COLUMN GENERATION FOR BI-OBJECTIVE VEHICLE PROBLEMS WITH A MIN-MAX OBJECTIVE

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MULTI-OBJECTIVE INTEGER PROGRAMMING

BI-OBJECTIVE VRP WITH A MIN-MAX OBJECTIVE

APPLICATION TO THE BOMCTP

COMPUTATIONAL RESULTS

CONCLUSIONS
Definition of a Multi-Objective Integer Program

\[
\text{(MOIP)} = \begin{cases} 
\min & \ F(x) = (f_1(x), f_2(x), \ldots, f_r(x)) \\
\text{s.t.} & \ x \in \mathcal{X} 
\end{cases}
\]

- \( r \geq 2 \) : number of objective functions
- \( F = (f_1, f_2, \ldots, f_r) \) : vector of objective functions
- \( \mathcal{X} \subseteq \mathbb{N}^n \) : feasible set of solutions
- \( \mathcal{Y} = F(\mathcal{X}) \) : feasible set in objective space
- \( x = (x_1, x_2, \ldots, x_n) \in \mathcal{X} \) : variable vector, variables
- \( y = (y_1, y_2, \ldots, y_r) \in \mathcal{Y} \) with \( y_i = f_i(x) \) : vector of objective function values
A solution $x^1$ dominates another solution $x^2$ if and only if
$\forall i \in \{1, \ldots, n\}, f_i(x^1) \leq f_i(x^2)$ and $\exists i \in \{1, \ldots, n\}$ such that
$f_i(x^1) < f_i(x^2)$.

A Pareto optimal solution is a solution dominated by no other feasible solution.

The image of a Pareto optimal solution in the objective space is said to be nondominated.
Lower Bound of a MOIP [Villarreal and Karwan, 1981]

A set of points (feasible or not) such that the image of every feasible solution is dominated by at least one of the points.

Upper Bound
A set of mutually nondominated feasible points.

- $f_1$
- $f_2$

- image of feasible solution
Lower Bound of a MOIP [Villarreal and Karwan, 1981]

Lower Bound
A set of points (feasible or not) such that the image of every feasible solution is dominated by at least one of the points.

Upper Bound
A set of mutually nondominated feasible points.

Diagram:
- $f_1$ and $f_2$ axes
- Ideal point
- Nadir point
- Image of feasible solutions
- Estimate of nondominated region
Lower Bound of a MOIP [Villarreal and Karwan, 1981]

**Lower Bound**
A set of points (feasible or not) such that the image of every feasible solution is dominated by at least one of the points.

- **image of feasible solution**
- **estimate of nondominated region**
- **member of lower bound**

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![Diagram of Lower Bound](image)
Lower Bound of a MOIP [Villarreal and Karwan, 1981]

**Lower Bound**
A set of points (feasible or not) such that the image of every feasible solution is dominated by at least one of the points.

**Upper Bound**
A set of mutually nondominated feasible points.
Computing Bound Sets for a BOIP

**Lower Bound Set**
- Transform the BOIP into single objective by using a scalarization method (weighted sum, $\varepsilon$-constraint, ...).
- Solve relaxations of transformed problem for several values of the parameters in order to generate a set of points satisfying the definition of a lower bound set.

**Upper Bound Set**
- Generally, obtained through heuristics and metaheuristics.
- The heuristic or metaheuristic need not depend on any particular scalarization method.
Column Generation for a BOVRPMMO

Minimize \( \sum_{k \in \Omega} c_k \theta_k \)

Minimize \( \Gamma_{\text{max}} \)

\[ \sum_{k \in \Omega} a_{ik} \theta_k \geq b_i \quad (i \in I) \]

\[ \Gamma_{\text{max}} \geq \sigma_k \theta_k \quad (k \in \Omega) \]

\[ \theta_k \in \{0, 1\} \quad (k \in \Omega) \]

Assumption

- There is a finite number of possible values for \( \sigma_k \) in a range \([\sigma_{\text{min}}, \sigma_{\text{max}}]\).

- Example: \( \sigma_k \) is an integer.
Minimize
\[ \sum_{k \in \Omega} c_k \theta_k \]
Subject to
\[ \sum_{k \in \Omega} a_{ik} \theta_k \geq b_i \] \hspace{0.5cm} (i \in I)
\[ -\Gamma_{\text{max}} \geq -\varepsilon \]
\[ \Gamma_{\text{max}} \geq \sigma_k \theta_k \] \hspace{0.5cm} (k \in \Omega)
\[ \theta_k \in \{0, 1\} \] \hspace{0.5cm} (k \in \Omega)

**Advantage**

- Each column is valid for all values of \( \varepsilon \) and so the subproblem algorithm does not need to keep track of this parameter.

**Disadvantage**

- Usually need to introduce extra variables and this can negatively affect the quality of a lower bound set.
Reformulation of a BOVRPMMO

Minimize

\[ \sum_{k \in \tilde{\Omega}} c_k \theta_k \]

Minimize \( \Gamma_{\text{max}} \)

\[ \sum_{k \in \tilde{\Omega}} a_{ik} \theta_k \geq b_i \quad (i \in I) \]

\[ \Gamma_{\text{max}} \geq \sigma_k \theta_k \quad (k \in \Omega) \]

\[ \theta_k \in \{0, 1\} \quad (k \in \tilde{\Omega}) \]

Idea Used

- \( \Omega \) is extended into a new set of columns \( \tilde{\Omega} \).
- The feasibility of a column \( k \in \tilde{\Omega} \) depends on \( \sigma_k \).
- We define \( \tilde{\Omega}^\varepsilon = \{ k \in \tilde{\Omega} : \sigma_k \leq \varepsilon \} \) when we need solutions satisfying \( \Gamma_{\text{max}} \leq \varepsilon \).
Reformulation of a BOVRPMMO

\[
\begin{align*}
\text{Minimize} & \quad \sum_{k \in \tilde{\Omega}} c_k \theta_k \\
\text{Minimize} & \quad \Gamma_{\text{max}} \\
\sum_{k \in \tilde{\Omega}} a_{ik} \theta_k & \geq b_i & (i \in I) \\
\Gamma_{\text{max}} & \geq \sigma_k \theta_k & (k \in \tilde{\Omega}) \\
\theta_k & \in \{0, 1\} & (k \in \tilde{\Omega})
\end{align*}
\]

**Idea Used**

- \(\Omega\) is extended into a new set of columns \(\tilde{\Omega}\).
- The feasibility of a column \(k \in \tilde{\Omega}\) depends on \(\sigma_k\).
- We define \(\tilde{\Omega}^\varepsilon = \{k \in \tilde{\Omega} : \sigma_k \leq \varepsilon\}\) when we need solutions satisfying \(\Gamma_{\text{max}} \leq \varepsilon\).
Minimize \[
\sum_{k \in \bar{\Omega}} c_k \theta_k
\]
\[
\sum_{k \in \bar{\Omega}} a_{ik} \theta_k \geq b_i \quad (i \in I)
\]
\[
\theta_k \in \{0, 1\} \quad (k \in \bar{\Omega})
\]

**Advantages**

- Both the master and the subproblem are single objective problems.
- No weakening of the linear relaxation for a given value of \(\varepsilon\).

**Disadvantage**

- Need to manage the parameter \(\varepsilon\) in both the master problem and the subproblem.
For any given value of $\varepsilon$, completely solve the linear relaxation of $\text{MP}(\bar{\Omega}^\varepsilon)$ by column generation.
For any given value of $\varepsilon$, completely solve the linear relaxation of $\text{MP}(\bar{\Omega}^\varepsilon)$ by column generation.

\begin{align*}
\max & \quad \varepsilon \\
\min & \quad \varepsilon
\end{align*}
For any given value of $\varepsilon$, completely solve the linear relaxation of $\text{MP}(\Omega^\varepsilon)$ by column generation.
For any given value of $\varepsilon$, completely solve the linear relaxation of $\text{MP}(\tilde{\Omega}^\varepsilon)$ by column generation.
For any given value of $\varepsilon$, completely solve the linear relaxation of $\text{MP}(\bar{\Omega}^{\varepsilon})$ by column generation.
At each column generation iteration in PPS, use heuristics (problem dependent) to search for other columns that are relevant for current value of $\varepsilon$ and may also be relevant for other values.

**Advantage**

- Can cheaply find a large number of columns once the price of a few columns have been paid.
- Tries to take advantage of the reformulation and the similar subproblems associated to the different values of $\varepsilon$.

**Disadvantage**

- No guarantee that a column found by a heuristic will be relevant for other values of $\varepsilon$ apart from the current one.
Find a set of routes on $V' \subseteq V$ with minimum total length and such that the nodes of $W$ are covered by those of $V'$.

- The number of nodes on each route cannot exceed $p$. 

![Diagram with nodes and edges representing the Multi-Vehicle Covering Tour Problem.

- Green circles: May be visited
- Red squares: MUST be covered
- Blue circle: Depot
- Dotted lines: Vehicle routes
- Black lines: Cover distance

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The Multi-Vehicle Covering Tour Problem [Hachicha et al., 2000]
Design of bi-level transportation networks \cite{Current_and_Schilling_1994}.

- **Aim:** Construct a primary route such that all points that are not on it can easily reach it.

The postbox location problem \cite{Labbé_and_Laporte_1986}.

- **Aim:** minimize the cost of a collection route through all post boxes and also ensure that every user is located within a reasonable distance from a post box.


- **Medical services can only be delivered to a subset of villages, but all users must be able to reach a visiting medical team.**
**The Bi-Objective MCTP**

**Problem**
Given graph $G = (V \cup W, E)$, design a set of routes on $V' \subseteq V$. $D = (d_{ij})$ is a distance matrix satisfying the triangle inequality.

**Objectives**
- Minimize the total length of the set of routes.
- Minimize the cover distance induced by the set of routes.

**Constraints**
- Each route must start and also end at the depot.
- The number of nodes on each route should not exceed $p$. 
THE COVER DISTANCE INDUCED BY A SET OF ROUTES
The Cover Distance Induced by a Set of Routes
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THE COVER DISTANCE INDUCED BY A SET OF ROUTES
The Cover Distance Induced by a Set of Routes
A “standard” $\varepsilon$-Constraint Formulation for the BOMCTP

Minimize \[ \sum_{k \in \Omega} c_k \theta_k \] (1)

\[ \sum_{v_i \in V \setminus \{v_0\}} z_{ij} \geq 1 \quad (w_j \in W) \] (2)

\[ \sum_{k \in \Omega} a_{ik} \theta_k - z_{ij} \geq 0 \quad (v_i \in V \setminus \{v_0\}, w_j \in W) \] (3)

\[ \Gamma_{\text{max}} - d_{ij}z_{ij} \geq 0 \quad (v_i \in V \setminus \{v_0\}, w_j \in W) \] (4)

\[ -\Gamma_{\text{max}} \geq -\varepsilon \] (5)

\[ \Gamma_{\text{max}} \geq 0 \] (6)

\[ z_{ij} \in \{0, 1\} \quad (v_i \in V \setminus \{v_0\}, w_j \in W) \] (7)

\[ \theta_k \in \{0, 1\} \quad (k \in \Omega) \] (8)
 reformulation: master problem

Minimize \[ \sum_{k \in \bar{\Omega}} c_k \theta_k \]

subject to: \[ \sum_{k \in \bar{\Omega}} a_{jk} \theta_k \geq 1 \quad (w_j \in W) \]

\[ \theta_k \in \{0, 1\} \quad (k \in \bar{\Omega}) \]

- A column \( k \in \bar{\Omega} \) is defined as a route \( R_k \) together with a subset \( \Psi_k \subseteq W \) of nodes it may cover.

- \( \sigma_k : \max\{d_{ij} : v_i \in R_k \text{ and } w_j \in \Psi_k\} \).

- \( a_{jk} : 1 \text{ if } w_j \in \Psi_k, \text{ and } 0 \text{ otherwise.} \)
Reformulation: Subproblem corresponding to $\varepsilon$

$$S(\varepsilon) = \min_{k \in \Omega \setminus \Omega_1} \left\{ c_k - \sum_{w_j \in W} a_{jk} \pi_j : \sigma_k \leq \varepsilon \right\}$$

- $S(\varepsilon)$ is associated with a dual vector $\pi$, and the value of $\varepsilon$ for which $\pi$ was computed.
- An elementary shortest path problem with resource constraints.
- $\Psi_k$ may only contain nodes of the set
  \[ \{ w_j \in W : \exists v_i \in R_k \text{ with } d_{ij} \leq \varepsilon \} . \]
- Need to modify dominance rule between labels.
IPPS Heuristic for the BOMCTP
IPPS Heuristic for the BOMCTP
Quality Measures \cite{Ehrgott2007}

$$
\mu_1 := \frac{d(L, U)}{\|y_{\text{max}} - y_{\text{min}}\|_2} \\
\mu_2 := \frac{A_L - A_U}{A_L}
$$
Instances

- $|V| + |W|$ random points in the $[0, 100] \times [0, 100]$ square.
- Depot is restricted to lie in the $[25, 75] \times [25, 75]$ square.
- Set $V$ taken as first $|V|$ points; Set $W$ takes remaining points.

Algorithms and coding

- All codes written in C/C++.
- RMP solved with CPLEX 12.4.
- Subproblem solved by DSSR algorithm [Boland et al. (2006), Righini and Salani (2008)].

Computer specifications

- Intel Core 2 Duo, 2.93 GHz, 2 GB RAM.
## Comparison of Approaches and Search Strategies

| $p$ | $|V|$ | $|W|$ | Standard $\mu_1\%$ | $\mu_2\%$ | PPS $\mu_1\%$ | $\mu_2\%$ | IPPS $\mu_1\%$ | $\mu_2\%$ |
|-----|-----|-----|------------------|---------|-------------|---------|----------------|---------|
| 5   | 40  | 80  | 6.41             | 23.22   | 1.07        | 7.69    | 1.01           | 6.09    |
| 5   | 40  | 120 | 4.67             | 23.35   | 0.91        | 11.92   | 0.92           | 10.33   |
| 5   | 50  | 100 | 4.74             | 21.94   | 0.70        | 9.48    | 0.68           | 7.85    |
| 5   | 50  | 150 | 4.33             | 26.44   | 1.20        | 16.67   | 1.20           | 15.41   |
| 8   | 40  | 80  | 8.38             | 25.86   | 0.32        | 3.04    | 0.38           | 2.46    |
| 8   | 40  | 120 | 6.48             | 21.70   | 0.40        | 4.50    | 0.40           | 3.70    |
| 8   | 50  | 100 | 6.13             | 21.83   | 0.32        | 3.83    | 0.31           | 3.65    |
| 8   | 50  | 150 | 5.00             | 18.70   | 0.31        | 4.30    | 0.36           | 3.66    |

**Table:** Quality of Bound Sets
COMPARISON OF APPROACHES AND SEARCH STRATEGIES

Figure: Computational Time (cpu seconds)
Comparison of Approaches and Search Strategies

**Figure:** Number of DSSR Iterations
Conclusions

▶ Methods and models for computing lower bounds are needed in multi-objective optimization.

▶ Application of column generation to multi-objective problems seems to have been overlooked.

▶ Column generation techniques and strategies for single objective problems can easily be extended to bi-objective problems.

▶ Study column generation approach based on other scalarization methods (eg. weighted sum method)
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| $p$ | $|V|$ | $|W|$ | Standard | | PPS | | IPPS |
|-----|-----|-----|---------|-----|-------|-------|
|     |     |     | time    | dsrr| time  | dsrr  | time  | dsrr  |
| 5   | 40  | 80  | 28      | 137 | 50    | 228   | 41    | 155   |
| 5   | 40  | 120 | 39      | 163 | 127   | 330   | 94    | 201   |
| 5   | 50  | 100 | 69      | 197 | 206   | 390   | 154   | 226   |
| 5   | 50  | 150 | 42      | 150 | 393   | 486   | 287   | 247   |
| 8   | 40  | 80  | 61      | 217 | 113   | 302   | 104   | 218   |
| 8   | 40  | 120 | 132     | 299 | 511   | 481   | 504   | 293   |
| 8   | 50  | 100 | 281     | 326 | 1344  | 522   | 1013  | 335   |
| 8   | 50  | 150 | 333     | 380 | 1525  | 672   | 1186  | 384   |

**Table:** Computational Time (cpu seconds)
**Figure**: Computational Time (cpu seconds)
**Figure**: Number of DSSR Iterations