Formulations and valid inequalities for the Vehicle Routing Problem with Intermediate Facilities

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• A mixed graph $G = (V, E \cup A)$ is given where $V = \{0\} \cup V'$
  • $V' = V_C \cup V_F$ ($n = |V'|$);
  • $V_C$: set of customers; $V_F$: set of intermediate facilities;
  • $\{0\}$ is the depot location.
  • $F_i \subseteq V_F$ subset of facilities to which customer $i$ can be assigned.
• $E = \{\{i, j\} : i, j \in V, i \neq j\}$ is the edge set where $e = \{i, j\}$
• $A = \{(i, j) : i \in V_C, j \in F_i\}$ is the arc set.
• A non-negative routing cost $r_e = r_{ij}$, is associated with each edge $\{i, j\} \in E$
• A non-negative assignment cost $d_{ij}$ is associated with each arc $\{i, j\} \in A$.
• A non-negative demand $q_i$ is associated with each customer $i \in V_C$ (we assume $q_0 = 0$ and $q_i = 0$, $\forall i \in V_F$).
• The cost of a route is equal to the sum of the costs of the edges it uses.
The aim of VRPIF is to design $m$ feasible routes such that:

- each customer is assigned to exactly one route,
- each intermediate facility is visited at most once,
- the total load of each route does not exceed the vehicle capacity $Q$.

The objective function consists in minimizing the sum of the costs of routes, where the cost of a route is the sum of the routing costs of the edges forming the route plus the sum of the assignment costs of the arcs in $A'$.

The number $m$ of routes in the network is known and given as an input parameter.
VRPIF: example of a solution
Related problems

VRPIF vs 2-echelon vehicle routing problem (2E-VRP)

Source: Perboli, Tadei, Vigo 2008
Related problems

VRPIF vs multiple traveling purchaser problem

Source: Riera-Ledesma, Salazar-Gonzalez 2012
Related problems

VRPIF vs school bus routing problem

Source: Schittekat, Sevaux, Sorensen, 2006
CmRSP = Capacitated $m$-Ring Star Problem:

- introduced by Baldacci et al in 2007
- problem of designing telecommunication networks
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R. Baldacci, M. Dell’Amico and J. Salazar-González.  
The Capacitated $m$-Ring-Star Problem.  

E. A. Hoshinoa and C. C. de Souza.  

A. Mauttone, S. Nesmachnow, A.O.F.Robledo Amoza.  
Solving a Ring Star Problem generalization.  
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A two-Index Formulation (TI)

Notation

For $S \subseteq V'$, we define

- $V_C(S) = S \cap V_C$,
- $V_F(S) = S \cap V_F$,
- $F_i(S) = V_F(S) \cap F_i$,
- $\overline{S} = V' \setminus S$.

For $S \subseteq V$, we denote

- $q(S) = \sum_{i \in S} q_i$ = total demand of nodes in $S$.
- $\delta(S) = \{\{i, j\} \in E : i \in S, j \notin S\}$
  - if $S = \{i\}$, $\delta(i)$ instead of $\delta(\{i\})$.
- $\overline{S}_0 = V \setminus S$.

Decision variables

- $x_e \in \{0, 1\}$, $\forall e \in E \setminus \{\{0, j\} : j \in V'\}$ and
- $x_e \in \{0, 1, 2\}$, $\forall e \in \{\{0, j\} : j \in V'\}$
- $z_{ij} = 1$ if and only if customer $i$ is assigned to node $j$, 0 otherwise
- $w_i = 1$ if and only if node $i$ is on a route, 0 otherwise
A two-Index Formulation (TI)

\[(TI) \min \sum_{e \in E} r_e x_e + \sum_{(i,j) \in A} d_{ij} z_{ij} \]

\[s.t. \sum_{e \in \delta(0)} x_e = 2m \]

\[\sum_{e \in \delta(i)} x_e = 2w_i, \quad \forall i \in V' \]

\[w_i + \sum_{j \in F_i} z_{ij} = 1, \quad \forall i \in V_C \]

\[\sum_{e \in \delta(S)} x_e \geq \frac{2}{Q} \left( \sum_{i \in V_C(S)} q_i w_i + \sum_{(i,j) \in A: j \in V_F(S)} q_i z_{ij} \right), \quad \forall S \subseteq V' : S \neq \emptyset \]

\[x_e \in \{0, 1\}, \quad \forall e \in E \setminus \{0, j : j \in V'\} \]

\[x_e \in \{0, 1, 2\}, \quad \forall e \in \{0, j : j \in V'\} \]

\[z_{ij} \in \{0, 1\}, \quad \forall (i,j) \in A \]

\[w_i \in \{0, 1\}, \quad \forall i \in V'. \]
A set-partitioning formulation (SP)

A tour $T = (0, i_1, \ldots, i_r, 0)$, with $r \geq 1$, is a simple cycle in $G$ with $V(T) \subseteq V$ that respects capacity constraints $\sum_{i \in V(T)} q_i \leq Q$.

Notation

- $R$ is the index set of all tours
- $E(\ell), \forall \ell \in R$ is the subset of edges of graph $G$ used by the tour.
- $a_{i\ell}, i \in V', \ell \in R$, is the binary coefficient equal to 1 if $i \in V(\ell)$ and 0 otherwise (we assume that $a_{0\ell} = 1, \forall \ell \in R$).

In addition, each tour $\ell \in R$ has

- a cost $c_\ell$ computed as the sum of the edge costs forming the tour.
- coefficients $\eta^\ell_{\{i,j\}}$ defined as follows:
  - if $\ell$ is a tour covering node $h$ only, then $\eta^\ell_{\{0,h\}} = 2$ and $\eta^\ell_{\{i,j\}} = 0$, $\forall \{i,j\} \in E \setminus \{0, h\}$;
  - if $\ell$ is not a single-node tour, then $\eta^\ell_{\{i,j\}} = 1$ for each edge $\{i,j\} \in E(T_\ell)$ and $\eta^\ell_{\{i,j\}} = 0, \forall \{i,j\} \in E \setminus E(T_\ell)$. 
and moreover \( \mathcal{S} = \{ S : S \subseteq V', |S| \geq 2, |S \cap V_F| \geq 1 \} \) and \( \mathcal{R}(S) = \{ \ell \in \mathcal{R} : V(T_\ell) \cap S \neq \emptyset \} \).

Decision variables:

- \( \xi_\ell, \ell \in \mathcal{R} \): binary variable equal to 1 if and only if tour \( \ell \) is in the optimal solution
- variables \( \{x_e\}, \{z_{ij}\} \) and \( \{w_i\} \) defined as for formulation \( TI \).
A set-partitioning formulation (SP)

\[(SP) \quad \min \sum_{\ell \in R} c_{\ell} \xi_{\ell} + \sum_{(i,j) \in A} d_{ij} z_{ij} \quad (10)\]

\[\text{s.t.} \quad \sum_{\ell \in R} a_{i\ell} \xi_{\ell} = w_i, \quad \forall i \in V' \quad (11)\]

\[\sum_{\ell \in R} \xi_{\ell} = m, \quad (12)\]

\[w_i + \sum_{j \in F_i} z_{ij} = 1, \quad \forall i \in V_C \quad (13)\]

\[x_e = \sum_{\ell \in R} \eta_{e\ell} \xi_{\ell}, \quad \forall e \in E \quad (14)\]

\[\sum_{e \in \delta(S)} x_e \geq \frac{2}{Q} \left( \sum_{i \in V_C(S)} q_i w_i + \sum_{(i,j) \in A : j \in V_F(S)} q_i z_{ij} \right), \quad \forall S \in \mathcal{P} \quad (15)\]

\[\xi_{\ell} \in \{0, 1\}, \quad \forall \ell \in R \quad (16)\]

\[x_e \in \{0, 1\}, \quad \forall e \in E \setminus \{\{0, j\} : j \in V'\} \quad (17)\]

\[x_e \in \{0, 1, 2\}, \quad \forall e \in \{\{0, j\} : j \in V'\} \quad (18)\]

\[z_{ij} \in \{0, 1\}, \quad \forall (i, j) \in A \quad (19)\]

\[w_i \in \{0, 1\}, \quad \forall i \in V'. \quad (20)\]
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Valid Inequalities for formulation $TI$

Preliminary computational results
Simple valid inequalities

\begin{align*}
    x\{i,j\} & \leq w_j, \quad \forall i \in V_C, \forall j \in V_C, i \neq j \\
    x\{i,j\} & \leq w_j, \quad \forall i \in V_F, j \in V', i \neq j \\
    x\{i,j\} + z_{ij} & \leq w_j, \quad \forall i \in V_C, \forall j \in F_i
\end{align*}
Connectivity Inequalities (CI)

\[
\sum_{e \in \delta(S)} x_e \geq 2 \left( w_i + \sum_{j \in V_F(S) \cap F_i} z_{ij} \right), \quad \forall S \subseteq V', \forall i \in V_C(S), S \neq \emptyset
\]  

The separation algorithm

1) Makes the right-hand-side constant

\[
\sum_{e \in \delta(S)} x_e + \sum_{j \in V_F(S) \cap F_i} z_{ij} \geq 2 \left( w_i + \sum_{j \in F_i} z_{ij} \right), \quad \forall S \subseteq V', \forall i \in V_C(S), S \neq \emptyset
\]

2) Looks for the min cut s-t on the auxiliary graph where: 0 = s, i = t and each arc \( x_e \) has a cost equal to the value of the optimal LP solution value \( x_e^* \) (or \( x_e^* + z_{ij}^* \))
Inequalities (26) dominate inequalities (5).

The separation algorithm:

We use the same algorithm used for inequalities (5)
Rounded Capacity Constraints

\[ \sum_{e \in \delta(S)} x_e \geq 2 \left[ \sum_{i \in S: F_i \subseteq S} \frac{q_i}{Q} \right], \quad \forall S \subseteq V', \ V_c(S) \neq \emptyset \quad (27) \]

The separation algorithm

We use a Multi-Start Local Search
These constraints are non linear.
Given the following lemma constraints (28) can be linearized:

**Lemma:**
Given three non negative integer values \(v\), \(y\) and \(b\), such that \(v > b\) and \(\text{mod} (v, b) \neq 0\), equation (29) holds.

\[
\left\lceil \frac{v - y}{b} \right\rceil \geq \left\lceil \frac{v}{b} \right\rceil - \frac{y}{\text{mod} (v, b)}.
\] (29)
Inequalities linearization

The rhs of (28) can be rewritten as follows:

\[
q(V_C) - \left( \sum_{i \in V_C(S)} q_i w_i + \sum_{(i,j) \in A: j \in V_F(S)} q_i z_{ij} \right) \quad \text{or} \quad q(V_C(S)) - \left( \sum_{(i,j) \in A: i \in V_C(S), j \in V_F(S)} q_i z_{ij} - \sum_{(i,j) \in A: i \in V_C(S), j \in V_F(S)} q_i z_{ij} \right)
\]  

(30)

Using the lemma we can write:

\[
\sum_{e \in \delta(S)} \frac{1}{2} x_e \geq \left\lceil \frac{q(V_C)}{Q} \right\rceil - \frac{1}{\text{mod}(q(V_C), Q)} \left( \sum_{i \in V_C(S)} q_i w_i + \sum_{(i,j) \in A: j \in V_F(S)} q_i z_{ij} \right)
\]  

or

\[
\sum_{e \in \delta(S)} \frac{1}{2} x_e \geq \left\lceil \frac{q(V_C(S))}{Q} \right\rceil - \frac{1}{\text{mod}(q(V_C(S)), Q)} \sum_{i \in V_C(S), j \in V_F(S)} q_i z_{ij}
\]  

(31)

(32)

There is no dominance relation between (31) and (32)
An example

Given \( S = \{1, 2, 4, 7\} \) customers \( \{1, 2, 4\} \) on a route while customers \( \{3, 5, 6\} \) assigned to facility node 7 of \( S \).

The right-hand side of (31) becomes \( \lceil 18/10 \rceil - 0 = 2 \), while the right-hand side of (32) has value \( \lceil 9/10 \rceil - 0 = 1 \).

Given \( S = \{1, 2, 3, 4, 5\} \) all visited on a route. Customer 6 is associated to a node in \( V' \setminus S \).

The right-hand side of (31) takes value \( \lceil 18/10 \rceil - 1/2 = 1.5 \), while the right-hand side of (32) becomes \( \lceil 16/10 \rceil - 0 = 2 \).
Further valid inequalities

\[ \sum_{e \in \delta(S)} \min \left\{ \frac{Q}{2 \mod(q(V_C), Q)}, 1 \right\} x_e \geq \left\lfloor \frac{q(V_C)}{Q} \right\rfloor - \]
\[ \sum_{i \in V_C(S)} \min \left\{ \frac{q_i}{\mod(q(V_C), Q)}, 1 \right\} w_i - \sum_{(i,j) \in A: j \in V_F(S)} \min \left\{ \frac{q_i}{\mod(q(V_C), Q)}, 1 \right\} z_{ij}. \] 

(33)

\[ \sum_{e \in \delta(S)} \min \left\{ \frac{Q}{2 \mod(q(V_C(S)), Q)}, 1 \right\} x_e \geq \left\lfloor \frac{q(V_C(S))}{Q} \right\rfloor - \]
\[ \sum_{(i,j) \in A: i \in V_C(S), j \in V_F(S)} \min \left\{ \frac{q_i}{\mod(q(V_C(S)), Q)}, 1 \right\} z_{ij} \] 

(34)
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Implementation details and Instances used

Implementation

- C + Cplex (Cutting Plane)
- Intel Core 2Duo, 2.66 GHz
- 4Go de RAM

Instances:

- CmRSP:
- VRPIF: generated using LRP instances
The results obtained are competitive with the literature, although no dedicated or strengthened cut for the CmRSP was used.
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We can expect the inequalities introduced for the VRPIF to improve significantly to the efficiency of the future column-and-cut generation algorithm.
Conclusions

We introduced the vehicle Routing Problem with intermediate facilities

The problem differs from the problems already proposed in the literature such as for example the school bus routing problem and the multiple traveling purchaser problem

The preliminary results show that the first formulation solved with a branch-and-cut algorithm is competitive with the literature when used for solving the Capacitated \( m \)-Ring Star Problem

Work in progress: new valid inequalities and a branch-and-cut-and-price based on the set partitioning like formulation.