Finding a Maximum Stable Flow in a Multi-agent Network with Controllable Capacities

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ROADEF'14 - 26-28 February 2014

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Outline

Outline

Introduction

Optimization in a multi-agent context

- A set of agents is involved in a common decision-making process.
- Every agent controls its own decision variables (strategy) and is interested in maximizing its profit.
- The agents' profit can depend on the strategies of the other agents.
  $\Rightarrow$ Which strategy to adopt in order to fulfill the social goal, provided agents' objectives are satisfied ?

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- A set of agents is involved in a common decision-making process.
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  $\Rightarrow$ Which strategy to adopt in order to fulfill the social goal, provided agents' objectives are satisfied ?

Multi-objective Optimization

$$\text{opt}\, F(x) = (Z_1(x), Z_2(x), \ldots, Z_m(x)) \text{ s.c. } x \in \omega$$

- $A = \{A_1, \ldots, A_m\}$ Agents set
- $x = (x_1, \ldots, x_m)$ Vector of agents' strategies.
- $Z_u(x)$ profit of agent A_u

Strategy requirement

Efficiency

- Only non-dominated strategies are of interest because every agent wants to maximize its profit.
- Multi-objective optimization : Pareto optima

Stability

- Given a strategy, no agent should be able to increase its profit by itself, while decreasing the profit of others.
- Game theory : Nash equilibria.

Efficiency Vs. Stability

- Price of cooperation
  - Price of anarchy : PA=(GOF value in the worst NE)/(OPT) [a]
  - Price of stability : PS=(GOF value in the best NE)/(OPT)

a. Global Objective Function (GOF).

Outline

Multi-agent Network with Controllable Capacities

Definition

Multi-agent Network $< G, \mathcal{A}, \underline{Q}, \overline{Q}, C, \pi, W >$:

- $G = (V, E)$ is a Network :
  - set of nodes V where $s, t \in V$ are the source and the sink nodes ;
  - set of arcs E each one having its controllable capacity and receiving a flow.
- $\mathcal{A} = \{A_1, \ldots, A_u, \ldots, A_m\}$ a set of m agents ;
  - Each arc (i, j) belongs to exactly one agent ;
  - E_u set of arcs belonging to agent A_u ;
- $\underline{Q} = \{\underline{q}_{i,j}\}_{(i,j)\in E}$ et $\overline{Q} = \{\overline{q}_{i,j}\}_{(i,j)\in E}$ referred as the vectors of normal and maximum arc capacities.
  - $q_{i,j} \in [\underline{q}_{i,j}, \overline{q}_{i,j}]$ is the capacity of arc (i, j).

Multi-agent Network with Controllable Capacities

Definition (2)

- $C = \{c_{i,j}\}$ is the vector of costs.
  - where $c_{i,j}$ is the unitary costs incurred by agent A_u for increasing $q_{i,j}$ by one unit ;
  - The total cost incurred by agent A_u for increasing its arc capacities is $\sum_{(i,j)\in E_u} c_{i,j}(q_{i,j} - \underline{q}_{i,j})$
- π the reward given by a final customer per unit of flow circulating in the Network.
- $W = \{w_u\}$ sharing policy of rewards among the agents.
  - The total reward for agent A_u is $w_u \times \pi \times F$
- F circulating flow in the Network.
  - $f_{i,j}$ flow circulating on arc (i, j) holds $f_{i,j} \leq q_{i,j}$ where $q_{i,j} \in [\underline{q}_{i,j}, \overline{q}_{i,j}]$

Multi-agent Network with Controllable Capacities

Multi-objective mathematical program

$$
\begin{array}{ll}
Max & (Z_1(S), Z_2(S), \ldots, Z_m(S)) \\
s.c. & \\
(i) & f_{i,j} \leq q_{i,j}, \forall (i,j) \in E \\
(ii) & \sum_{(i,j)\in E} f_{i,j} = \sum_{(j,i)\in E} f_{ji}, \forall j \in V\{s,t\} \\
(iii) & \underline{q}_{i,j} \leq q_{i,j} \leq \overline{q}_{i,j}, \forall (i,j) \in E \\
 & f_{i,j} \geq 0, \forall (i,j) \in E
\end{array}
$$

- Profit of agent A_u is
  $Z_u(S) = w_u \, \pi \, (F(S) - \underline{F}) - \sum_{(i,j)\in E_u} c_{i,j}(q_{i,j} - \underline{q}_{i,j})$
- $S = (Q_1, \ldots, Q_m)$ is the vector of individual strategies of all agents, where :
  $\underline{Q} \leq Q_u \leq \overline{Q}$ is the vector of the capacities chosen by agent A_u for the arcs belonging to him.

Multi-agent Network with Controllable Capacities

Social goal

- The customer wants to maximize the flow in the Network and give a reward to agents.

Strategies of the agents

- The capacities chosen by the agent A_u for its arcs corresponds to the individual strategy of A_u.

Type of the game

- Non-cooperative game between the agents where each of them aims to maximize its profit.

Example

Example of MA-MCF Network

- Network $G(V, U)$ with controllable capacities
- Two agents : Blue (A_B) and Green (A_G)
- Reward and sharing policy : $\pi = 120$ and $w_B = w_G = \frac{1}{2}$

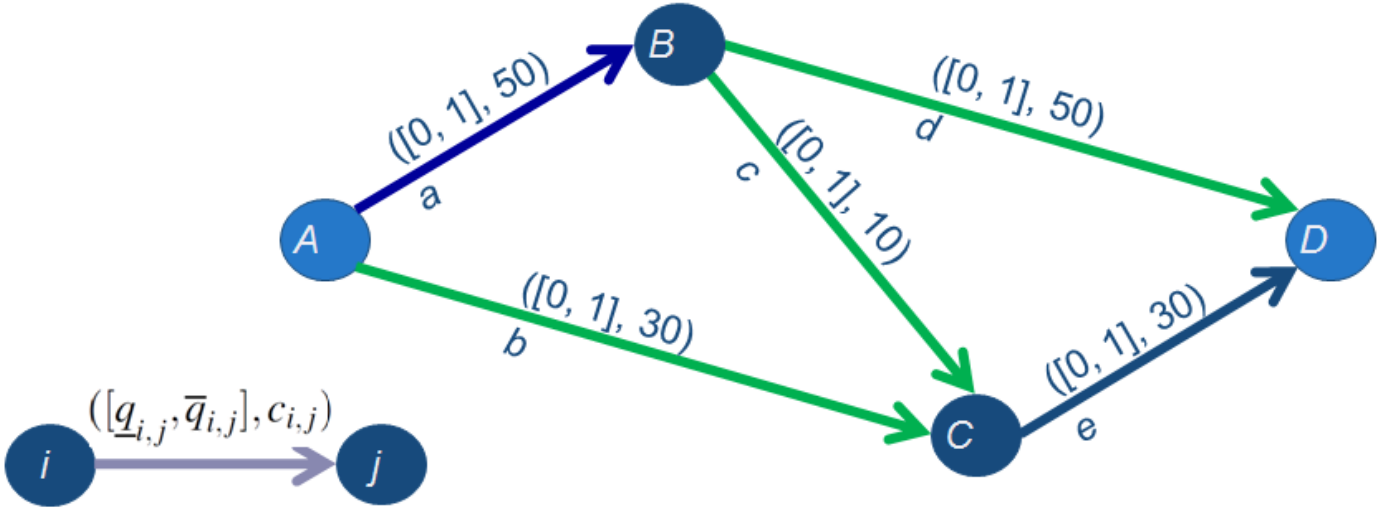


FIGURE: Example of MA-MCF Network with two agents

Example (2)

Increasing the flow

Find an increasing path P in the residual graph such that $cost_u(P) < w_u \times \pi$ for all A_u, with :

- $cost_u(P) = \sum_{(i,j)\in P^+\cap E_u} c_{i,j} - \sum_{(i,j)\in P^-\cap E_u} c_{i,j}$

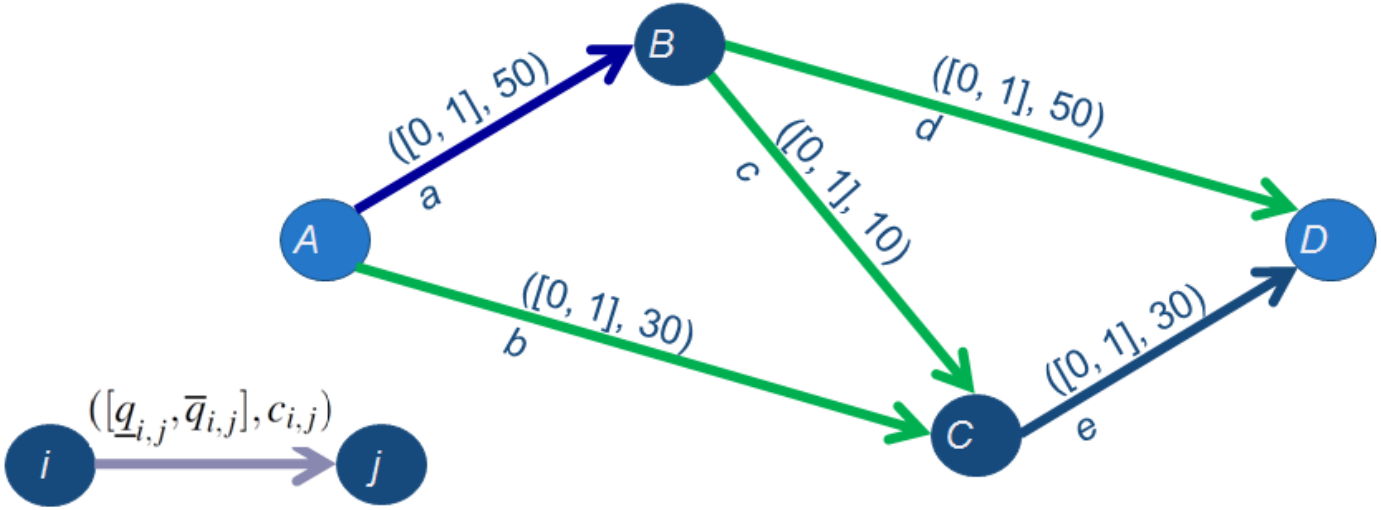


FIGURE: Example of MA-MCF Network with two agents

Example (3)

Decreasing the flow

Find a decreasing path $\overline{P}$ in the residual graph such that $profit_u(\overline{P}) > w_u \times \pi$ for all A_u, with :

- $profit_u(\overline{P}) = \sum_{(i,j)\in\overline{P}^- \cap E_u} c_{i,j} - \sum_{(i,j)\in\overline{P}^+ \cap E_u} c_{i,j}$

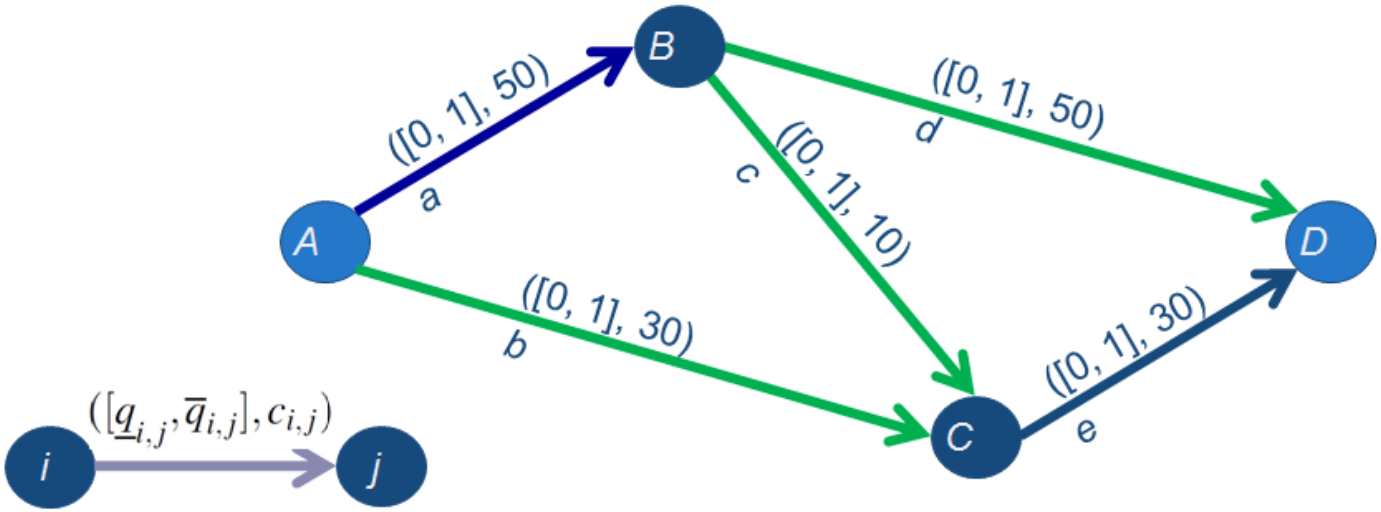


FIGURE: Example of MA-MCF Network with two agents

Example (4)

Efficiency Vs. Stability

Strategy	Flow	Reward	$cost_G$	$cost_B$	Z_G	Z_B
S_0	0	0	0	0	0	0

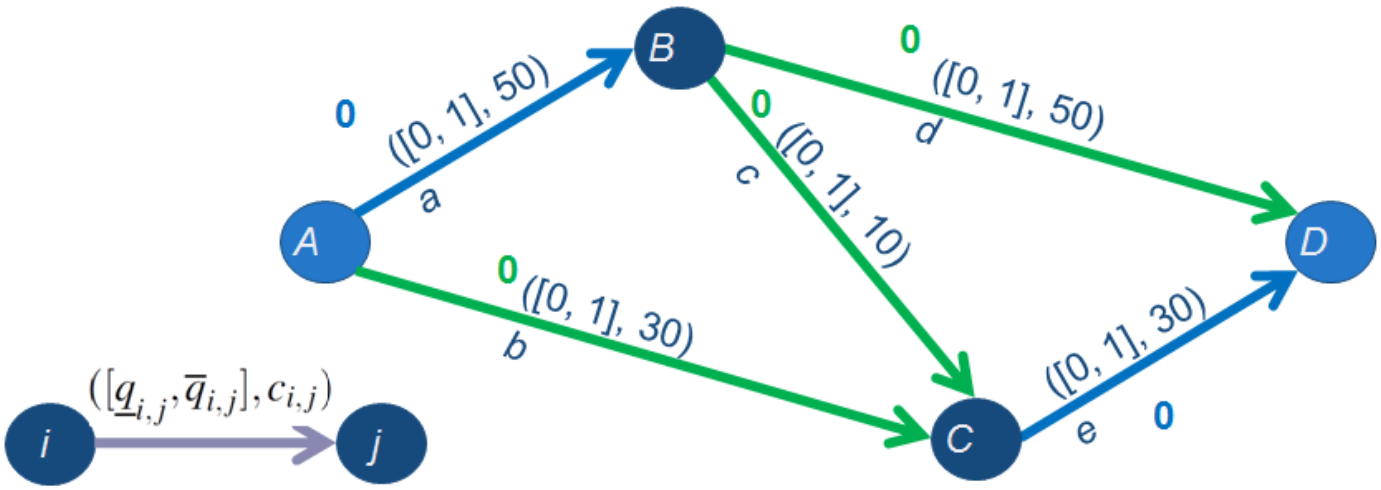


FIGURE: Strategy S_0

Example (5)

Efficiency Vs. Stability

Strategy	Flow	Reward	$cost_G$	$cost_B$	Z_G	Z_B
S_0	0	0	0	0	0	0
S_1	1	120	30	30	30	30

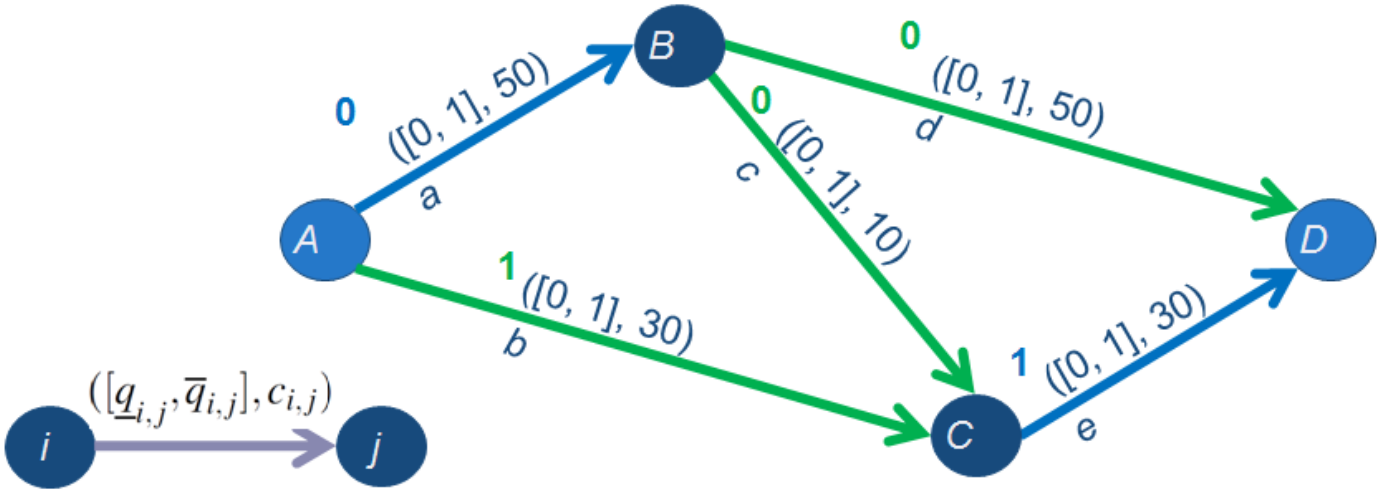


FIGURE: Strategy S_1

Example (6)

Efficiency Vs. Stability

Strategy	Flow	Reward	$cost_G$	$cost_B$	Z_G	Z_B
S_0	0	0	0	0	0	0
S_1	1	120	30	30	30	30
S_2	2	240	80	80	40	40

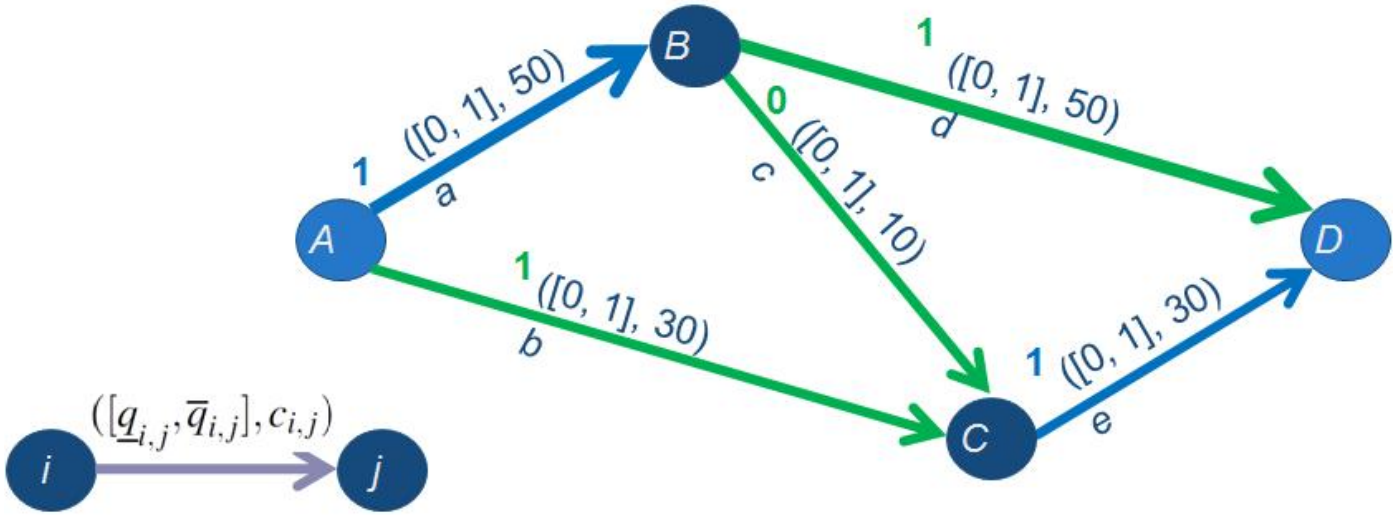


FIGURE: Strategy S_2

Example (7)

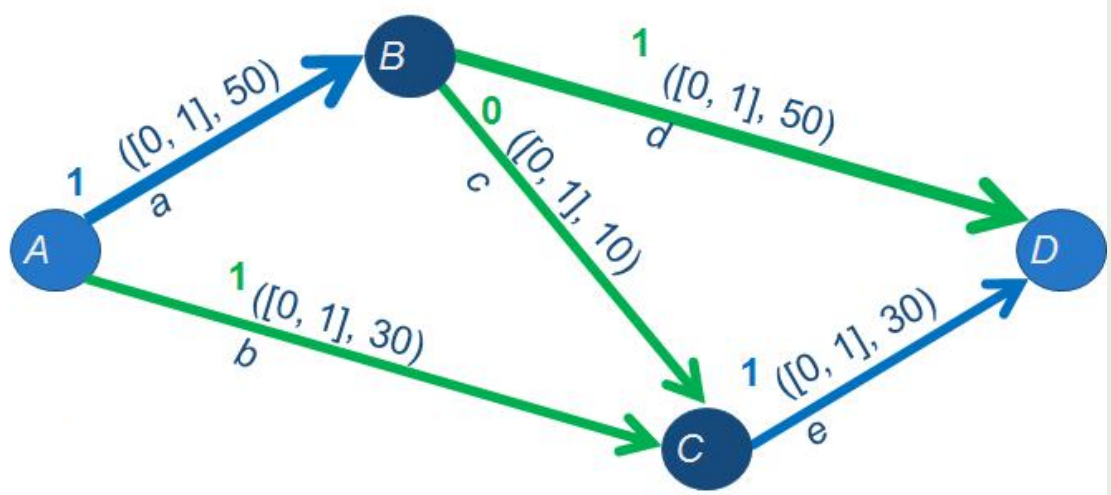
Efficiency Vs. Stability

Strategy	Flow	Reward	$cost_G$	$cost_B$	Z_G	Z_B
S_0	0	0	0	0	0	0
S_1	1	120	30	30	30	30
S_2	2	240	80	80	40	40

Efficiency Vs. Stability

- S_2 maximizes the flow.
- Is S_2 Nash-stable ?

Strategy S_2



Example (8)

Efficiency Vs. Stability

Strategy	Flow	Reward	$cost_G$	$cost_B$	Z_G	Z_B
S_0	0	0	0	0	0	0
S_1	1	120	30	30	30	30
S_2	2	240	80	80	40	40

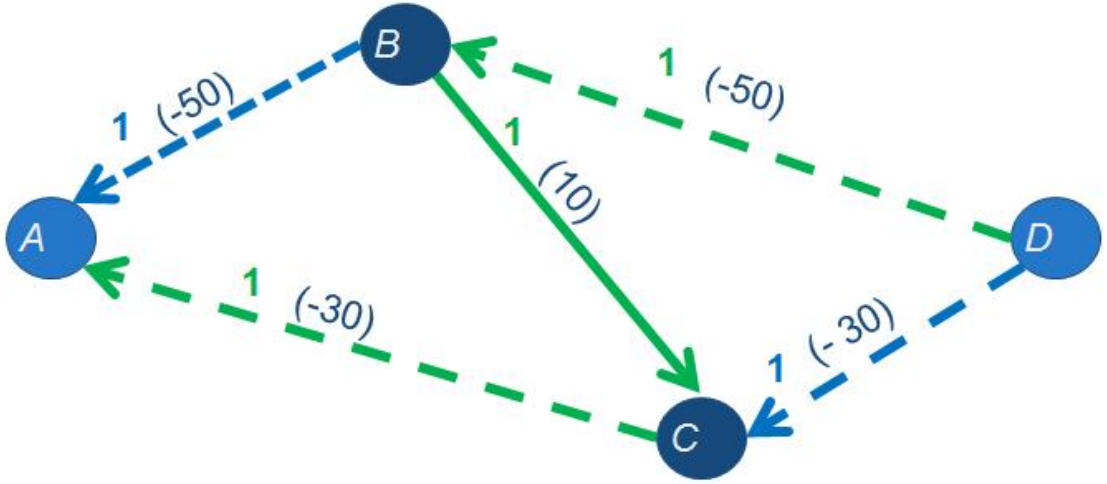


FIGURE: Residual graph corresponding to the strategy S_2

Example (9)

Efficiency Vs. Stability

Strategy	Flow	Reward	$cost_G$	$cost_B$	Z_G	Z_B
S_0	0	0	0	0	0	0
S_1	1	120	30	30	30	30
S_2	2	240	80	80	40	40

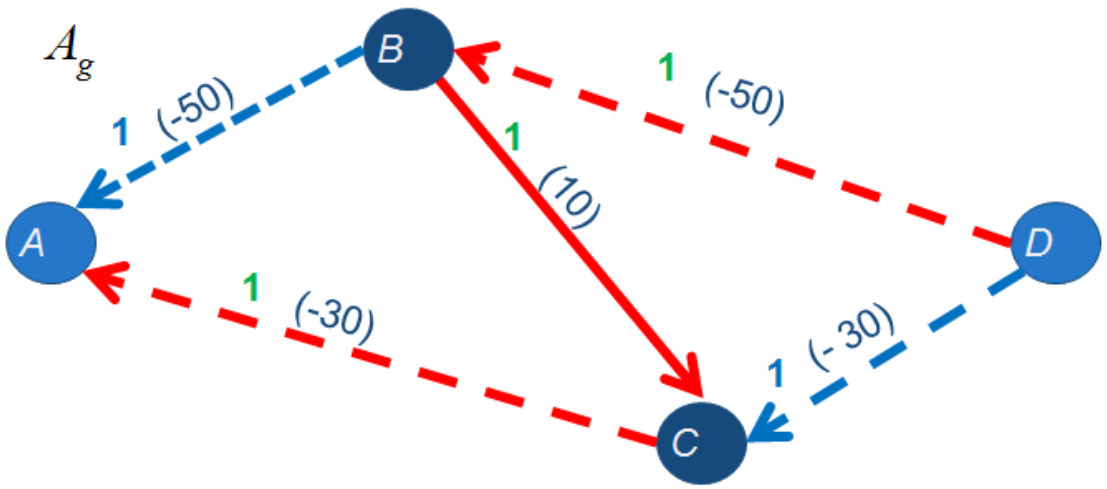


FIGURE: Profitable decreasing path for agent A_G

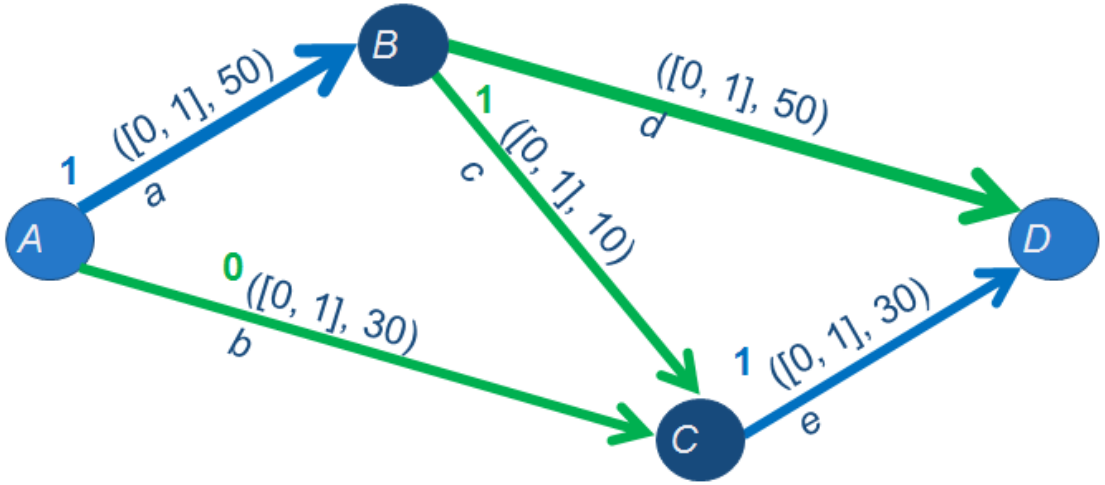
Example (10)

Efficiency Vs. Stability

Strategy	Flow	Reward	$cost_G$	$cost_B$	Z_G	Z_B
S_0	0	0	0	0	0	0
S_1	1	120	30	30	30	30
S_2	2	240	80	80	40	40

Agent A_G can improve its own profit by modifying unilaterally its strategy.

Strategy	Flow	Reward	$cost_G$	$cost_B$	Z_G	Z_B
S_2'	1	120	10	80	50	-20



Example (11)

Efficiency Vs. Stability

- S_2 is a Pareto Optimum but not a Nash equilibrium
- S_1 is not Pareto Optimum but a Nash equilibrium

Objective

- Find a strategy that maximizes the flow, which is a Nash equilibrium.

Outline

Characterization of Nash equilibrium

Nash equilibrium

- A non-poor strategy S is a Nash equilibrium if and only if there is no profitable path P such that :
  - $\forall A_u \in \mathcal{A}, \forall P \in \mathcal{P}$ such that $(i, j) \in \mathcal{E}_u$

$$cost_u(P) > w_u \times \pi \tag{1}$$

  - $\forall A_u \in \mathcal{A}, \forall \overline{P} \in \overline{\mathcal{P}}$

$$profit_u(\overline{P}) < w_u \times \pi \tag{2}$$

Problem complexity

Is there a Nash equilibrium strategy S such that $F(S) > \varphi$?

- Strongly NP-complete.

Reduction from 3-partition problem

- Consider a set $\zeta = \{a_1, \ldots, a_K\}$ of $K = 3 \times k$ integer, such that[a] :
  - each integer $a_i \in]B/4, B/2]$, for all $i = 1, \ldots, K$
  - $\sum_{i=1}^{K} a_i = k \times B$
- Deciding whether ζ can be partitioned into k subsets so that the sum of integers in each subset is equal to B ?

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a. Garey, M.R. and Johnson, D.S. (1979). Computers and Intracability : A guide to the Theory of NP-Completeness. W.H. Freeman and Co., New York, USA.

Problem complexity

Proof : Reduction from a 3-partition problem

- Reduce an instance of the 3-partition problem to an instance of MA-MCF problem.
- Build up a network G from the 3-partition problem instance, such that $K = 9$, $k = 3$, $B = 24$ and $\zeta = \{7, 8, 7, 7, 7, 8, 9, 10, 9\}$.
  - Multi-agent Network flow with $k = 3$ agents
  - each agent owns $K = 9$ arcs
  - each integer a_i is represented by 3 parallel arcs with capacity $q_{i,j} \in [0, 1]$ and cost $c_{i,j} = a_i$
  - flow $= 0$
  - Reward of the agent A_u : $w_u \times \pi = B + \epsilon$ where ϵ is small positive number.

Problem complexity

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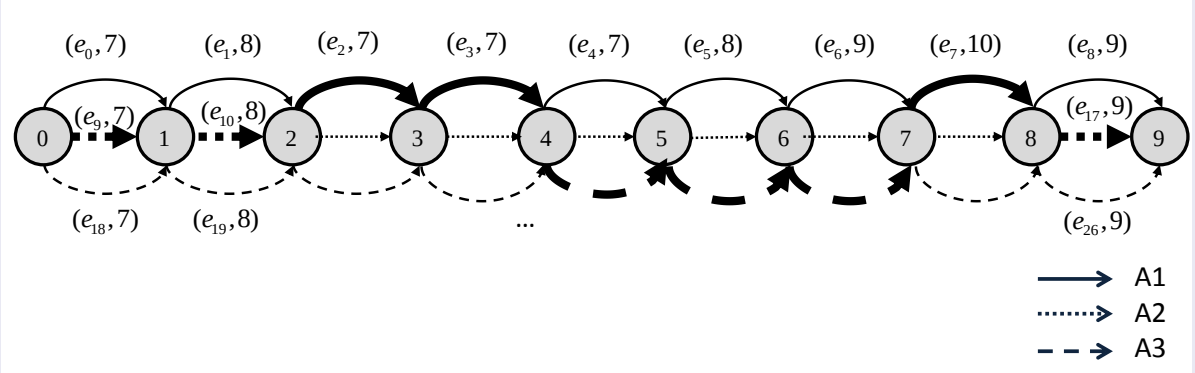
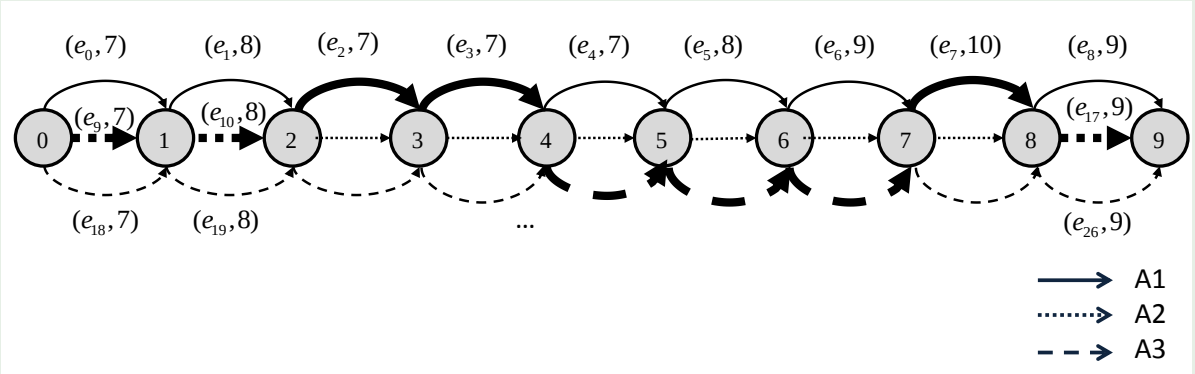


FIGURE: Reduction from a 3-partition instance problem with $k = 3$

Proof : Reduction from a 3-partition problem (2)

- Find whether it exists a Nash strategy such that the flow is strictly greater than 0 ?
- Find a profitable path such that the cost for each agent must not exceed the reward $B + \epsilon = 24 + \epsilon$



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Problem complexity

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- Find a profitable path such that the cost for each agent must not exceed the reward $B + \epsilon = 24 + \epsilon$

Problem complexity

Finding a Nash Equilibrium that maximizes the flow in a multi-agent Network with controllable capacities is NP$-$hard.

Mathematical formulation

Finding a NE maximizing the flow

$$
\begin{array}{lll}
Max & F & \\
s.t. & & \\
(1) & \sum_{(i,j)\in E} f_{i,j} - \sum_{(j,i)\in E_r} f_{j,i} = \begin{cases} 0 \ \forall i \neq s,t \\ F \ , \ i = s \\ -F \ , \ i = t \end{cases} , \forall i \in V & \\
(iii) & \underline{q}_{i,j} \leq q_{i,j} \leq \overline{q}_{i,j}, \ \forall (i,j) \in E & \\
(iv) & profit_u(\overline{P}) < w_u \times \pi, \ \forall \overline{P} \in \overline{G}_f & \text{Stability constraints} \\
 & f_{i,j} \geq 0, \ \forall (i,j) \in E &
\end{array}
$$

Mathematical formulation

How to formulate the stability constraints ?

Linearize the Nash stability constraints as constraints of MILP.

(iv) $\quad profit_u(\overline{P}) < w_u \times \pi, \forall \overline{P} \in \overline{G}_f$ $\qquad$ Stability constraints

$f_{i,j} \geq 0, \forall (i,j) \in E$

- Formulating the constraints based on the identification of decreasing path $\overline{P} \in \overline{G}_f$ having the maximum profit $profit_u(\overline{P})$.

MILP formulation

Constraint reformulation

- Constraint reformulation is based on identification of path $\overline{P} \in \overline{\mathcal{P}}(S)$ having maximal $profit_u(\overline{P})$.

$$profit_u(\overline{P}) < w_u \times \pi, \forall \overline{P} \in \overline{\mathcal{P}}(S) \Rightarrow \max_{\forall \overline{P} \in \overline{\mathcal{P}}} profit_u(\overline{P}) < w_u \times \pi$$

- All path have to be computed.
- Computing the longest path for a given strategy using primal/dual constraints.

MILP formulation

Using primal/dual constraints :

Primal : longest path

$$
\begin{array}{ll}
Min & t^u_{n+1} \\
s.t. & \\
(i) & t^u_j - t^u_i \geq \delta^{i,j,u}_F, \forall (i,j) \in E, \forall A_u \in \mathcal{A} \\
(ii) & t^u_i - t^u_j \geq \delta^{j,i,u}_B, \forall (i,j) \in E, \forall A_u \in \mathcal{A} \\
 & t^u_i \geq 0, \forall i \in V
\end{array}
$$

Dual

$$
\begin{array}{ll}
Max & \sum_{(i,j)\in E} \varphi^u_{i,j} \times \delta^{i,j,u}_F + \sum_{(i,j)\in E_r} \varphi^u_{i,j} \times \delta^{j,i,u}_B \\
s.t. & \\
(i) & \sum_{(i,j)\in E\cup E_r} \varphi^u_{i,j} - \sum_{(i,j)\in E\cup E_r} \varphi^u_{j,i} = \begin{cases} 0 \ \forall i \neq s,t \\ -1 \ , \ i = s \\ 1 \ , \ i = t \end{cases} , \forall i \in V, \forall A_u \in \mathcal{A} \\
 & \phi^u_{i,j} \in \{0,1\} \forall (i,j) \in E, \forall A_u \in \mathcal{A}
\end{array}
$$

Outline

Conclusion

Multi-agent min cost flow

Multi-agent min cost flow problem with controllable capacities.

- Efficiency and stability notions.
- Optimization problem = Find a Nash equilibrium that maximizes the network flow.
  - NP-hard in the strong sense.

Conclusion

Multi-agent min cost flow

Multi-agent min cost flow problem with controllable capacities.

- Efficiency and stability notions.
- Optimization problem = Find a Nash equilibrium that maximizes the network flow.
  - NP-hard in the strong sense.

Perspectives

- Work in progress
  - Mixed Integer Linear Programming (MILP) in order to find the best Nash equilibrium that maximizes the flow.
- Future work
  - Distributed approach to find efficient strategies that are Nash-stable.

Thank you for your attention !