The Bi-objective Multi-Vehicle Covering Tour Problem (BOMCTP): formulation and lower-bound computation

Boadu Mensah SARPONG

Directors:
Nicolas JOZEFOWIEZ
Christian ARTIGUES

09/02/2012
1 Methods and Approaches
- Multi-objective optimization problems
- Column generation

2 The Bi-Objective Multi-Vehicle Covering Tour Problem
- Motivation and description
- A set-covering model for the BOMCTP
- Definition of sub-problem

3 Conclusion
Definition of a multi-objective optimization problem

\[(MOP) = \begin{cases} 
\min F(x) = (f_1(x), f_2(x), \ldots, f_n(x)) \\
\text{s.t. } x \in \Omega 
\end{cases}\]

where:

- \( n \geq 2 \): number of objective functions
- \( F = (f_1, f_2, \ldots, f_n) \): vector of objective functions
- \( \Omega \subseteq \mathbb{R}^m \): feasible set of solutions
- \( \mathcal{Y} = F(\Omega) \): feasible set in objective space
- \( x = (x_1, x_2, \ldots, x_m) \in \Omega \): variable vector, variables
- \( y = (y_1, y_2, \ldots, y_n) \in \mathcal{Y} \) with \( y_i = f_i(x) \): vector of objective function values
**Dominance and Pareto Optimality**

A solution \( x \) dominates (\( \preceq \)) another solution \( y \) if and only if \( \forall i \in \{1, \ldots, n\}, f_i(x) \leq f_i(y) \) and \( \exists i \in \{1, \ldots, n\} \) such that \( f_i(x) < f_i(y) \).

**Pareto optimal solution**

A solution is said to be Pareto optimal if no other feasible solution dominates it.
Lower bound of a MOP
Lower bound of a MOP
Lower bound of a MOP

\[ f_2 \]

\[ f_1 \]

\[ \text{lb}_1 \]
Lower bound of a MOP

\[ f_1 \leq lb_1 \]
\[ f_2 \leq lb_2 \]
LOWER BOUND OF A MOP
Lower bound of a MOP

\[ f_1 \text{lb}_1 \]

\[ f_2 \]

\[ \text{ideal point} \]

\[ \text{lb}_2 \]

\[ \text{lb}_1 \]
LOWER BOUND OF A MOP
**What is column generation?**

A method for solving LPs in which there are very large number of variables, without having to enumerate all the variables a priori.

- Based on the principles of LP decomposition
- Useful as a heuristic for computing the lower bound

**Where has it been applied?**

- Vehicle routing problems
- Multi-commodity flow problems
- Cutting stock problems
- Binary cutting stock problems
- Crew rostering
- Optical telecommunications design
- etc.
**Definitions**

**Master Problem (MP)**

The original LP with a huge number of columns (variables).

**Restricted Master Problem (RMP)**

A linear relaxation of MP with only a subset of the original columns.

- Initial columns must be chosen so that the RMP is feasible.

**Sub-problem**

A problem solved to determine which columns are beneficial to add to RMP.
APPROACH 1: POINT-BY-POINT SEARCH
APPRAOCH 1: POINT-BY-POINT SEARCH
APPROACH 1: POINT-BY-POINT SEARCH
APPROACH 1: POINT-BY-POINT SEARCH
APPROACH 1: POINT-BY-POINT SEARCH
Approach 1: point-by-point search
Approach 1: Point-by-Point Search
APPROACH 1: POINT-BY-POINT SEARCH
APPROACH 1: POINT-BY-POINT SEARCH
APPROACH 1: POINT-BY-POINT SEARCH
APPROACH 1: POINT-BY-POINT SEARCH
APPROACH 1: POINT-BY-POINT SEARCH
APPROACH 1: POINT-BY-POINT SEARCH
APPROACH 1: POINT-BY-POINT SEARCH
Approach 1: point-by-point search
Approach 1: point-by-point search
Approach 1: point-by-point search
APPRAOCH 1: POINT-BY-POINT SEARCH
APPROACH 2: SEQUENTIAL SEARCH
**Approach 2: Sequential Search**

\[
\begin{align*}
\varepsilon_0 & \quad f_2 \\
\varepsilon_1 & \quad f_1
\end{align*}
\]

\[\text{Graph showing points at } \varepsilon_0 \text{ and } \varepsilon_1 \text{ on the } f_1 \text{ axis.} \]
**Approach 2: Sequential Search**
Approach 2: sequential search
**Approach 2: Sequential Search**
**Approach 2: Sequential Search**

- $f_2$
- $f_1$
- $\varepsilon_0$
- $\varepsilon_k$
- $\varepsilon_{k-1}$
- $\varepsilon_2$
- $\varepsilon_1$
- $\varepsilon_3$

generate $m/k$ columns for RMP

generate $m/k$ columns for RMP

generate $m/k$ columns for RMP

generate $m/k$ columns for RMP

generate $m/k$ columns for RMP

generate $m/k$ columns for RMP
Approach 2: Sequential Search
Approach 2: sequential search
APPROACH 2: SEQUENTIAL SEARCH
Approach 2: sequential search
**Approach 2: Sequential Search**
Approach 2: sequential search
**Approach 2: Sequential Search**

```
generate m columns 
starting with those that 
 affect the most points
```
APPROACH 2: SEQUENTIAL SEARCH
APPROACH 2: SEQUENTIAL SEARCH
Approach 2: sequential search
**Approach 2: Sequential Search**

![Graph showing sequential search process with labels ε_0, ε_1, ε_2, and ε_k along the y-axis and f1 along the x-axis. The graph illustrates the sequential search with points marked at ε_0, ε_1, ε_2, and ε_k.]
Approach 2: Sequential Search
**The Multi-Vehicle CTP** [Hodgson *et al.*, 1998]

Find a set of at most $m$ tours on $V' \subseteq V$, having minimum total length and such that the nodes of $W$ are covered by those of $V'$.

- Each route must be of length less than a preset value $p$.
- The number of vertices on each route cannot exceed a preset value $q$.

![Diagram of vehicle routes and covered area]

- **May be visited**
- **MUST be visited**: $T$
- **MUST be covered**: $W$
- **Vehicle routes**
- **Cover distance**
**Description of the BOMCTP**

**Problem**

Given a graph $G = (V \cup W, E)$ with $T \subseteq V$, design a set of at most $m$ vehicle routes on $V' \subseteq V$.

**Objectives**

- Minimize the total length of the set of routes.
- Minimize the cover distance induced by the set of routes.

**Constraints**

- Each vertex of $T$ must belong to a vehicle route.
- Each vertex of $W$ must be covered.
- Each route must be of length less than a preset value $p$.
- The number of vertices on each route cannot exceed a preset value $q$. 
A set-covering model for the BOMCTP

Variables

- \( \Omega \): set of all feasible routes
- \( r_k \in \Omega \): feasible route \( k \)
- \( c_k \): cost of route \( r_k \)
- \( \theta_k \): 1 if route \( r_k \) is selected in solution and 0 otherwise
- \( z_{ij} \): 1 if vertex \( v_j \in V \) is used to cover vertex \( w_i \in W \) and 0 otherwise
- \( a_{ik} \): 1 if \( r_k \) uses vertex \( v_i \in V \) and 0 otherwise
- \( Cov_{max} \): cover distance induced by a set of routes

Objective functions

\[
\text{minimize } \sum_{r_k \in \Omega} c_k \theta_k
\]
\[
\text{minimize } Cov_{max}
\]
A set-covering model for the BOMCTP

**Constraints**

\[-z_{ij} + \sum_{r_k \in \Omega} a_{jk} \theta_k \geq 0 \quad (w_i \in W, v_j \in V)\]

\[\sum_{r_k \in \Omega} a_{jk} \theta_k \geq 1 \quad (v_j \in T)\]

\[\text{Cov}_{\text{max}} - c_{ij} z_{ij} \geq 0 \quad (w_i \in W, v_j \in V)\]

\[\sum_{v_j \in V} z_{ij} \geq 1 \quad (w_i \in W)\]

\[\text{Cov}_{\text{max}} \geq 0\]

\[z_{ij} \in \mathbb{N} \quad (w_i \in W, v_j \in V)\]

\[\theta_k \in \mathbb{N} \quad (r_k \in \Omega)\]
The Restricted Master Problem (RMP)

minimize $\sum_{r_k \in \Omega_1} c_k \theta_k$

Constraints

$- z_{ij} + \sum_{r_k \in \Omega_1} a_{jk} \theta_k \geq 0 \quad (w_i \in W, v_j \in V)$

$\sum_{r_k \in \Omega_1} a_{jk} \theta_k \geq 1 \quad (v_j \in T)$

$Cov_{\text{max}} - c_{ij} z_{ij} \geq 0 \quad (w_i \in W, v_j \in V)$

$\sum_{v_j \in V} z_{ij} \geq 1 \quad (w_i \in W)$

$- Cov_{\text{max}} \geq -\varepsilon$
The Restricted Master Problem (RMP)

\[
\text{minimize } \sum_{r_k \in \Omega_1} c_k \theta_k
\]

Constraints

\[
\begin{align*}
- z_{ij} &+ \sum_{r_k \in \Omega_1} a_{jk} \theta_k \geq 0 & (w_i \in W, v_j \in V) & \alpha_{ij} \\
\sum_{r_k \in \Omega_1} a_{jk} \theta_k &\geq 1 & (v_j \in T) & \varphi_j \\
Cov_{\text{max}} - c_{ij} z_{ij} &\geq 0 & (w_i \in W, v_j \in V) & \gamma_{ij} \\
\sum_{v_j \in V} z_{ij} &\geq 1 & (w_i \in W) & \beta_i \\
- Cov_{\text{max}} &\geq -\varepsilon & & \lambda
\end{align*}
\]
Dual of RMP

\[
\text{maximize } -\varepsilon \lambda + \sum_{w_i \in W} \beta_i + \sum_{v_j \in T} \varphi_j \\
\text{subject to: }
\sum_{w_i \in W} \sum_{v_j \in T} a_{jk} \alpha_{ij} + \sum_{v_j \in T} a_{jk} \varphi_j \leq c_k \quad (r_k \in \Omega_1)
\]

\[
-\lambda + \sum_{w_i \in W} \sum_{v_j \in V} \gamma_{ij} \leq 0
\]

\[
-c_{ij} \gamma_{ij} + \beta_i - \alpha_{ij} \leq 0 \quad (w_i \in W, v_j \in V)
\]
**Definition of sub-problem**

Find routes such that \( c_k - \sum_{w_i \in W} a_{jk} \alpha_{ij} - \sum_{v_j \in V} a_{jk} \varphi_j < 0. \)

- Let \( \alpha_{ij}^* = \alpha_{hi} \) if \( v_j \in V \) and 0 otherwise.
- Let \( \varphi_j^* = \varphi_j \) if \( v_j \in T \) and 0 otherwise.
- Let \( A \) be the set of arcs formed between two nodes of \( V \).
- Let \( x_{ijk} = 1 \) if route \( r_k \) uses arc \((v_i, v_j)\) and 0 otherwise.

**Note:** \( c_k = \sum_{(v_i, v_j) \in A} x_{ijk} c_{ij} \) and \( a_{jk} = \sum_{\{v_i \in V | (v_i, v_j) \in A\}} x_{ijk} \)

So \( \sum_{(v_i, v_j) \in A} c_{ij} x_{ijk} - \sum_{(v_i, v_j) \in A} \sum_{v_h \in W} \alpha_{hj}^* x_{ijk} - \sum_{(v_i, v_j) \in A} \varphi_j^* x_{ijk} < 0. \)
**Definition of sub-problem**

\[
\sum_{(v_i, v_j) \in A} \left( c_{ij} - \varphi_j^* - \sum_{v_h \in W} \alpha_{hj}^* \right) \chi_{ijk} < 0.
\]

**Sub-problem**

Find elementary paths from the depot to the depot with a negative cost, satisfying the constraints of length and maximum number of vertices on a path. Costs are set to

\[
c_{ij} - \varphi_j^* - \sum_{v_h \in W} \alpha_{hj}^*.
\]

- An ESPPRC
Conclusion

Work done

- Master problem
- Reduced costs matrix
- Sub-problem (MOESPPRC)
- Column generation (point-by-point search, sequential search)

Preliminary results

\[ p = 8, \ q = +\infty, \ T = \{ v_0 \} \]

| \( |V| \) | \( |W| \) | Nb points in lb set | nb columns generated | time (sec) | Nb columns generated | time (sec) |
|-------|-------|----------------|---------------------|------------|---------------------|------------|
| 40    | 160   | 52             | 8                   | 8.56       | 9                   | 39.28      |
| 60    | 140   | 45             | 5                   | 15.35      | 6                   | 62.04      |
| 80    | 120   | 51             | 11                  | 37.33      | 13                  | 169.05     |
THANK YOU FOR YOUR ATTENTION.