq 1, \quad i = 1, \dots, m \end{array}$$



## SDP representability and singular values

Let

$$\Sigma_k(X) = \sum_{i=1}^{k} \sigma_i(X), \quad k = 1, \dots, n$$

denote the sum of the $k$ largest singular values of $X \in \mathbb{R}^{n \times n}$

Then the epigraph

$$\{X \in \mathbb{S}^n, t \in \mathbb{R} \,:\, \Sigma_k(X) \leq t\}$$

is SDP representable since

$$\boxed{\sigma_i(X) = \lambda_i\left(\left[\begin{array}{cc} 0 & X' \\ X & 0 \end{array}\right]\right)}$$

and the sum of largest eigenvalues of a symmetric matrix is SDP representable

## Nonlinear matrix ineqalities (1)
## Schur complement

We can use the Schur complement to convert a non-linear matrix inequality into an LMI

$$A(x) - B(x)C^{-1}(x)B'(x) \succeq 0$$
$$C(x) \succ 0$$

$$\iff$$

$$\begin{bmatrix} A(x) & B(x) \\ B(x) & C(x) \end{bmatrix} \succeq 0$$
$$C(x) \succ 0$$

Issai Schur
(1875 Mogilyov - 1941 Tel Aviv)

## Nonlinear matrix ineqalities (2)
## Elimination lemma

To remove decision variables we can use the elimination lemma

$$A(x) + B(x)XC(x) + C'(x)X'B'(x) > 0$$

$$\Longleftrightarrow$$

$$B^{\perp}(x)A(x)B^{\perp\prime}(x) > 0 \quad C'^{\perp}(x)A(x)C'^{\perp\prime}(x) > 0$$

where $B^{\perp}$ and $C'^{\perp}$ are orthogonal complements of $B$ and $C'$ respectively, and $x$ is a decision variable independent of matrix $X$

Can be shown with SDP duality and theorem of alternatives

## LMIR and Positive polynomials (1)

The set of univariate polynomials that are positive on the real axis is a convex set that is LMI representable

Can be proved with cone duality (Nesterov) or with theory of moments (Lasserre)

The even polynomial

$$p(s) = p_0 + p_1 s + \cdots + p_{2n} s^{2n}$$

satisfies $p(s) \geq 0$ for all $s \in \mathbb{R}$ if and only if

$$\begin{array}{rcll} p_k & = & \displaystyle\sum_{i+j=k} X_{ij}, & k = 0, 1, \ldots, 2n \\ & = & \text{trace}\, H_k X & \end{array}$$

for some matrix $X = X' \succeq 0$

## LMIR and Positive polynomials (2)
## Sum-of-squares decomposition

The expression of $p_k$ with Hankel matrices $H_k$ comes from

$$p(s) = [1 \quad s \quad \cdots \quad s^n] X [1 \quad s \quad \cdots \quad s^n]^\star$$

hence $X \succeq 0$ naturally implies $p(s) \geq 0$

Conversely, existence of $X$ for any polynomial $p(s) \geq 0$ follows from the existence of a sum-of-squares decomposition (with at most two elements) of

$$\boxed{p(s) = \sum_k q_k^2(s) \geq 0}$$

Matrix $X$ has entries $X_{ij} = \sum_k q_{k_i} q_{k_j}$

## Optimizing over polynomials (1)
## Primal and dual formulations

Global minimization of polynomial

$$p(s) = \sum_{k=0}^{n} p_k s^k$$

Global optimum $p^*$: maximum value of $\widehat{p}$ such that $p(s) - \widehat{p} \geq 0$ for all $s \in \mathbb{R}$

Primal LMI

$$\begin{array}{ll} \max & \widehat{p} = p_0 - \text{trace}\, H_0 X \\ \text{s.t.} & \text{trace}\, H_k X = p_k, \quad k = 1, \ldots, n \\ & X \succeq 0 \end{array}$$

Dual LMI

$$\begin{array}{ll} \min & p_0 + \displaystyle\sum_{k=1}^{n} p_k y_k \\ \text{s.t.} & H_0 + \displaystyle\sum_{k=1}^{n} H_k y_k \succeq 0 \end{array}$$

with Hankel structure (moment matrix)

# Optimizing over polynomials (2)
## Example

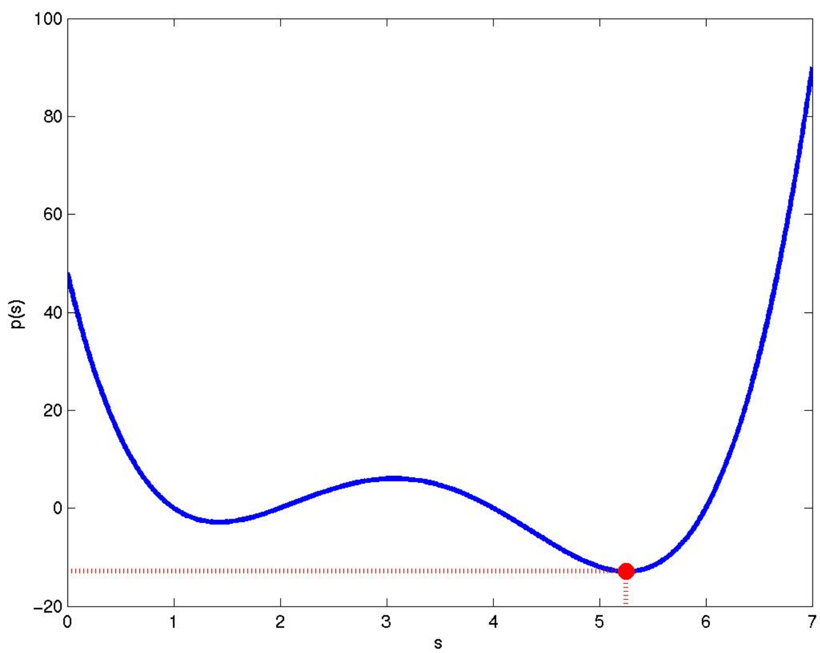
Global minimization of the polynomial

$$p(s) = 48 - 92s + 56s^2 - 13s^3 + s^4$$

We just have to solve the dual LMI

$$\begin{array}{ll} \min & 48 - 92y_1 + 56y_2 - 13y_3 + y_4 \\ \text{s.t.} & \left[\begin{array}{ccc} 1 & y_1 & y_2 \\ y_1 & y_2 & y_3 \\ y_2 & y_3 & y_4 \end{array}\right] \succeq 0 \end{array}$$

to obtain $p^* = p(5.25) = -12.89$



## Complex LMIs

The complex valued LMI

$$F(x) = A(x) + jB(x) \succeq 0$$

is equivalent to the real valued LMI

$$\boxed{\begin{bmatrix} A(x) & B(x) \\ -B(x) & A(x) \end{bmatrix} \succeq 0}$$

If there is a complex solution to the LMI then there is a real solution to the same LMI

Note that matrix $A(x) = A'(x)$ is symmetric whereas $B(x) = -B'(x)$ is skew-symmetric

## Rigid convexity

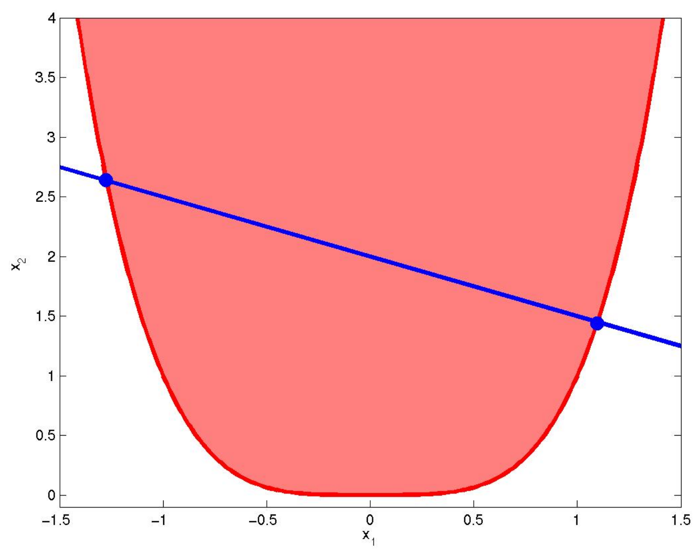
Helton & Vinnikov showed that a convex 2D set

$$\mathcal{F} = \{x \in \mathbb{R}^2 \,:\, p(x) \geq 0\}$$

defined by a polynomial $p(x)$ of minimum degree $d$ is LMI representable without lifting variables iff $\mathcal{F}$ is rigidly convex, meaning that

> for every point $x \in X$ and almost every line through $x$ then the line intersects $p(x) = 0$ in exactly $d$ points

Example: $\mathcal{F} = \left\{(x_1, x_2 \in \mathbb{R}^2 \,:\, p(x) = x_2 - x_1^4 \geq 0\right\}$
with 2 line intersections
is not rigidly convex because $2 < d = 4$




.. but it is LMI representable with lifting variables
see the previous construction for even power monomials