Finding a Maximum Stable Flow in a Multi-agent Network with Controllable Capacities

Nadia CHAABANE\textsuperscript{(1)} Cyril Briand \textsuperscript{(1)} Marie-José Huguet\textsuperscript{(1)}

\textsuperscript{(1)}LAAS-CNRS, University of Toulouse

ROADEF’14 - 26-28 February 2014
Outline

1. Introduction

2. Multi-agent Network with Controllable Capacities
   - Problem statement
   - Example

3. Finding a maximum-flow stable strategy
   - Characterization of Nash equilibria
   - Problem complexity
   - MILP formulation

4. Conclusion
1. **Introduction**

2. **Multi-agent Network with Controllable Capacities**
   - Problem statement
   - Example

3. **Finding a maximum-flow stable strategy**
   - Characterization of Nash equilibria
   - Problem complexity
   - MILP formulation

4. **Conclusion**
Optimization in a multi-agent context

- A set of agents is involved in a common decision-making process.
- Every agent controls its own decision variables (strategy) and is interested in maximizing its profit.
- The agents’ profit can depend on the strategies of the other agents.

⇒ Which strategy to adopt in order to fulfill the social goal, provided agents’ objectives are satisfied?
Introduction

Optimization in a multi-agent context

- A set of agents is involved in a common decision-making process.
- Every agent controls its own decision variables (strategy) and is interested in maximizing its profit.
- The agents’ profit can depend on the strategies of the other agents.

⇒ Which strategy to adopt in order to fulfill the social goal, provided agents’ objectives are satisfied?

Multi-objective Optimization

\[
\text{opt } F(x) = (Z_1(x), Z_2(x), \ldots, Z_m(x)) \ \text{s.c. } x \in \omega
\]

- \( A = \{A_1, \ldots, A_m\} \) Agents set
- \( x = (x_1, \ldots, x_m) \) Vector of agents’ strategies.
- \( Z_u(x) \) profit of agent \( A_u \)
**Strategy requirement**

### Efficiency
- Only non-dominated strategies are of interest because every agent wants to maximize its profit.
- Multi-objective optimization: Pareto optima

### Stability
- Given a strategy, no agent should be able to increase its profit by itself, while decreasing the profit of others.
- Game theory: Nash equilibria.

### Efficiency Vs. Stability
- Price of cooperation
  - Price of anarchy: \( PA = \frac{\text{GOF value in the worst NE}}{\text{OPT}} \)
  - Price of stability: \( PS = \frac{\text{GOF value in the best NE}}{\text{OPT}} \)

---

*a. Global Objective Function (GOF)*
1. Introduction

2. Multi-agent Network with Controllable Capacities
   - Problem statement
   - Example

3. Finding a maximum-flow stable strategy
   - Characterization of Nash equilibria
   - Problem complexity
   - MILP formulation

4. Conclusion
Multi-agent Network with Controllable Capacities

Definition

Multi-agent Network \( < G, A, Q, \overline{Q}, C, \pi, W > \):

- \( G = (V, E) \) is a Network:
  - set of nodes \( V \) where \( s, t \in V \) are the source and the sink nodes;
  - set of arcs \( E \) each one having its controllable capacity and receiving a flow.

- \( A = \{A_1, \ldots, A_u, \ldots, A_m\} \) a set of \( m \) agents;
  - Each arc \( (i, j) \) belongs to exactly one agent;
  - \( E_u \) set of arcs belonging to agent \( A_u \);

- \( Q = \{q_{i,j}\}_{(i,j) \in E} \) et \( \overline{Q} = \{\overline{q}_{i,j}\}_{(i,j) \in E} \) referred as the vectors of normal and maximum arc capacities.
  - \( q_{i,j} \in [\underline{q}_{i,j}, \overline{q}_{i,j}] \) is the capacity of arc \( (i, j) \).
**Definition (2)**

- \( C = \{ c_{i,j} \} \) is the vector of costs.
  - where \( c_{i,j} \) is the unitary costs incurred by agent \( A_u \) for increasing \( q_{i,j} \) by one unit;
  - The total cost incurred by agent \( A_u \) for increasing its arc capacities is
    \[ \sum_{(i,j) \in E_u} c_{i,j}(q_{i,j} - q_{i,j}) \]
- \( \pi \) the reward given by a final customer per unit of flow circulating in the Network.
- \( W = \{ w_u \} \) sharing policy of rewards among the agents.
  - The total reward for agent \( A_u \) is \( w_u \times \pi \times F \)
- \( F \) circulating flow in the Network.
  - \( f_{i,j} \) flow circulating on arc \((i,j)\) holds \( f_{i,j} \leq q_{i,j} \) where \( q_{i,j} \in [\underline{q}_{i,j}, \bar{q}_{i,j}] \)
Multi-objective mathematical program

\[
\begin{align*}
\text{Max} & \quad (Z_1(S), Z_2(S), \ldots, Z_m(S)) \\
\text{s.c.} & \quad (i) \quad f_{i,j} \leq q_{i,j}, \forall (i, j) \in E \\
& \quad (ii) \quad \sum_{(i,j) \in E} f_{i,j} = \sum_{(j,i) \in E} f_{ji}, \forall j \in V \{s, t\} \\
& \quad (iii) \quad q_{i,j} \leq q_{i,j} \leq \bar{q}_{i,j}, \forall (i, j) \in E \\
& \quad f_{i,j} \geq 0, \forall (i, j) \in E
\end{align*}
\]

- Profit of agent \(A_u\) is
  \[
  Z_u(S) = w_u \pi (F(S) - F) - \sum_{(i,j) \in E_u} c_{i,j}(q_{i,j} - q_{i,j})
  \]

- \(S = (Q_1, \ldots, Q_m)\) is the vector of individual strategies of all agents, where:
  - \(\underline{Q} \leq Q_u \leq \bar{Q}\) is the vector of the capacities chosen by agent \(A_u\) for the arcs belonging to him.
Multi-agent Network with Controllable Capacities

Social goal
- The customer wants to maximize the flow in the Network and give a reward to agents.

Strategies of the agents
- The capacities chosen by the agent $A_u$ for its arcs corresponds to the individual strategy of $A_u$.

Type of the game
- Non-cooperative game between the agents where each of them aims to maximize its profit.
**Example**

**Example of MA-MCF Network**

- Network $G(V, U)$ with controllable capacities
- Two agents: Blue ($A_B$) and Green ($A_G$)
- Reward and sharing policy: $\pi = 120$ and $w_B = w_G = \frac{1}{2}$

**Figure**: Example of MA-MCF Network with two agents
Increasing the flow

Find an increasing path $P$ in the residual graph such that $cost_u(P) < w_u \times \pi$ for all $A_u$, with:

- $cost_u(P) = \sum_{(i,j) \in P^+ \cap E_u} c_{i,j} - \sum_{(i,j) \in P^- \cap E_u} c_{i,j}$

**Figure:** Example of MA-MCF Network with two agents
Decreasing the flow

Find a decreasing path $\overline{P}$ in the residual graph such that $profit_u(\overline{P}) > w_u \times \pi$ for all $A_u$, with:

$$profit_u(\overline{P}) = \sum_{(i,j) \in \overline{P}^- \cap E_u} c_{i,j} - \sum_{(i,j) \in \overline{P}^+ \cap E_u} c_{i,j}$$

**Figure:** Example of MA-MCF Network with two agents
Example (4)

### Efficiency Vs. Stability

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Flow</th>
<th>Reward</th>
<th>$cost_G$</th>
<th>$cost_B$</th>
<th>$Z_G$</th>
<th>$Z_B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_0$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Figure**: Strategy $S_0$
Example (5)

Efficiency Vs. Stability

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Flow</th>
<th>Reward</th>
<th>( cost_G )</th>
<th>( cost_B )</th>
<th>( Z_G )</th>
<th>( Z_B )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_0 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( S_1 )</td>
<td>1</td>
<td>120</td>
<td>30</td>
<td>30</td>
<td>30</td>
<td>30</td>
</tr>
</tbody>
</table>

**FIGURE**: Strategy \( S_1 \)
Example (6)

Efficiency Vs. Stability

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Flow</th>
<th>Reward</th>
<th>cost_\text{G}</th>
<th>cost_\text{B}</th>
<th>Z_\text{G}</th>
<th>Z_\text{B}</th>
</tr>
</thead>
<tbody>
<tr>
<td>S_0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>S_1</td>
<td>1</td>
<td>120</td>
<td>30</td>
<td>30</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>S_2</td>
<td>2</td>
<td>240</td>
<td>80</td>
<td>80</td>
<td>40</td>
<td>40</td>
</tr>
</tbody>
</table>

\textbf{FIGURE:} Strategy S_2
Example (7)

### Efficiency Vs. Stability

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Flow</th>
<th>Reward</th>
<th>$cost_G$</th>
<th>$cost_B$</th>
<th>$Z_G$</th>
<th>$Z_B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_0$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$S_1$</td>
<td>1</td>
<td>120</td>
<td>30</td>
<td>30</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>$S_2$</td>
<td>2</td>
<td>240</td>
<td>80</td>
<td>80</td>
<td>40</td>
<td>40</td>
</tr>
</tbody>
</table>

- $S_2$ maximizes the flow.
- Is $S_2$ Nash-stable?
Example (8)

### Efficiency Vs. Stability

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Flow</th>
<th>Reward</th>
<th>$cost_G$</th>
<th>$cost_B$</th>
<th>$Z_G$</th>
<th>$Z_B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_0$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$S_1$</td>
<td>1</td>
<td>120</td>
<td>30</td>
<td>30</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>$S_2$</td>
<td>2</td>
<td>240</td>
<td>80</td>
<td>80</td>
<td>40</td>
<td>40</td>
</tr>
</tbody>
</table>

**Figure**: Residual graph corresponding to the strategy $S_2$
### Example (9)

#### Efficiency Vs. Stability

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Flow</th>
<th>Reward</th>
<th>$cost_G$</th>
<th>$cost_B$</th>
<th>$Z_G$</th>
<th>$Z_B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_0$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$S_1$</td>
<td>1</td>
<td>120</td>
<td>30</td>
<td>30</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>$S_2$</td>
<td>2</td>
<td>240</td>
<td>80</td>
<td>80</td>
<td>40</td>
<td>40</td>
</tr>
</tbody>
</table>

**Figure**: Profitable decreasing path for agent $A_G$
Example (10)

### Efficiency Vs. Stability

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Flow</th>
<th>Reward</th>
<th>$cost_G$</th>
<th>$cost_B$</th>
<th>$Z_G$</th>
<th>$Z_B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_0$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$S_1$</td>
<td>1</td>
<td>120</td>
<td>30</td>
<td>30</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>$S_2$</td>
<td>2</td>
<td>240</td>
<td>80</td>
<td>80</td>
<td>40</td>
<td>40</td>
</tr>
</tbody>
</table>

Agent $A_G$ can improve its own profit by modifying unilaterally its strategy.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Flow</th>
<th>Reward</th>
<th>$cost_G$</th>
<th>$cost_B$</th>
<th>$Z_G$</th>
<th>$Z_B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S'_2$</td>
<td>1</td>
<td>120</td>
<td>10</td>
<td>80</td>
<td>50</td>
<td>-20</td>
</tr>
</tbody>
</table>
Efficiency Vs. Stability

- $S_2$ is a Pareto Optimum but not a Nash equilibrium
- $S_1$ is not Pareto Optimum but a Nash equilibrium

Objective

- Find a strategy that maximizes the flow, which is a Nash equilibrium.
Outline

1 Introduction

2 Multi-agent Network with Controllable Capacities
   - Problem statement
   - Example

3 Finding a maximum-flow stable strategy
   - Characterization of Nash equilibria
   - Problem complexity
   - MILP formulation

4 Conclusion
A non-poor strategy $S$ is a Nash equilibrium if and only if there is no profitable path $P$ such that:

1. $\forall A_u \in A, \forall P \in \mathcal{P}$ such that $(i, j) \in \mathcal{E}_u$
   \[
   \text{cost}_u(P) > w_u \times \pi \quad (1)
   \]

2. $\forall A_u \in A, \forall \overline{P} \in \overline{P}$
   \[
   \text{profit}_u(\overline{P}) < w_u \times \pi \quad (2)
   \]
Problem complexity

Is there a Nash equilibrium strategy $S$ such that $F(S) > \varphi$?

- Strongly NP-complete.

Reduction from 3-partition problem

- Consider a set $\zeta = \{a_1, \ldots, a_K\}$ of $K = 3 \times k$ integer, such that $a$:
  - each integer $a_i \in ]B/4, B/2]$, for all $i = 1, \ldots, K$
  - $\sum_{i=1}^{K} a_i = k \times B$

- Deciding whether $\zeta$ can be partitioned into $k$ subsets so that the sum of integers in each subset is equal to $B$?

---

Problem complexity

Proof: Reduction from a 3-partition problem

- Reduce an instance of the 3-partition problem to an instance of MA-MCF problem.
- Build up a network $G$ from the 3-partition problem instance, such that $K = 9$, $k = 3$, $B = 24$ and $\zeta = \{7, 8, 7, 7, 8, 9, 10, 9\}$.
  - Multi-agent Network flow with $k = 3$ agents
  - each agent owns $K = 9$ arcs
  - each integer $a_i$ is represented by 3 parallel arcs with capacity $q_{i,j} \in [0, 1]$ and cost $c_{i,j} = a_i$
  - flow = 0
  - Reward of the agent $A_u$: $w_u \times \pi = B + \epsilon$ where $\epsilon$ is small positive number.
Problem complexity

Proof: Reduction from a 3-partition problem

- Build up a network $G$ from the 3-partition problem instance, where $K = 9$, $k = 3$, $B = 24$ and $\zeta = \{7, 8, 7, 7, 7, 8, 9, 10, 9\}$.
- Multi-agent Network flow with $k = 3$ agents
- each agent owns $K = 9$ arcs
- each integer $a_i$ is represented by 3 parallel arcs with capacity $q_{i,j} \in [0, 1]$ and cost $c_{i,j} = a_i$
- flow = 0
- Reward of the agent $A_u$: $w_u \times \pi = B + \epsilon$ where $\epsilon$ is small positive number.

![Figure: Reduction from a 3-partition instance problem with $k = 3$](image)
Problem complexity

Proof: Reduction from a 3-partition problem (2)

- Find whether it exists a Nash strategy such that the flow is strictly greater than 0?
- Find a profitable path such that the cost for each agent must not exceed the reward $B + \epsilon = 24 + \epsilon$

**Figure:** Reduction from a 3-partition instance problem with $k = 3$
Problem complexity

Proof: Reduction from a 3-partition problem (2)

- Find, whether it exists, a Nash strategy such that the flow is strictly greater than 0?
- Find a profitable path such that the cost for each agent must not exceed the reward $B + \epsilon = 24 + \epsilon$

Problem complexity

Finding a Nash Equilibrium that maximizes the flow in a multi-agent Network with controllable capacities is NP-hard.
Mathematical formulation

Finding a NE maximizing the flow

Max \( F \)

s.t.

\[
\sum_{(i,j) \in E} f_{i,j} - \sum_{(j,i) \in E_r} f_{j,i} = \begin{cases} 
0 & \forall i \neq s, t \\
F & i = s, \forall i \in V \\
-F & i = t 
\end{cases}
\]

\( (iii) \quad q_{i,j} \leq q_{i,j} \leq \overline{q}_{i,j}, \forall (i,j) \in E \)

\( (iv) \quad profit_u(P) < w_u \times \pi, \forall P \in G_f \quad \text{Stability constraints} \)

\( f_{i,j} \geq 0, \forall (i,j) \in E \)
How to formulate the stability constraints?

Linearize the Nash stability constraints as constraints of MILP.

\[(iv) \quad \text{profit}_u(\overline{P}) < w_u \times \pi, \quad \forall \overline{P} \in \overline{G}_f \]  
Stability constraints

\[f_{i,j} \geq 0, \quad \forall (i, j) \in E\]

- Formulating the constraints based on the identification of decreasing path \(\overline{P} \in \overline{G}_f\) having the maximum profit \(\text{profit}_u(\overline{P})\).
MILP formulation

Constraint reformulation

- Constraint reformulation is based on identification of path $P \in \mathcal{P}(S)$ having maximal $\text{profit}_u(P)$.

$$\text{profit}_u(P) < w_u \times \pi, \ \forall P \in \mathcal{P}(S) \Rightarrow \max_{\forall P \in \mathcal{P}} \text{profit}_u(P) < w_u \times \pi$$

- All path have to be computed.
- Computing the longest path for a given strategy using primal/dual constraints.
MILP formulation

Using primal/dual constraints:

Primal: longest path

\begin{align*}
\text{Min} & \quad t_{n+1}^u \\
\text{s.t.} & \quad t_j^u - t_i^u \geq \delta_{i,j}^{i,j,u}, \forall (i,j) \in E, \forall A_u \in A \\
(i) & \quad t_j^u - t_i^u \geq \delta_{F}^{i,j,u}, \forall (i,j) \in E, \forall A_u \in A \\
(ii) & \quad t_i^u \geq 0, \forall i \in V
\end{align*}

Dual

\begin{align*}
\text{Max} & \quad \sum_{(i,j) \in E} \varphi_{i,j}^u \times \delta_{F}^{i,j,u} + \sum_{(i,j) \in E_r} \varphi_{i,j}^u \times \delta_{B}^{i,j,u} \\
\text{s.t.} & \quad \sum_{(i,j) \in E \cup E_r} \varphi_{i,j}^u - \sum_{(i,j) \in E \cup E_r} \varphi_{j,i}^u = \begin{cases} 
0 & \forall i \neq s, t \\
-1, i = s & , \forall i \in V, \forall A_u \in A \\
1, i = t & 
\end{cases} \\
\phi_{i,j}^u & \in \{0, 1\} \forall (i,j) \in E, \forall A_u \in A
\end{align*}
Outline

1. Introduction

2. Multi-agent Network with Controllable Capacities
   - Problem statement
   - Example

3. Finding a maximum-flow stable strategy
   - Characterization of Nash equilibria
   - Problem complexity
   - MILP formulation

4. Conclusion
Conclusion

Multi-agent min cost flow

Multi-agent min cost flow problem with controllable capacities.

- Efficiency and stability notions.
- Optimization problem = Find a Nash equilibrium that maximizes the network flow.
  - NP-hard in the strong sense.

Perspectives

- Work in progress
  - Mixed Integer Linear Programming (MILP) in order to find the best Nash equilibrium that maximizes the flow.
- Future work
  - Distributed approach to find efficient strategies that are Nash-stable.
Conclusion

Multi-agent min cost flow

Multi-agent min cost flow problem with controllable capacities.

- Efficiency and stability notions.
- Optimization problem = Find a Nash equilibrium that maximizes the network flow.
  - NP-hard in the strong sense.

Perspectives

- Work in progress
  - Mixed Integer Linear Programming (MILP) in order to find the best Nash equilibrium that maximizes the flow.
- Future work
  - Distributed approach to find efficient strategies that are Nash-stable.
Thank you for your attention!